Basic Logic and Fregean Set Theory[1]

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1 Introduction

It is tempting to conclude that classical mathematics is vindicated by its success, in particular by its applicability to the natural sciences. Constructive mathematics and intuitionism, on the other hand, lost by being too puritanical, and by making it too hard to prove results constructively that are easily shown to be true by classical means. Its sole rôle is that of servant to classical mathematics through model theory or topos theory, and through its assistance to (classical) computer algebra and their likes (such a serving rôle need not be interpreted as degrading). Some careful observations, however, reveal a more complicated picture.

When one looks at the applications of classical mathematics to the natural sciences and in engineering, then a significant part appears to be completely constructive, and even essentially computational, in nature. At that level distinctions between constructive and classical mathematics are fictitious. This observation is partly an argument in favor of classical mathematics, since it allows for simpler methods while obtaining the same results as constructivism. But the sameness disappears at a deeper level. For instance, in [13] and [14] (see also [1]) Pour-El and Richards show that a certain recursively initialized boundary value problem in physics has a solution that is not recursive. Some interpret this as saying that nature is not bound by recursion theory. There is also another interpretation. The recursively initialized boundary value problem of Pour-El and Richards is equivalent to a boundary value problem in constructive mathematics for which one cannot constructively show the existence of a solution. Maybe constructive mathematics signals that such a solution is not physically feasible or, more likely, that the classical model used for the physical phenomenon is an imperfect approximation of physics that fails in this extreme case, as shown by the lack of a constructively derivable solution. Although constructivism was not introduced for this purpose, we cannot exclude the possibility that better models of nature are obtained through some theory involving a constructive logic. From a formal point of view classical mathematics is a theory extending constructive mathematics with principles including the Principle of the Excluded Third $A \vee \neg A$. So the model using classical logic may be a close approximation of a 'correct' but more cumbersome model using some form

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of constructive logic, only failing in cases of the kind mentioned above. In areas like computer algebra constructive logic may perform relatively more prominent functions. The idea of using models of nature with a logic different from the classical one is not new. Quantum logic has been used to model quantum mechanical phenomena. In this paper we restrict ourselves to constructive logics.

The pleasing image of a constructive logic and theory that is 'more correct' but less efficient than its narrow extension in classical mathematics, suggests a new possibility: Are there extensions of constructive logic that may be just as valuable as classical logic, but that are relatively inconsistent with classical logic? The answer is a resounding yes, and examples date back to Brouwer himself, who introduced 'new' principles in intuitionistic mathematics with which one can prove that all functions on the Cauchy reals with compact support are uniformly continuous. Similar results in differential geometry, accompanied by interesting models, are discussed in [8] and [11]. These theories include first-order fragments satisfying principles like $\neg\forall x(Ax \vee \neg Ax)$ for certain Ax. These extensions are not out of reach of the classical mathematician. By the completeness theorems, through reinterpretation of intuitionistic statements into classical mathematics using realizability, sheaf models, and so on, classical mathematics is able to model these intuitionistic theories. As important difference remains that the original intuitionistic theories tend to select the interesting structures from the chaff of these models, and often lead to more efficient and more natural derivations of the relevant properties.

It appears that classical mathematics and logic is just one of several useful extensions of constructive logic. There are of course also variations on constructive logic, if only by just making up arbitrary rules for the logical constants. Most, but not all, of these are utterly useless. Logics that are motivated by interesting philosophical or technical principles have a better chance of being worthy of our consideration. In this paper we restrict our attention to (first-order) logics that are motivated by some form of constructivism. How arbitrary or unique is constructive logic? There are several schools of constructive mathematics, all with their own philosophies. Although they differ from one another in principles as well as in some parts of their mathematics, they all agree, be it sometimes grudgingly, on the same first-order constructive logic: Intuitionistic logic. This logic, and some further principles on the existence of natural numbers and sets, is recognized by all leading schools, and may be considered the basic constructive logic mentioned in the paragraphs above. This convergence of forces, and the additional connection with topos theory, shows that it is a 'right' generalization of classical logic and mathematics [10, page 103]. The picture as sketched suggests that there is essentially one constructive logic, intuitionistic logic, of which classical logic is one of several interesting extensions. However, first impressions may deceive us again.

For many constructivists constructive logic is an afterthought, being secondary to constructive mathematics itself. Logic is useful in clarifying constructive principles, but is itself supposed not to contribute to the principles of constructivism. This complacent approach forms the weak link between constructivism and logic. There are some attempts at more precisely clarifying the logical connectives. The most well-known is the proof interpretation of Heyting and Kolmogorov [6], and a variation on it that was once proposed, for different purposes, by Kreisel [9]. The proof interpretation of intuitionistic first-order logic is generally considered an explanation rather than an interpretation of the first-order logical connectives. Many details of it have been the subject of debate and criticism, and the existence of the variation by Kreisel already indicates that constructivists generally feel that the explanation is not completely satisfactory. The weakest spots are associated with the interpretations of implication and universal quantification. Usually constructivists expect that a more detailed study of the logical operations will result in an improved interpretation that will confirm, or at least support, intuitionistic first-order logic. Bishop, for example, questioned the contemporary understanding of implication [2], but found that wherever he used it in his own work on constructive analysis, he was able to give additional numerical justification for its application. Significantly, Bishop did not express any expectation that the rules of intuitionistic logic themselves should be questioned from his constructive point of view. In this paper we show that a different approach to the proof interpretation results in a new constructive logic that is a proper subsystem of intuitionistic logic.

In Section 2 we mention the proof interpretation of Heyting and Kolmogorov, and the variation by Kreisel, and compare these with a new proof interpretation. In Section 3 we describe a new first-order logic, Basic Predicate Calculus BQC, that is motivated by the new proof interpretation, and that is a proper subsystem of Intuitionistic Predicate Calculus IQC. Both IQC and Formal Predicate Calculus FQC are introduced as extensions of BQC. The extension FQC is associated with the provability logic PrL of [17] (called G in [4]) in the same way that IQC is associated with the modal logic S4 (and BQC with K4). The theories FQC and IQC are relatively inconsistent. In Section 4 we introduce Fregean set theories over BQC. Over F the traditional proof of the Russell Paradox turns into a derivation of the additional axiom schema of FQC. This result illustrates that logic sometimes doesn't precede set theory. Variations on F also allow for proofs of the additional axiom schema. Fregean set theories may be new useful extensions of BQC (and FQC) that are not extensions of intuitionistic logic IQC.

2 The Proof Interpretation

In classical mathematics one assumes a logic for a mathematical world where each statement is either true or false. This is motivated by our experience with mathematical statements about finite structures, and with certain elementary statements about the natural numbers, and consequently generalized to other mathematical structures. For finite discrete structures, and for certain fragments of number theory (hence also for certain fragments of the theories of integers, rationals, and so on), this understanding is almost universally accepted, though maybe for varying philosophical reasons. Several mathematicians and philosophers have questioned the generalizations. One alternative has been the introduction of notions of mathematical truth based on proofs, thereby introducing more rigorous, hence more reliable, 'definitions' that can more safely be extended to new mathematical structures. An alternative approach is restricting the number of possible mathematical structures, but we don't expect to ever reduce their number so much that logic becomes simple. So the constructive variations on the classical notion of truth have priority.

Heyting, and independently Kolmogorov, initiated the first serious attempts at explaining the logical constants of first-order logic in terms of proofs. Their explanations are generally considered equivalent. Heyting based his version on the use of the logical constants in the intuitionistic literature, in particular on their use by Brouwer. The resulting Brouwer-Heyting-Kolmogorov BHK interpretation, or explanation [19], expresses the provability, hence validity, of first-order statements in terms of proofs and constructions concerning their parts. The validity of atomic sentences is supposed to be provided with the model, say $\mathcal{D}$.

- $\top$ is true by itself. that is, the empty proof suffices. There is no proof for $\bot$.
- A proof p of $A \wedge B$ consists of a pair of proofs q, r such that q is a proof of A and r is a proof of B.
- A proof p of $A \vee B$ consists of a pair n, q with n an integer such that either $n = 0$ and q is a proof of A, or $n = 1$ and q is a proof of B.
- A proof p of $\neg A$ is a construction that converts hypothetical proofs of A into a proof of $\bot$.
- A proof p of $A \to B$ is a construction that converts proofs of A into proofs of B.
- A proof p of $\forall x A x$ is a construction which for each construction c of an element d of the domain of $\mathcal{D}$ produces a proof $p(c)$ of Ad.
- A proof p of $\exists x A x$ consists of a pair c, q, where c is the construction of an element d of the domain of $\mathcal{D}$, and q is a proof of Ad.

The clause for negation is commonly considered a special case of implication under the usual translation of $\neg A$ into $A \to \bot$. A statement is *true* if there exists a proof for it. The BHK interpretation does not express how to verify a proof when we see one. This may be obtained if we accept the following modifications to the interpretation, as introduced by Kreisel [9]: Proofs of implications and universal quantifications need additional evidence that they 'work'.

- A proof p of $A \to B$ is a pair q, r such that q is a construction that converts proofs of A into proofs of B, and r is a proof that q is such a construction.
- A proof p of $\forall x A x$ is a pair q, r such that q is a construction which for each construction c of an element d of the domain of $\mathcal{D}$ produces a proof $q(c)$ of Ad, and r is a proof that q is such a construction.

The proof interpretation leaves several questions. What is the nature of proof versus that of construction? Do the explanations for implication imply that there is a universe of all possible proofs? Can we apply the proof interpretation to this 'universe' of proofs and have an intuitionistic mathematics of proofs [9]? Is the additional evidence proposed by Kreisel of the same nature as the original proofs or constructions?

The proof interpretation is not reductive. The meaning of the logical constants is not explained in simpler terms. One way to make it so is by assuming the existence of a universe U of proofs. To make the interpretation reductive, this universe should in general not be more complicated than the theory of which it is the universe of proofs. Otherwise the interpretations of implication and universal quantification, where one quantifies over the elements of U, would create an interpretation more complicated than the theory that we started with. Kleene's realizability interpretation for Heyting

Arithmetic may be considered a proof interpretation in which there is a universe of 'proofs' encoded in numbers. However, a formula of first-order arithmetic is realized by a number if and only if that formula is derivable from HA plus a formalized version of Church's Thesis [20, page 196]. It appears inevitable that if a theory is sufficiently strong, then the existence of a universe of proofs requires some kind of Church's Thesis.

In the absence of a universe of proofs we must look for a different approach to make the interpretation reductive. Below we change the proof interpretation in such a way that we no longer quantify over all proofs. On one hand this removes the need for additional evidence as proposed by Kreisel. On the other hand we need a new constructive logic. The following is the interpretation that naturally follows.

- $\top$ is true by itself. that is, the empty proof suffices. There is no proof for $\bot$.
- A proof p of $A \wedge B$ consists of a pair of proofs q, r such that q is a proof of A and r is a proof of B.
- A proof p of $A \vee B$ consists of a pair n, q with n an integer such that either $n = 0$ and q is a proof of A, or $n = 1$ and q is a proof of B.
- A proof p of $\neg A$ is a construction that uses the assumption A to produce a proof of $\bot$.
- A proof p of $A \rightarrow B$ is a construction that uses the assumption A to produce a proof of B.
- A proof p of $\forall x A x$ is a construction that uses the assumption that an element d has been constructed to produce a proof of Ad.
- A proof p of $\exists x A x$ consists of a pair c, q, where c is the construction of an element d of the domain of $\mathcal{D}$, and q is a proof of Ad.

Negation is a special case of implication. Using assumption A, rather than a proof of A, to produce a proof of B avoids the need for converting proofs as in the BHK interpretation. It also makes it harder to prove B, since less information is provided. Assumptions are like boxes with on the outside a description of their contents. We may pretend that the boxes contain what the outside says, as long as we are not required to actually open them and use their contents to proceed. If the outside of the box describes a more involved logical structure, then we are allowed to use the proof interpretation of its logical connectives as a more detailed description of the contents of the box. So a box that is assumed to contain a proof of $A \wedge B$ may be assumed to consist of two boxes, one with a proof of A, and one with a proof of B. A box that is assumed to contain a proof of $A \vee B$ contains a box with a number that equals 0 or 1, and a box that contains a proof of A or B depending on whether the number equals 0 or 1 (we cannot look inside either one of these boxes). If the box is assumed to contain a proof of $(A \vee B) \rightarrow C$, then it is assumed to contain a construction p that uses an assumed proof of $A \vee B$ to produce a proof of C. This construction p, therefore, assumes that there is a pair n, q, where n equals 0 or 1 and so on, to produce a proof of C, but without being able to know whether n equals 0 or 1. So the assumed proof p is a construction that must enable us to produce a proof of C from A, and to produce a proof from B. So we can produce a box that is supposed to prove $(A \rightarrow C) \wedge (B \rightarrow C)$. So

$$((A \vee B) \rightarrow C) \rightarrow ((A \rightarrow C) \wedge (B \rightarrow C))$$

holds. From assumption A it is easy to produce a proof of A, namely the trivial proof of A by assumption. So the implication $A \to (\top \to A)$ holds (and so does $A \to A$). But A need not follow from the assumption $\top \to A$, for a proof of $(\top \to A) \to A$ is a construction p that uses the assumption that there is a construction q that produces a proof of A to produce a construction that produces a proof of A. However, the construction q is assumed to exist rather than explicitly given. So we may not be able to deliver p without the assumption $\top \to A$ being 'satisfied', that is, being proven. Following the metaphor above, we would have had to open the assumption 'box' before proceeding. A proof p of $(\top \to A) \to (\top \to A)$, by contrast, can use the assumed construction q for a proof of A to construct a construction, that is supposed to produce a proof of A, by just echoing q.

In general $A \to (B \to C)$ and $B \to (A \to C)$ are not equivalent, and also essentially weaker than $(A \wedge B) \to C$. This has significant consequences for universal quantification. The meaning of $\forall x A$ is essentially equivalent to the meaning of an implication $Ex \to A$, where E is the extent operator of Heyting and Scott [16]. So $\forall x \forall y A$ is essentially equivalent to $Ex \to (Ey \to A)$. We wish to also have a universal quantification that is essentially equivalent to $(Ex \wedge Ey) \to A$. It suffices to include universal quantification over pairs of elements $\forall (x, y) A$, and similarly over longer finite sequences of variables. We will often use boldface characters to represent finite sequences. Another enrichment is obtained by a quantification that allows for equivalence with implications $A \to B$ in general. Therefore our new universal quantification looks like $\forall \mathbf{x} : A.B$, which is essentially equivalent to $(Ex_1 \wedge \ldots \wedge Ex_n \wedge A) \to B$.

The new proof interpretation resolves a drawback that the BHK interpretation experiences with the axiomatic method in mathematics. With the new interpretation a new axiom, say A, is an assumption that works like a sealed box with a proof that is only needed as far as the logical structure of A is concerned. So there is no reason to worry about the further origins of the assumed proof. With the BHK interpretation, however, a proof of, say, $A \to B$ may use every aspect of the proof of the axiom A in deriving B. There may be some hidden structure among all possible proofs of A that can be used to prove B, without our knowing of the existence of this hidden structure. Thus $A \to B$ may be provable without our being able to recognize its proof. Kreisel tried to resolve this issue by requiring additional evidence, but the exact form of this additional evidence may itself never become clear.

3 Basic Logic

The proof interpretation of the previous Section is the basis for our axiomatization of Basic Predicate Calculus BQC. The proof interpretation of first-order logic without implication and universal quantification is the same for both the BHK interpretation and the new interpretation. Therefore we start with the axiomatization of the fragment of BQC built up from the atomic formulas and $\top$ and $\bot$ with the logical connectives $\wedge$, $\vee$, and $\exists$ only. Since we give an axiomatization using sequents $A \Rightarrow B$, this theory turns out to be exactly geometric logic. A sequent $A \Rightarrow B$ also represents that there exists a proof of B from the assumption A. Structurally geometric logic essentially behaves like a distributive lattice, with indexed colimits that commute with finite limits. For the rules a single horizontal line means that if the sequents above the line hold, then so do the ones below the line. A double line means the same, but in both directions.

$$A \Rightarrow A$$

$$\frac{A \Rightarrow B \quad B \Rightarrow C}{A \Rightarrow C}$$

$$A \Rightarrow \top \qquad\qquad \bot \Rightarrow A$$

$$\frac{A \Rightarrow B \quad A \Rightarrow C}{A \Rightarrow B \wedge C} \qquad \frac{B \Rightarrow A \quad C \Rightarrow A}{B \vee C \Rightarrow A}$$

$$\frac{Ax \Rightarrow Bx}{At \Rightarrow Bt}\ \dagger$$

$$\frac{B \Rightarrow A}{\exists x B \Rightarrow A}\ \ddagger$$

$$A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$$

$$A \wedge \exists x B \Rightarrow \exists x(A \wedge B)\ \ddagger$$

We allow the substitution of new variables for bound variables. In case †, the term t does not contain a variable bound by a quantifier of A or B; in cases ‡, the variable x is not free in A. As theory of equality we have

$$\top \Rightarrow x = x$$

$$x = y \wedge Ax \Rightarrow Ay\ *$$

In case $*$, the variables x, y are not bound by a quantifier of A. The theory above is called *geometric logic*. A theory of sequents from this sublanguage is called a *geometric theory*. From [15] we see that even classical logic CQC with Excluded Middle, and with its additional rules for implication and universal quantification, is conservative over geometric logic. Intuitionistic Predicate Calculus and Basic Predicate Calculus diverge in their rules for implication and universal quantification. BQC only satisfies:

$$\frac{A \wedge B \Rightarrow C}{A \Rightarrow B \to C}$$

$$\frac{A \Rightarrow B \to C}{A \Rightarrow \forall \mathbf{x} : B.C}\ \ddagger$$

We must also add the 'formalized' versions of some of the rules of $\Rightarrow$ to make $\to$ its 'faithful' reflection:

$$(A \to B) \wedge (B \to C) \Rightarrow A \to C$$

$$(A \to B) \wedge (A \to C) \Rightarrow A \to (B \wedge C)$$

$$(B \to A) \wedge (C \to A) \Rightarrow (B \vee C) \to A$$

$$A \to (B \to C) \Rightarrow A \to \forall \mathbf{x} : B.C\ \ddagger$$

$$B \to A \Rightarrow \exists x B \to A\ \ddagger$$

In cases ‡, the variable x and the variables in the finite sequence $\mathbf{x}$ are not free in A. This completes the axiomatization of BQC. The expressions $\neg A$ and $A \leftrightarrow B$ are abbreviations of $A \to \bot$ and $(A \to B) \wedge (B \to A)$ respectively. We write $\Rightarrow A$ for $\top \Rightarrow A$, and $A \Leftrightarrow B$ as short for $A \Rightarrow B$ plus $B \Rightarrow A$.

Unless specified differently, theories extending BQC are formed by adding axioms in the form of sequents. It is also possible to form extensions by adding new rules. A theory S is a subtheory of a theory T if T not only satisfies the new sequents of S, but also satisfies the new rules of S. BQC, for example, is closed under all instances of the rule

$$\frac{\Rightarrow \top \to A}{\Rightarrow A}$$

but does not satisfy the rule, and it also has an extension that is not closed under this rule [15]. We write $T \vdash (A \Rightarrow B)$ if the sequent $A \Rightarrow B$ follows from BQC augmented with the additional sequents and rules of T. Intuitionistic Predicate Calculus IQC is axiomatized by the additional sequent schema $\top \to A \Rightarrow A$. This is equivalent to adding the rule of Modus Ponens [15]

$$\frac{A \Rightarrow B \to C}{A \wedge B \Rightarrow C}.$$

Classical Predicate Calculus CQC equals IQC extended with the sequent schema $\top \Rightarrow A \vee \neg A$. Formal Predicate Calculus FQC is axiomatized by the additional sequent schema (Löb's Axiom) $(\top \to A) \to A \Rightarrow \top \to A$. This is equivalent to adding Löb's Rule [21]

$$\frac{A \wedge (\top \to B) \Rightarrow B}{A \Rightarrow B}.$$

Proposition 3.4 below implies that FQC with Löb's Axiom, and with its additional rules for implication and universal quantification, is conservative over geometric logic. There is a strong completeness theorem for BQC in the same way that there is a strong completeness theorem for IQC, except that the class of Kripke models is such that the underlying collection of nodes (or: worlds) is provided with a transitive binary relation $\prec$ that need not be reflexive [15].

In the absence of Modus Ponens there is a different schema to reflect that implication $\to$ behaves properly relative to derivability. A theory T is called *faithful* if

$$T \vdash \bigwedge_i (\forall \mathbf{x}_i : A_i.B_i) \Rightarrow \forall \mathbf{x} : A.B$$

implies

$$T \cup \{A_i \Rightarrow B_i\}_i \vdash (A \Rightarrow B),$$

for all A_i, B_i and A, B such that all their free variables are among the $\mathbf{x}_i$ and $\mathbf{x}$ respectively. The reverse of the implication always holds for theories T that are axiomatizable with just sequents. Such theories always have a minimal 'faithful closure' $F(T)$ which is also axiomatizable by sequents. The theory T is called weakly inconsistent if $F(T)$ is inconsistent. Define $\bot_0$ to be $\bot$, and $\bot_{n+1}$ to be $\top \to \bot_n$, for all n. Then T is weakly inconsistent exactly when $T \vdash (\Rightarrow \bot_n)$ for some n. BQC, FQC, and all extensions of IQC, are faithful [15]. A Kripke model is called *rooted* if there is a node α such that $\alpha \prec \beta$ for all $\beta \neq \alpha$.

Proposition 3.1 *Geometric theories are faithful.*

Proof. Let T be a geometric theory, and let A_i, B_i, A, and B be such that

$$T \cup \{A_i \Rightarrow B_i\}_i \nvdash (A \Rightarrow B).$$

There is a rooted Kripke model $\mathbf{K}$ of $T \cup \{A_i \Rightarrow B_i\}_i$ with root α such that $\mathbf{K} \not\models A \Rightarrow B$. Construct a new model $\mathbf{K'}$ by adding a new root $\alpha_0 \prec \alpha$ with structure $D\alpha_0$ and interpretations borrowed from α. Then $\mathbf{K'} \models T$, and $\mathbf{K'} \not\models \bigwedge_i (\forall \mathbf{x}_i : A_i.B_i) \Rightarrow \forall \mathbf{x} : A.B$. $\dashv$

Certain properties of BQC and extensions are best expressed through substitution schemas of formulas inside formulas. We prefer to stay with first-order logic, and introduce formula substitution in the metalanguage. So we don't have 'logical' variables in our language, but instead use the notion of *logical context* $C[\ ]$ or $C[P]$, where P is a placeholder for a logical substitution. Logical contexts are easily described as adding a new propositional symbol, say P, to the existing collection of atomic formulas, and closing the language under the usual logical operations. Multi-parameter logical contexts $C[P, Q]$ are defined similarly. A formula is the same as a logical context without new propositional symbols. Illustration: BQC satisfies the Substitution Schema

$$\frac{A \wedge B \Rightarrow C \quad A \wedge C \Rightarrow B}{A \wedge D[B] \Rightarrow D[C]},$$

where the lower sequent satisfies the usual variable condition that the free variables of B and C are not bound by a quantifier of D. A substitution parameter P occurs *formally* in a logical context $C[P]$ if P occurs solely inside implication subformulas or within range of universal quantifiers; P occurs *strictly informally* if it does not occur inside implication subformulas or within range of a universal quantifier. A logical context is *formal* if all atomic subformulas and parameters occur formally. A logical context is *strictly informal* if it does not contain implications or universal quantifications. Note that for each formula A there exists an essentially unique strictly informal logical context $C[P_1, \ldots, P_n]$, and formulas $A_1, \ldots, A_n$ with each A_i either of the form $B \to C$ or of the form $\forall \mathbf{x} : B.C$, such that A is equal to $C[A_1, \ldots, A_n]$. If A is formal, then $C[\ ]$ has no atomic subformulas. If A is strictly informal, then $n = 0$ and A equals C. Each time when we substitute, we assume the usual variable restrictions. These are easily enforced by renaming bound variables.

Lemma 3.2 (Monotonicity) *Let P be strictly informal in the logical context $C[P]$, and let A and B be such that no bound variable of $C[\ ]$ is free in A or B. Then BQC satisfies*

$$\frac{A \Rightarrow B}{C[A] \Rightarrow C[B]}.$$

Proof. All occurrences of P are in strictly positive positions. $\dashv$

Lemma 3.3 *Let P be formal in the logical context $C[P]$, and let A and B be such that no bound variable of $C[\ ]$ is free in A or B. Then BQC satisfies*

$$(A \leftrightarrow B) \wedge C[A] \Rightarrow C[B].$$

Proof. The result is immediate for implications $D[\ ] \to E[\ ]$ and universal quantifications $\forall \mathbf{x} : D[\ ].E[\ ]$. The general case then follows by a straightforward induction on the size of $C[\ ]$. $\dashv$

With Lemmas 3.2 and 3.3 we easily derive the following small generalization of the Fixed Point Theorem of [21]. Let $C[\ ]$ be a logical context in BQC. By substitution BQC satisfies $C[\top] \Rightarrow C[C[\top]]$. We can write $C[P]$ as $D[P, P]$, where $D[Q, R]$ is a logical context in which Q occurs only formally, and R occurs strictly informally. Then BQC satisfies $D[C[\top], C[\top]] \Rightarrow D[C[\top], \top]$ by Monotonicity, and $D[C[\top], \top] \wedge (\top \to C[\top]) \Rightarrow D[\top, \top]$ by Lemma 3.3. So BQC satisfies $C[C[\top]] \wedge (\top \to C[\top]) \Rightarrow C[\top]$. Over FQC, using Löb's Rule, we therefore have $C[\top] \Leftrightarrow C[C[\top]]$.

A theory is *formal* if it is axiomatizable by sequents $A \Rightarrow B$ with B formal. Formal theories are consistent, though sometimes barely, since they are contained in the theory axiomatized by $\{\Rightarrow \top \to \bot\}$.

Proposition 3.4 *Let T be a formal theory, and let $U \cup \{A \Rightarrow B\}$ be geometric sequents. Then $T \cup U \vdash (A \Rightarrow B)$ if and only if $U \vdash (A \Rightarrow B)$.*

Proof. Suppose that $U \not\vdash (A \Rightarrow B)$. Then there is a one-node irreflexive model $\mathbf{K}$ of U such that $\mathbf{K} \not\models (A \Rightarrow B)$. But $\mathbf{K} \models (\Rightarrow \top \to \bot)$, so $\mathbf{K}$ is a model of T. $\dashv$

A theory T is said to satisfy the *weak* Completeness Theorem with respect to a class of Kripke models $\mathcal{K}$ if for all sequents γ it satisfies: $T \vdash \gamma$ if and only if for all $\mathbf{K} \in \mathcal{K}$, if $\mathbf{K} \models T$, then $\mathbf{K} \models \gamma$. The *strong* Completeness Theorem is equivalent to the weak Completeness Theorem applied to all extensions of T. A Kripke model is called a *tree model* if the reflexive closure $\preceq$ of $\prec$ on the set of nodes is such that the collection of predecessors of each node is a finite set linearly ordered by $\preceq$. Recall that a Kripke model is called rooted if there is a node α such that $\alpha \prec \beta$ for all $\beta \neq \alpha$. A node α is called *reflexive* if it satisfies $\alpha \prec \alpha$; otherwise it is called *irreflexive*.

Proposition 3.5 *BQC satisfies the strong completeness theorem with respect to rooted tree models.*

Proof. Obviously, BQC is strongly complete with respect to rooted Kripke models. From an existing model $\mathbf{K} = (P, D, I)$ with root α we construct a rooted tree model with as nodes all finite ascending sequences $(\alpha, \alpha_1, \ldots, \alpha_n)$, $n \geq 0$, of the original model, and $(\alpha, \alpha_1, \ldots, \alpha_n) \prec (\alpha, \beta_1, \ldots, \beta_m)$ exactly when the first sequence is an initial segment of the second and, additionally, $\alpha_n \prec \beta_m$. As domain and structure above $(\alpha_1, \ldots, \alpha_n)$ we choose $D\alpha_n$ and its structure from $\mathbf{K}$. Then $(\alpha, \alpha_1, \ldots, \alpha_n) \Vdash A$ in the new model, if and only if $\alpha_n \Vdash A$ in the original model. $\dashv$

Lemma 3.6 *Let T be a formal theory, and let U be a geometric theory. Then $T \cup U$ satisfies the weak completeness theorem with respect to rooted tree models with irreflexive root.*

Proof. Suppose that $T \cup U \not\vdash \gamma$. There exists a rooted tree model $\mathbf{K}$ of $T \cup U$ such that $\mathbf{K} \not\models \gamma$. We may assume that its root α satisfies $\alpha \prec \alpha$. Construct a new model $\mathbf{K}'$ by adding a new irreflexive bottom node α_0 to $\mathbf{K}$ and by setting its domain $D\alpha_0$ and the interpretations for the atomic sentences equal to $D\alpha$ and the interpretations at α. Clearly, $\mathbf{K}' \not\models \gamma$ and $\mathbf{K}' \models U$. It suffices to show that $\alpha_0 \Vdash \varphi$ for all sequents $\varphi \in T$. But this immediately follows from φ being formal: The truth of an implication or universal quantification above α_0 only depends on its interpretation above all nodes $\beta \succ \alpha_0$. And this follows from the reflexivity of α and from $\mathbf{K} \models \varphi$. $\dashv$

A geometric sequent $A \Rightarrow B$ is a *production sequent* if B is built up from $\top$, $\bot$, atomic formulas, and conjunction $\wedge$ only.

Proposition 3.7 (Disjunction Property) *Let T be a formal theory, and U be a theory of production sequents. Then $T \cup U \vdash (\Rightarrow A \vee B)$ implies $T \cup U \vdash (\Rightarrow A)$ or $T \cup U \vdash (\Rightarrow B)$, for all sentences A and B.*

Proof. Suppose $T \cup U \nvdash (\Rightarrow A)$ and $T \cup U \nvdash (\Rightarrow B)$. There exist rooted Kripke models $\mathbf{K}_A$ and $\mathbf{K}_B$ of $T \cup U$ with irreflexive roots such that $\mathbf{K}_A \not\models A$ and $\mathbf{K}_B \not\models B$. Form a new model $\mathbf{K}$ by removing the root nodes α_A and α_B from $\mathbf{K}_A$ and $\mathbf{K}_B$, and adding a new irreflexive root node α below the disjoint union of the two remaining parts. In $D\alpha$ we only put the closed terms of the language (allow one variable in case the language has no constant symbols), and map these into the remaining nodes of $\mathbf{K}_A$ and $\mathbf{K}_B$ as required. Above α we force the structure as required by the production sequents of U and by equality. Obviously, the new model satisfies $T \cup U$. It suffices to show that $\alpha \not\Vdash A$ (and so by symmetry $\alpha \not\Vdash B$). But A equals $C[A_1, \ldots, A_n]$ for some strictly informal logical context C and implication and universal quantifier subformulas A_i. So for all i, $\alpha \Vdash A_i$ if and only if $\alpha_A \Vdash A_i$ and $\alpha_B \Vdash A_i$. By the Monotonicity Lemma 3.2, since $\alpha_A \not\Vdash A$, we have $\alpha \not\Vdash A$. $\dashv$

Proposition 3.8 (Explicit Definability) *Let T be a formal theory, and U be a theory of production sequents, over a language with at least one constant symbol. Then, for all sentences $\exists x Ax$, $T \cup U \vdash (\Rightarrow \exists x Ax)$ implies $T \cup U \vdash (\Rightarrow At)$ for a closed term t.*

Proof. Suppose $T \cup U \nvdash At$ for all closed terms. There are rooted Kripke models $\mathbf{K}_t$ of $T \cup U$ with irreflexive roots α_t such that $\mathbf{K}_t \not\models At$. Form a new model $\mathbf{K}$ by removing the root nodes α_t from the models $\mathbf{K}_t$, and adding a new irreflexive root node α below the disjoint union of the remaining parts. In $D\alpha$ we only put the closed terms of the language, and map these into the remaining nodes of the $\mathbf{K}_t$ as required. Above α we force the structure as required by the production sequents of U and by equality. So the new model satisfies $T \cup U$. It suffices to show $\alpha \not\Vdash At$ for any closed term t. But Ax equals $C[A_1 x, \ldots, A_n x]$ for some strictly informal logical context C and implication and universal quantifier subformulas $A_i x$. So for all i, $\alpha \Vdash A_i t$ if and only if $\alpha_u \Vdash A_i t$ for all u. By the Monotonicity Lemma 3.2, since $\alpha_t \not\Vdash At$, we have $\alpha \not\Vdash At$, for all t. So $\alpha \not\Vdash \exists x Ax$. $\dashv$

Formal Provability Calculus FQC is a formal theory since it is axiomatizable by Löb's Axiom Schema $(\top \to A) \to A \Rightarrow \top \to A$. So Propositions 3.7 and 3.8 give new proofs that FQC satisfies the Disjunction Property and Explicit Definability.

For each set of sequents U, define $U^\top$ to be the set of sequents $\Rightarrow A \to B$ with $A \Rightarrow B$ from U. So all sequents of $U^\top$ are formal. Obviously, $U \vdash \gamma$ for all $\gamma \in U^\top$.

Proposition 3.9 *Let $T \cup \{\gamma\}$ be a set of formal sequents, and let U be a set of production sequents. If $T \cup U \vdash \gamma$, then $T \cup U^\top \vdash \gamma$.*

Proof. Suppose $T \cup U^\top \nvdash \gamma$. There exists a model $\mathbf{K}$ of $T \cup U^\top$ with irreflexive root such that $\mathbf{K} \not\models \gamma$. Form a new model $\mathbf{K}'$ from $\mathbf{K}$ by replacing the structure above the irreflexive root by only the closed terms as required by the language (allow one variable if the language has no constant symbols), and then forcing a structure as required by equality and the production sequents of U. Clearly, by $T \cup \{\gamma\}$ being formal, $\mathbf{K}' \not\models \gamma$ and $\mathbf{K}' \models T$. Since $\mathbf{K} \models U^\top$ we have $\mathbf{K}' \models U$. $\dashv$

4 Fregean Set Theory

Basic logic allows for consistent interesting set theories that are not available to classical mathematicians or intuitionists. Despite Brouwer's Continuity Theorem and Bishop's recognition of the relevance of classical mathematics to constructive mathematics [3, page 3], it has once been argued [18, page 16] that constructivists tend to be secure-minded, while the daring are found among the classical mathematicians. In particular, classical mathematicians dare to use more powerful principles and are able to prove more results, at the risk of inconsistency. In [15] we dared to introduce a Frege-style set theory based on the very constructive BQC. It is inconsistent. The set theory F below is an improved version.

The Fregean set theory F is built in a first-order language with binary relation symbols $=$ and $\in$ for equality and membership, and where for each formula A we have a term $\{x \mid A\}$ in the language. The number of different free variables of A, minus x if present, determine the arity of $\{x \mid A\}$. We write $\{x \mid A(x_1,\ldots,x_n)\}$ instead of the more standard $\{x \mid A\}(x_1,\ldots,x_n)$. We identify terms that are equal up to a renaming of the bound variables. F is axiomatized by BQC with the usual axioms for $=$. The hiding of structure inside single terms requires the axiom schema of *special equality*

$$t = u \Rightarrow \{y \mid At\} = \{y \mid Au\},$$

where no free variable of t or u is bound by a quantifier of Ax. Besides this we need one technical axiom schema that involves a certain ambiguity in our notation. Let Ax be a formula, and let t be a term whose variables are not among the bound variables of Ax. We can construct the primitive term $\{y \mid At\}$ directly from At, or we can obtain a similarly looking composite term $\{x \mid At\}$ obtained from the primitive term $\{y \mid Ax\}$ after substitution of t for x. To avoid such confusion, we temporarily write composition as $\{y \mid Ax\}(t/x)$. The additional axiom schema of *composition* says that we usually can ignore the ambiguity:

$$\Rightarrow \{y \mid At\} = \{y \mid Ax\}(t/x).$$

Finally, for $\in$, we have the Frege-style axiom schema of *β-conversion*:

$$x \in \{x \mid B\} \Leftrightarrow B.$$

This completes the axiomatization of F.

Define $V = \{x \mid \top\}$ and $\emptyset = \{x \mid \bot\}$. Replacing $\bot$ by $\emptyset = V$, and the axiom schema $\bot \Rightarrow A$ by the axiom $x \in \emptyset \Rightarrow x \in y$, gives us a system that is equivalent to F: One easily shows $\emptyset = V \Rightarrow x \in \{x \mid A\}$, thus $\emptyset = V \Rightarrow A$. So F can be axiomatized without $\bot$.

The axiom schema of β-conversion has been a major source of paradoxes, most notably Russell's Paradox, which uses classical (even intuitionistic) logic to show that the set $\{x \mid x \in x \to \bot\}$ is member of itself exactly when it is not. In [15] we converted the traditional proof of Russell's Paradox into a useful theorem. The following improved version avoids $=$. Let A be a formula in which x does not occur. Define $\lceil A \rceil = \{x \mid x \in x \to A\}$.

Lemma 4.1 *F satisfies the schema* $\lceil A \rceil \in \lceil A \rceil \Rightarrow \top \to A$.

Proof. Use

$$\lceil A \rceil \in \lceil A \rceil \Rightarrow (\top \to \lceil A \rceil \in \lceil A \rceil) \wedge (\lceil A \rceil \in \lceil A \rceil \to A)$$

and the transitivity of $\to$. $\dashv$

Corollary 4.2 *F satisfies the schema* $\Rightarrow \lceil A \rceil \in \lceil \top \to A \rceil$.

Theorem 4.3 *F satisfies Löb's Axiom Schema* $(\top \to A) \to A \Rightarrow \top \to A$.

Proof. Let A be a formula and let x be a variable that does not occur in A. Then $(x \in x \to (\top \to A)) \wedge ((\top \to A) \to A) \Rightarrow x \in x \to A$. So $x \in \lceil \top \to A \rceil \wedge ((\top \to A) \to A) \Rightarrow x \in \lceil A \rceil$. Substitution of $\lceil A \rceil$ for x gives $(\top \to A) \to A \Rightarrow \lceil A \rceil \in \lceil A \rceil$. Thus, by Lemma 4.1, $(\top \to A) \to A \Rightarrow \top \to A$. $\dashv$

We show the consistency of F by means of Kripke models. Rather than constructing a single model, we show that the class of Kripke models of F is closed under a certain operation: For each set of Kripke models $\mathcal{K}$ of F we construct a new model $\mathbf{K}_\omega(\mathcal{K})$ of F that is the limit of a sequence of models $\mathbf{K}_n(\mathcal{K})$, $n < \omega$. Among the $\mathbf{K} \in \mathcal{K}$ we allow the 'inconsistent' model $\mathbf{E}$ with empty set of underlying nodes. So $\mathbf{E}$ satisfies $\Rightarrow \bot$. Each model $\mathbf{K}_\lambda$, $\lambda \leq \omega$, is constructed by adding a new irreflexive bottom node α_λ below the disjoint union of the models $\mathbf{K} \in \mathcal{K}$. The underlying sets $D\alpha_\lambda$ equal the set of closed terms, that is, the set of terms without free variables and up to renaming of the bound variables, modulo some equivalence relation. So their mapping into the nodes of the models $\mathbf{K} \in \mathcal{K}$ is uniquely determined. The precise structure above α_{n+1} is constructed from the structure above α_n, and the structure above α_ω is set as the limit of the structures above α_n. Construct a sequence of theories $T_0 \subseteq T_1 \subseteq \ldots$ as follows. T_0 is the theory axiomatized by the set U_0 of special equality and composition sequents, plus the set T of sequents $\Rightarrow A \to B$ for which $\mathbf{K} \models A \Rightarrow B$ for all $\mathbf{K} \in \mathcal{K}$. Once T_n has been constructed, let T_{n+1} be the extension of T_n axiomatized by T plus the set U_{n+1} which equals U_n plus the sequents $\Rightarrow t \in \{x \mid Ax\}$ for which $T_n \vdash (\Rightarrow At)$. Set $U_\omega = \cup_n U_n$, and $T_\omega = T \cup U_\omega$. For all $\lambda \leq \omega$ set $t = u$ in $D\alpha_\lambda$ if and only if $T_\lambda \vdash (\Rightarrow t = u)$, and set $\alpha_\lambda \Vdash t \in u$ if and only if $T_\lambda \vdash (\Rightarrow t \in u)$. Obviously, $\mathbf{K} \models U_\omega$ for all $\mathbf{K} \in \mathcal{K}$, so the models $\mathbf{K}_\lambda$ are Kripke models, for all $\lambda \leq \omega$.

Lemma 4.4 *For all* $\lambda \leq \omega$, $\mathbf{K}_\lambda(\mathcal{K})$ *is a Kripke model such that* $\alpha_\lambda \Vdash \varphi$ *if and only if* $T_\lambda \vdash (\Rightarrow \varphi)$.

Proof. Each theory T_λ is axiomatized by the union of a formal theory T and a geometric theory of production sequents U_λ. We proceed by induction on the complexity of φ. For atomic sentences we obviously have $\alpha_\lambda \Vdash \varphi$ if and only if $T_\lambda \vdash (\Rightarrow \varphi)$. We have $\alpha_\lambda \Vdash \varphi \vee \psi$ if and only if $\alpha_\lambda \Vdash \varphi$ or $\alpha_\lambda \Vdash \psi$, while by Proposition 3.7 $T_\lambda \vdash (\Rightarrow \varphi \vee \psi)$ if and only if $T_\lambda \vdash (\Rightarrow \varphi)$ or $T_\lambda \vdash (\Rightarrow \psi)$. A similar argument, using Proposition 3.8, applies in case of existential quantification. So the induction easily passes through $\vee$ and $\exists$. Conjunction $\wedge$ is always simple. Suppose $\alpha_\lambda \Vdash \varphi \to \psi$. Then $\mathbf{K} \models (\varphi \Rightarrow \psi)$, for all $\mathbf{K} \in \mathcal{K}$. So $T_\lambda \supseteq T \vdash (\Rightarrow \varphi \to \psi)$. Conversely, if $T_\lambda \vdash (\Rightarrow \varphi \to \psi)$, then $T \cup U_\lambda \vdash (\Rightarrow \varphi \to \psi)$, so by Proposition 3.9 $T \cup U_\lambda^\top \vdash (\Rightarrow \varphi \to \psi)$. But $U_\lambda^\top \subseteq T$, so $\alpha_\lambda \Vdash \varphi \to \psi$. The case for universal quantification $\forall$ is similar. $\dashv$

Proposition 4.5 *If* $\mathcal{K}$ *is a collection of Kripke models of F, then* $\mathbf{K}_\omega(\mathcal{K})$ *is a Kripke model of F.*

Proof. It suffices to verify β-conversion. Let $\alpha_\omega \Vdash At$, where no variable of t is bound by a quantifier of Ax. Then $\alpha_n \Vdash At$ for some n, hence

$\alpha_{n+1} \Vdash t \in \{x \mid Ax\}$. So $\alpha_\omega \Vdash t \in \{x \mid Ax\}$. Conversely, suppose $\alpha_\omega \Vdash t \in \{x \mid Ax\}$. Then $\alpha_{n+1} \Vdash t \in \{x \mid Ax\}$, hence $T_{n+1} \vdash (\Rightarrow t \in \{x \mid Ax\})$, for some n. By Proposition 3.4 $U_{n+1} \vdash (\Rightarrow t \in \{x \mid Ax\})$. So we must have $T_n \vdash (\Rightarrow At)$. So $\alpha_\omega \Vdash At$. $\dashv$

Theorem 4.6 *The theory F is consistent, faithful, and satisfies the Disjunction Property and Explicit Definability.*

Proof. Exercise. Use the model construction $\mathbf{K}_\omega(\mathcal{K})$, and compare with the proofs of Propositions 3.1, 3.7, and 3.8. $\dashv$

Lemma 4.1 also follows from the Fixed Point Theorem below. Because of the peculiar way by which terms are introduced in the language of F, we can distinguish two different versions of context. The limited version is that of logical context as defined in Section 3. The notion of *generalized context* $C[\ ]$ or $C[P]$ is defined by adding a propositional symbol, say P, to the language, but then closing off not only under the usual logical operations but also under the formation of terms $\{x \mid \ \}$. For both logical and generalized contexts we could also introduce substitution parameters for contexts themselves, see the example below Theorem 4.7. Substitution context parameters are used only once, so we don't feel the need for a special definition.

Theorem 4.7 (Fixed Point Theorem) *Let $C[\ ]$ be a generalized context in which no bound variable occurs as a free variable. Then there is a formula W with the same free variables as $C[\ ]$ such that F satisfies*

$$W \Leftrightarrow C[W].$$

Proof. Let x be a variable that does not occur in $C[\ ]$, and let w be the term $\{x \mid C[x \in x]\}$. Set W equal to $w \in w$. $\dashv$

A formula W as in Theorem 4.7 is called an (explicit) *Fixed Point* of the context $C[\ ]$. From this Theorem we derive the following universal fixed point formula context. Consider the two-parameter context $P[Q]$, where P is a context parameter. Let w_P be the expression $\{x \mid P[x \in x]\}$. Then F satisfies all substitution instances of $w_P \in w_P \Leftrightarrow P[w_P \in w_P]$ by contexts that do not contain x, and whose bound variables do not occur as free variables. Let Ω be the term $\{x \mid x \in x\}$. Then, by the Fixed Point Theorem, $W \equiv \Omega \in \Omega$ is a fixed point of the logical context $C[P] \equiv P$. But $W \Leftrightarrow C[W]$ ($\equiv W$) is a substitution instance of β-conversion as well. In the model construction preceding Lemma 4.4 we therefore have $U_{n+1} \vdash (\Rightarrow W)$ exactly when $T_n \vdash (\Rightarrow W)$, exactly when $U_n \vdash (\Rightarrow W)$ (by Proposition 3.4). So $T_\omega \vdash (\Rightarrow W)$ if and only if $U_0 \vdash (\Rightarrow W)$. So $\mathbf{K}_\omega(\mathcal{K}) \not\models (\Rightarrow W)$, hence $\mathrm{F} \nvdash (\Rightarrow W)$. A straightforward verification shows that $\mathrm{EF} \nvdash (\Rightarrow W)$, for the system EF below. This example shows that the fixed points W of Theorem 4.7 need not be equivalent to the 'maximal' fixed points $C[\top]$. In certain circumstances, however, they are. Recall the definition, in Section 3, of formal parameters in logical contexts.

Proposition 4.8 (Uniqueness) *Let $C[P]$ be a logical context in some first-order language in which all occurrences of P are formal, and let D be a fixed point of $C[\ ]$ over FQC. Then FQC satisfies*

$$D \Leftrightarrow C[\top].$$

Proof. As for FPC, see [21]: By the Substitution Schema of Section 3 we have $D \Rightarrow C[\top]$. Conversely, by Lemma 3.3, we have $C[\top] \wedge (\top \to D) \Rightarrow C[D]$, so $C[\top] \wedge (\top \to D) \Rightarrow D$. Thus, by Löb's Rule, $C[\top] \Rightarrow D$. $\dashv$

Examples: Let $C[\ \ ]$ be the context $(\ \ ) \to A$. Then w equals $\lceil A \rceil$, and

$$\lceil A \rceil \in \lceil A \rceil \Leftrightarrow \top \to A.$$

Let $C[\ \ ]$ be the context $A \to (\ \ )$. Then w equals $\{x \mid A \to x \in x\}$, and Proposition 4.8 implies $\Rightarrow w \in w$. Let $C[\ \ ]$ equal the context $A \leftrightarrow (\ \ )$. Then w equals $\{x \mid A \leftrightarrow x \in x\}$, and F satisfies $\top \to A \Leftrightarrow w \in w$.

The system F lacks any form of equality among its sets except for what is generated by special equality and composition. One may wish to include some rule of extensionality like strong extensionality [15]:

$$\frac{A \wedge x \in y \Rightarrow x \in z \quad A \wedge x \in z \Rightarrow x \in y}{A \Rightarrow y = z},$$

where x is not free in A. However, a refinement of Russell's Paradox applies. In fact, the refined Paradox below even follows when we extend F by all sequents

$$\Rightarrow \{x \mid A\} = \{x \mid B\}$$

for which $A \Leftrightarrow B$ holds in F. By the Fixed Point Theorem 4.7 there is for each sentence A a sentence W_A such that

$$W_A \Leftrightarrow \{y \mid W_A\} = \{y \mid W_A \wedge A\}.$$

Then $W_A \Rightarrow A$, so also $W_A \Leftrightarrow W_A \wedge A$. If we had the additional sequents, then this would imply $\Rightarrow \{y \mid W_A\} = \{y \mid W_A \wedge A\}$, and so $\Rightarrow W_A$. But then also $\Rightarrow A$. So in particular, using $W_\perp$, we could have derived $\Rightarrow \perp$. One easily produces mild variations of this result. They show that even a weak equivalent of a general extensionality rule may result in unwanted inconsistencies or almost-inconsistencies. Still, when two formulas A and B essentially say the same, it ought to be possible to consider the terms $\{x \mid A\}$ and $\{x \mid B\}$ equal. This is the motivation behind the following partial version of extensionality. The system E is the theory axiomatized by special equality, composition, and by all sequents

$$\Rightarrow \{x \mid A\} = \{x \mid B\}$$

for which $A \Leftrightarrow B$ holds in FQC. We write EF as short for $\mathrm{E} \cup \mathrm{F}$.

Proposition 4.9 *The theory EF is consistent, faithful, and satisfies the Disjunction Property and Explicit Definability.*

Proof. We map the language of EF into itself in such a way that equivalent terms are mapped to identical terms. The map is a composition of two other maps, t and u. Map u replaces all compositions $\{x \mid Ax\}(t/x)$ inside formulas by $\{x \mid At\}$. The relation $\mathrm{E} \vdash (A \Leftrightarrow B)$ is an equivalence on the set of function symbols $\{x \mid A\}$, $\{x \mid B\}$ of the language. From each equivalence class pick a representative with the least number of free variables in such a way that this picking function commutes with renaming of free variables. Map t is a projection that maps each formula in the image of u to the formula where all function symbols have been replaced by the representatives of their equivalence classes. Form the set T of sequents $A \Rightarrow B$ such that $tuA \Rightarrow tuB$ holds in F. One easily verifies that T is closed under the rules and sequents of BQC. Special equality, composition, and β-conversion for T follow from those for F. Since EF $\subseteq T$, we have consistency for EF. But EF satisfies $A \Leftrightarrow tuA$, so EF and T are equal. Finally, T inherits the Disjunction Property and Explicit Definability from F, so EF satisfies these too. $\dashv$

The extension EF of F is fairly simple. For the following further extension of EF, called EF$^+$, we don't have a consistency proof. To avoid the paradox with extensionality, we only allow it for formal formulas. For that purpose we present extensionality in the form

$$\frac{A \wedge B \Rightarrow C \quad A \wedge C \Rightarrow B}{A \Rightarrow \{x \mid B\} = \{x \mid C\}},$$

where x is not free in A. To maintain $\rightarrow$ as a proper reflection of $\Rightarrow$ we must add the matching sequent schema [15], in this case

$$(A \wedge B \rightarrow C) \wedge (A \wedge C \rightarrow B) \Rightarrow A \rightarrow \{x \mid B\} = \{x \mid C\},$$

where x is not free in A. Now EF$^+$ is the extension of EF with the above rule and sequent of extensionality under the additional restriction that B and C are formal. So the paradoxical proof with extensionality is blocked. A further possible extension of EF$^+$ includes adding η-conversion

$$\Rightarrow y = \{x \mid x \in y\}.$$

An alternative to above solution to the conflict between β-conversion and extensionality is weakening β-conversion instead. There are many possible approaches along such lines, and below we sketch only a few. For instance, consider the system F$^-$ that is axiomatized by special equality and composition, and by the weakened schema of β-conversion

$$x \in \{x \mid B\} \Leftrightarrow \top \rightarrow B,$$

(Exercise: The 'weakening' $\top \rightarrow x \in \{x \mid B\} \Leftrightarrow B$ is inconsistent.) Over this limited system generalized contexts $C[\ ]$ have weak fixed points W only, that is, formulas W such that

$$W \Leftrightarrow \top \rightarrow C[W].$$

The construction of W is identical to the one in the proof of the Fixed Point Theorem 4.7: Let w be $\{x \mid C[x \in x]\}$, x a new variable, and set W equal to $w \in w$. Remarkably, Löb's Axiom Schema still follows from a, slightly modified, 'Russell Proof.' Let A be a formula in which x does not occur, and let $\lceil A \rceil$ be short for $\{x \mid x \in x \rightarrow A\}$. By weak β-conversion we have

$$\lceil A \rceil \in \lceil A \rceil \Rightarrow \top \rightarrow (\top \rightarrow A).$$

So

$$((\top \rightarrow A) \rightarrow A) \wedge \lceil A \rceil \in \lceil A \rceil \Rightarrow \top \rightarrow A,$$

and thus $(\top \rightarrow A) \rightarrow A \Rightarrow \lceil A \rceil \in \lceil \top \rightarrow A \rceil$. Now $(\top \rightarrow (x \in x \rightarrow (\top \rightarrow A))) \wedge ((\top \rightarrow A) \rightarrow A) \Rightarrow \top \rightarrow (x \in x \rightarrow A)$. So $x \in \lceil \top \rightarrow A \rceil \wedge ((\top \rightarrow A) \rightarrow A) \Rightarrow x \in \lceil A \rceil$. Substitution of $\lceil A \rceil$ for x gives $(\top \rightarrow A) \rightarrow A \Rightarrow \lceil A \rceil \in \lceil A \rceil$. Thus $(\top \rightarrow A) \rightarrow A \Rightarrow \top \rightarrow A$.

We could add full extensionality to the system F$^-$, but we may obtain stronger systems if we first embed F$^-$ into the system G$^-$ below. This embedding illustrates that the notation $\{x \mid B\}$ hides the syntactic form B

on the inside from the first-order logic on the outside. Let G$^-$ be the system axiomatized by special equality, composition, and the usual β-conversion for formulas of the form $\top \to B$ only:

$$x \in \{x \mid \top \to B\} \Leftrightarrow \top \to B.$$

The system F$^-$ is easily seen to be embeddable into the system G$^-$: Just map formulas A to A^t in such a way that $(\quad)^t$ commutes with all logical operations, equality, and membership, and such that terms $\{x \mid B\}^t$ equal $\{x \mid \top \to B^t\}$. Clearly, $(F^-)^t \subseteq$ G$^-$, and G$^-$ satisfies Löb's Axiom. We conjecture that this embedding is conservative: $A \Rightarrow B$ holds in F$^-$ if and only if $A^t \Rightarrow B^t$ holds in G$^-$.

Let G be the extension of G$^-$ axiomatized by special equality, composition, and β-conversion for formal formulas only:

$$x \in \{x \mid B\} \Leftrightarrow B$$

for all formal formulas B. Let EG be the extension of G by full extensionality.

$$\frac{A \wedge B \Rightarrow C \quad A \wedge C \Rightarrow B}{A \Rightarrow \{x \mid B\} = \{x \mid C\}},$$

where x is not free in A. Again, to maintain $\to$ as a proper reflection of $\Rightarrow$, we add the matching sequent schema

$$(A \wedge B \to C) \wedge (A \wedge C \to B) \Rightarrow A \to \{x \mid B\} = \{x \mid C\},$$

where x is not free in A. This completes the axiomatization of EG.

Over G and EG formal generalized contexts $C[\quad]$ have fixed points W, that is, formulas W such that G (or EG) satisfies

$$W \Leftrightarrow C[W].$$

The paradox that came with extensionality now turns into a second proof of Löb's Axiom Schema for EG. Let A be a formula in which x does not occur. By the weak fixed point result there is a sentence W_A such that

$$W_A \Leftrightarrow \top \to \{x \mid \top \to W_A\} = \{x \mid \top \to W_A \wedge A\}.$$

So we have $W_A \Rightarrow \top \to (\top \to A)$, so $\Rightarrow W_A \to (\top \to (\top \to A))$. Then

$$(\top \to A) \to A \Rightarrow W_A \to A;$$

$$(\top \to A) \to A \Rightarrow W_A \leftrightarrow W_A \wedge A;$$

$$(\top \to A) \to A \Rightarrow (\top \to W_A) \leftrightarrow (\top \to W_A \wedge A);$$

$$(\top \to A) \to A \Rightarrow \top \to \{x \mid \top \to W_A\} = \{x \mid \top \to W_A \wedge A\};$$

$$(\top \to A) \to A \Rightarrow W_A; \text{ and thus}$$

$$(\top \to A) \to A \Rightarrow \top \to A.$$

We don't have a consistency proof for EG. We conjecture that consistency of EG and EF$^+$ may be proven using Church-Rosser techniques from the λ-calculus.

All Fregean set theories that we showed to be consistent, or that we conjecture to be so, satisfy FQC. Each formula of FQC is equivalent to a logical context $C[P_1, \ldots, P_n]$ in the obvious way by replacing all propositional constants by new context symbols. A formula *holds* in F (or EF, or G, and so on) if all substitution instances of its corresponding logical context in the language of F (or EF, etc.) are derivable from F (or EF, etc.). One can show, by combining the construction preceding Lemma 4.4 with [21, Theorem 2.2], that the collection of propositional sequents that hold in F forms exactly the theory FPC. We conjecture that the same holds, with FQC, for all Fregean set theories considered.

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