What makes some latarres so special?

Mohammad Ardeshir

Wim Ruitenburg

Latarres are essentially defined as LATtices with an ARRow. Language $(\sqcap, \sqcup, \rightarrow)$. A lattice with respect to $\sqcap$ and $\sqcup$. Arrow properties:

$$
\begin{aligned}
&x \rightarrow y = (x \sqcup y) \rightarrow y \\
&x \rightarrow y = x \rightarrow (x \sqcap y) \\
&y \trianglelefteq z \text{ implies } x \rightarrow y \trianglelefteq x \rightarrow z \\
&y \trianglelefteq z \text{ implies } z \rightarrow x \trianglelefteq y \rightarrow x \\
&(x \rightarrow y) \sqcap (y \rightarrow z) \trianglelefteq x \rightarrow z
\end{aligned}
$$

where $\trianglelefteq$ is the definable order.
Axiomatize with equations. Language $(\sqcap, \sqcup, \varepsilon, \rightarrow)$, with 'technical' ε. Universal algebra axioms are lattice axioms plus:

N1. $x \rightarrow y = (x \sqcup y) \rightarrow y$

N2. $x \rightarrow y = x \rightarrow (x \sqcap y)$

N3. $x \rightarrow (x \sqcap y \sqcap z) \trianglelefteq x \rightarrow (x \sqcap y)$

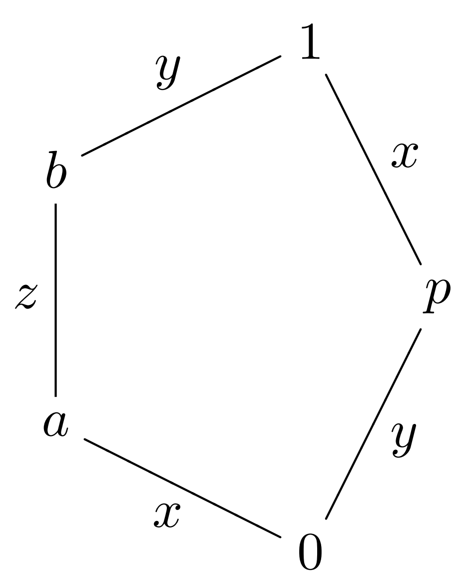
N4. $y \rightarrow (y \sqcap z) \trianglelefteq (x \sqcap y) \rightarrow (x \sqcap y \sqcap z)$

N5. $(x \multimap (x \sqcap y)) \sqcap ((x \sqcap y) \multimap (x \sqcap y \sqcap z)) \trianglelefteq x \multimap (x \sqcap y \sqcap z)$

N6. $\varepsilon \multimap \varepsilon = \varepsilon$

A latarre if *unitary* if the lattice has a top 1 and $\varepsilon = 1$.

As example, define a unitary latarre on lattice N_5 as follows. In the diagram, labels x, y, and z mean that we set $1 \multimap b = y$, set $b \multimap a = z$, and so on. The letters x, y and z are values to be chosen freely from among the set of elements $\{0, a, b, p, 1\}$ with the only restrictions that $x \trianglelefteq z$ and $y \trianglelefteq z$.



We inductively define $\nabla^0 x = x$ and $\nabla^{n+1} x = \varepsilon \multimap \nabla^n x$. An x occurs at *depth* $n \geq 0$ in term $t(x)$ if x occurs n levels deep inside implication subformulas of implication subformulas and so on (so x occurs at depth n in $\nabla^n x$). An x occurs *informally* if depth $n = 0$, otherwise x occurs *formally*. Obviously informal occurrences are always positive.

Proposition 0.1. *Let a, b, c, and d be elements of a latarre $\mathfrak{A}$. Then*

1. $a \rightarrowtriangle (b \sqcap c) = (a \rightarrowtriangle b) \sqcap (a \rightarrowtriangle c)$
2. $(b \sqcup c) \rightarrowtriangle a = (b \rightarrowtriangle a) \sqcap (c \rightarrowtriangle a)$
3. $(a \rightarrowtriangle b) \sqcap (b \rightarrowtriangle c) = (a \sqcup b) \rightarrowtriangle (b \sqcap c)$
4. $(a \sqcup b) \trianglerighteq c \trianglerighteq b$ *implies* $(a \sqcup b) \rightarrowtriangle c = a \rightarrowtriangle (a \sqcap c) = a \rightarrowtriangle c$
5. $a \trianglerighteq c \trianglerighteq a \sqcap b$ *implies* $c \rightarrowtriangle (a \sqcap b) = (c \sqcup b) \rightarrowtriangle b = c \rightarrowtriangle b$
6. $a \rightarrowtriangle b \trianglelefteq \varepsilon$
7. $a \rightarrowtriangle a = \varepsilon$
8. $a \trianglelefteq b$ *implies* $a \rightarrowtriangle b = \varepsilon$
9. $a \rightarrowtriangle b = \varepsilon$ *implies* $c \rightarrowtriangle a \trianglelefteq c \rightarrowtriangle b$ *and* $b \rightarrowtriangle c \trianglelefteq a \rightarrowtriangle c$
10. *Suppose* $c \rightarrowtriangle a \trianglelefteq (a \rightarrowtriangle b) \sqcap (b \rightarrowtriangle c)$. *Then* $(c \rightarrowtriangle a) = (c \rightarrowtriangle b) \sqcap (b \rightarrowtriangle a)$. *In particular, if* $c \trianglerighteq b \trianglerighteq a$, *then* $(c \rightarrowtriangle a) = (c \rightarrowtriangle b) \sqcap (b \rightarrowtriangle a)$

11. $b \multimap c \trianglelefteq (a \sqcap b) \multimap (a \sqcap c)$

12. $(b \multimap a) \sqcap ((a \sqcap b) \multimap (a \sqcap c)) = (b \multimap a) \sqcap (b \multimap c)$

13. $d \sqcap \varepsilon = d \sqcap (b \multimap a)$ *if and only if* $\mathfrak{A}$ *satisfies schema* $d \sqcap ((a \sqcap b) \multimap (a \sqcap x)) = d \sqcap (b \multimap x)$

14. $a \sqcap b \multimap c = \varepsilon$ *implies* $b \multimap a \trianglelefteq b \multimap c$, *so also* $a \sqcap b \trianglelefteq c$ *implies* $b \multimap a \trianglelefteq b \multimap c$

15. $\nabla^n (a \sqcap b) = \nabla^n a \sqcap \nabla^n b$, *for all* n

16. $a \trianglelefteq b \multimap c$ *implies* $a \sqcap (d \multimap b) \trianglelefteq d \multimap c$, *in particular* $a \trianglelefteq b \multimap c$ *implies* $a \sqcap \nabla b \trianglelefteq \nabla c$

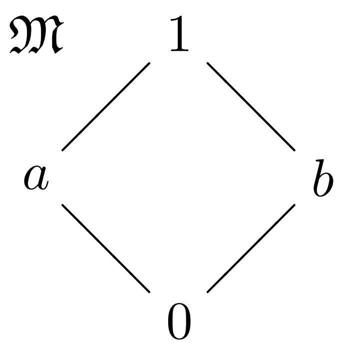
17. $b \multimap \varepsilon = \varepsilon$ *implies* $\nabla a \sqcap ((a \sqcap b) \multimap (a \sqcap c)) = \nabla a \sqcap (b \multimap c)$

18. $\mathfrak{A}$ *satisfies schema* $a \sqcap \varepsilon \trianglelefteq z \multimap a$ *if and only if* $\mathfrak{A}$ *satisfies schema* $a \sqcap ((a \sqcap x) \multimap (a \sqcap y)) = a \sqcap (x \multimap y)$

19. $b \multimap \varepsilon = \varepsilon$ *plus* $a \sqcap b \trianglelefteq c$ *implies* $\nabla a \trianglelefteq b \multimap c$

Proposition 0.2. *Let* $t(x)$ *be a term over a latarre* $\mathfrak{A}$. *If* x *is only positive in* $t(x)$, *then* $x \trianglelefteq y$ *implies* $t(x) \trianglelefteq t(y)$. *If* x *is only negative in* $t(x)$, *then* $x \trianglelefteq y$ *implies* $t(y) \trianglelefteq t(x)$.

We do not always have that x positive in $t(x)$ implies $x \rightarrowtriangle y \trianglelefteq t(x) \rightarrowtriangle t(y)$. For otherwise with $t(x) = \nabla x$ it would imply $x \rightarrowtriangle y \trianglelefteq (\varepsilon \rightarrowtriangle x) \rightarrowtriangle (\varepsilon \rightarrowtriangle y)$, so in particular $\nabla y \trianglelefteq \nabla^2 y$. Here is a counterexample to this last equation. Consider the Boolean lattice $\mathfrak{M}$.



We can construct a (unique) unitary latarre on $\mathfrak{M}$ with $\varepsilon \rightarrowtriangle a = 1 \rightarrowtriangle a = b$ and $1 \rightarrowtriangle b = a$. So $\nabla b = a$ and $\nabla^2 b = b$.

Proposition 0.3. *Let $t(x)$ be a term over a latarre $\mathfrak{A}$ and $n \geq 0$ be such that x only occurs at depth n in $t(x)$. If x is only positive in $t(x)$, then $\nabla^n(x \rightarrowtriangle y) \trianglelefteq t(x) \rightarrowtriangle t(y)$. If x is only negative in $t(x)$, then $\nabla^n(x \rightarrowtriangle y) \trianglelefteq t(y) \rightarrowtriangle t(x)$.*

Proposition 0.4. *Let $t(x)$ be a term over a latarre $\mathfrak{A}$ in which x occurs only at depths at least n in $t(x)$, for some $n \geq 1$. Let $a, b \in A$ be such that $\nabla^{n-1}(a \rightarrowtriangle b) = \varepsilon$. If x is only positive in $t(x)$, then $t(a) \trianglelefteq t(b)$. If x is only negative in $t(x)$, then $t(b) \trianglelefteq t(a)$.*

Construct new latarres from old ones. Given a latarre $\mathfrak{A}$, relation $x \sim y$ defined by $x \leftrightarrow y = \varepsilon$, is a congruence. Write x' for the equivalence class of x. $\mathfrak{A}' = (A', \sqcap', \sqcup', \varepsilon', \rightarrow')$ is a latarre, and map $x \mapsto x'$ is a latarre morphism from $\mathfrak{A}$ onto $\mathfrak{A}'$.

Repeat this construction and form $\mathfrak{A}'' = \mathfrak{A}^{(2)}$. Continuing in this way, we get a chain

$$\mathfrak{A} = \mathfrak{A}^{(0)} \to \mathfrak{A}^{(1)} \to \mathfrak{A}^{(2)} \to \mathfrak{A}^{(3)} \to \dots$$

with for all $a, b \in A$ and $n \geq 1$ we have $a^{(n)} = b^{(n)}$ in $\mathfrak{A}^{(n)}$ exactly when $\nabla^{n-1}(a \leftrightarrow b) = \varepsilon$.

Semilatarres exist over language $(\sqcap, \varepsilon, \rightarrow)$. Drop the axioms involving $\sqcup$, but add

$$(x \sqcap y) \rightarrow (x \sqcap y) \trianglelefteq y \rightarrow y$$

Proposition 0.5. *Let $\mathfrak{A}$ be a semilatarre. Let $\mathfrak{D} = \mathfrak{D}(\mathfrak{A})$ be the usual topological space of downward closed subsets. Define $\rightarrow$ by $U \rightarrow V = \{z \in A : z \trianglelefteq \varepsilon \wedge \forall x \in U \, \exists y \in V \, (z \trianglelefteq x \rightarrow y\}$. Then $\mathfrak{D}$ is a distributive latarre. The map $\delta(a) = \langle a]$ is a semilatarre embedding of $\mathfrak{A}$ into $\mathfrak{D}$.*

On latarre $\mathfrak{D}(\mathfrak{A})$ we can define the 'usual' Heyting arrow.

Proposition 0.6. *Let $t(x)$ be a term over a distributive latarre $\mathfrak{A}$ and $n \geq 1$ be such that x only occurs at depth n in $t(x)$. If*

x is only positive in $t(x)$, then $t(x) \sqcap \nabla^{n-1}(x \rightarrow y) \trianglelefteq t(y)$. If x is only negative in $t(x)$, then $t(y) \sqcap \nabla^{n-1}(x \rightarrow y) \trianglelefteq t(x)$.

Proposition 0.7. *Let $\mathfrak{A}$ be a latarre. Let $\mathfrak{I} = \mathfrak{I}(\mathfrak{A})$ be the substructure of $\mathfrak{D}(\mathfrak{A})$ of ideals (only $\sqcup$ changes). Then $\mathfrak{I}$ is a latarre with an algebraic complete lattice. The map $\delta(a) = \langle\, a]$ is a latarre embedding of $\mathfrak{A}$ into $\mathfrak{I}$.*

From here on essentially all (semi)latarres are unitary.

A *CJ latarre* is a unitary distributive latarre, where CJ stands for Celani and Jansana.

A *Visser latarre* is a distributive latarre satisfying the schema $x \trianglelefteq \nabla x$ of *arrow persistence*. Arrow persistence implies being unitary, since $\nabla x \trianglelefteq \varepsilon$ for all x.

A *Heyting latarre* is a latarre satisfying the schema $x = \nabla x$ of *arrow balance*. We show below that Heyting latarres are distributive.

A *Boolean latarre* is a latarre satisfying the schema $(x \rightarrow y) \rightarrow y = x \sqcup y$. We show below that Boolean latarres are Heyting.

Similar definitions for CJ semilatarres, Visser semilatarres, Heyting semilatarres, and Boolean semilatarres.

Proposition 0.8. *The following are equivalent for a latarre.*

1. *The latarre is arrow persistent*
2. $(x \sqcap y \rightharpoondown z) = \varepsilon$ *implies* $x \trianglelefteq y \rightharpoondown z$*, for all* x*,* y*, and* z
3. $x \sqcap y \trianglelefteq z$ *implies* $x \trianglelefteq y \rightharpoondown z$*, for all* x*,* y*, and* z

The following are equivalent for a latarre.

4. *The latarre is arrow balanced (or: Heyting)*
5. $x \sqcap y \trianglelefteq z$ *if and only if* $x \trianglelefteq y \rightharpoondown z$*, for all* x*,* y*, and* z

The last schema implies distributivity, so all Heyting latarres are distributive.

A latarre satisfying schema $(x \rightharpoondown y) \rightharpoondown y = x \sqcup y$ *(or: Boolean) is arrow balanced (or: Heyting).*

Proposition 0.9. *The following are equivalent for a latarre* $\mathfrak{A}$*.*

1. $\mathfrak{A}$ *is arrow balanced (or: Heyting).*
2. $\mathfrak{A}$ *satisfies schema* $x \sqcap (x \rightharpoondown y) = x \sqcap y$*.*

Proof. Suppose item 2. Setting $x = y$ in the schema shows that the latarre is unitary. So we write 1 for ε. Setting $x = 1$ in the schema shows arrow balance.

Conversely, suppose item 1. Then $\mathfrak{A}$ is unitary. So $x \sqcap y = x \sqcap (1 \multimap y) \trianglelefteq x \sqcap (x \multimap y)$, and by the previous Proposition, $x \multimap y \trianglelefteq x \multimap y$ implies $x \sqcap (x \multimap y) \trianglelefteq x \sqcap y$. □

Proposition 0.10. *Let a, b, and c be elements of a Heyting semilatarre $\mathfrak{A}$, and let a and b have a least upper bound d. Then $(a \multimap c) \sqcap (b \multimap c) = d \multimap c$.*

Proof. The semilatarre satisfies schema

$$(a \trianglelefteq x) \wedge (b \trianglelefteq x) \ \leftrightarrow \ (d \trianglelefteq x),$$

and $(d \multimap c) \trianglelefteq (a \multimap c) \sqcap (b \multimap c)$. Write e as short for $(a \multimap c) \sqcap (b \multimap c)$. We have the following derivation.

$e \trianglelefteq a \multimap c$ and $e \trianglelefteq b \multimap c$.
$e \sqcap a \trianglelefteq c$ and $e \sqcap b \trianglelefteq c$.
$a \trianglelefteq e \multimap c$ and $b \trianglelefteq e \multimap c$.
$d \trianglelefteq e \multimap c$.
$d \sqcap e \trianglelefteq c$.
$e \trianglelefteq d \multimap c$.

That is, $(a \multimap c) \sqcap (b \multimap c) \trianglelefteq d \multimap c$. □

A *Boolean* semilatarre satisfies $(x \multimap (x \sqcap y)) \multimap (x \sqcap y) = x$.

Proposition 0.11. *The following are equivalent for latarre $\mathfrak{A}$.*

1. *$\mathfrak{A}$ satisfies* $(x \rightarrowtriangle y) \rightarrowtriangle y = x \sqcup y$ *(or: Boolean)*

2. *$\mathfrak{A}$ satisfies* $(x \rightarrowtriangle (x \sqcap y)) \rightarrowtriangle (x \sqcap y) = x$

3. *$\mathfrak{A}$ satisfies* $x \sqcup (x \rightarrowtriangle y) = \varepsilon$ *and* $x \sqcap (x \rightarrowtriangle y) = x \sqcap y$

Proof. The schema of item 1 turns into the schema of item 2 when we substitute $x \sqcap y$ for y. The schema of item 2 turns into the schema of item 1 when we substitute $x \sqcup y$ for x. So items 1 and 2 are equivalent.

Finally, the equivalence with item 3. Set $x = y$ in item 1 to get $\nabla y = y$, so ε is top (we may write $\varepsilon = 1$). Next, $x \sqcap y = x \sqcap \nabla y \trianglelefteq x \sqcap (x \rightarrowtriangle y) = x \sqcap \nabla x \sqcap (x \rightarrowtriangle y) \trianglelefteq x \sqcap \nabla y = x \sqcap y$. Finally, $x \sqcup (x \rightarrowtriangle y) = (x \rightarrowtriangle (x \rightarrowtriangle y)) \rightarrowtriangle (x \rightarrowtriangle y) = (x \rightarrowtriangle (x \sqcap (x \rightarrowtriangle y))) \rightarrowtriangle (x \rightarrowtriangle y) = (x \rightarrowtriangle (x \sqcap y)) \rightarrowtriangle (x \rightarrowtriangle y) = (x \rightarrowtriangle y) \rightarrowtriangle (x \rightarrowtriangle y) = \varepsilon$.

Conversely, suppose item 3. The second schema implies that $\mathfrak{A}$ is a Heyting latarre. We establish item 2 as follows. $(x \rightarrowtriangle (x \sqcap y)) \rightarrowtriangle (x \sqcap y) = (x \rightarrowtriangle y) \rightarrowtriangle (x \sqcap (x \rightarrowtriangle y)) = (x \rightarrowtriangle y) \rightarrowtriangle x = (x \sqcup (x \rightarrowtriangle y)) \rightarrowtriangle x = \nabla x = x$. □

Proposition 0.12. *Let $\mathfrak{A} = (\mathfrak{M}, \rightarrowtriangle)$ be a Boolean semilatarre. Define*

$$x \sqcup^* y = ((x \rightarrowtriangle (x \sqcap y)) \sqcap (y \rightarrowtriangle (x \sqcap y))) \rightarrowtriangle (x \sqcap y).$$

Then $\mathfrak{A}^* = (\mathfrak{M}, \sqcup^*, \multimap)$ *is a Boolean latarre.*

Proof. From the definitions we see that Boolean semilatarres satisfy schema

$$x \trianglerighteq y \quad \rightarrow \quad (x \multimap y) \multimap y = x.$$

Since $(x \multimap (x \sqcap y)) \sqcap (y \multimap (x \sqcap y)) \trianglelefteq (x \multimap (x \sqcap y))$, we have $x = (x \multimap (x \sqcap y)) \multimap (x \sqcap y) \trianglelefteq x \sqcup^* y$. By symmetry we also have $y \trianglelefteq x \sqcup^* y$. So $x \sqcup^* y$ is an upper bound of x and y. Suppose $x \trianglelefteq z$ and $y \trianglelefteq z$. Then

$$\begin{aligned}
&z \multimap (x \sqcap y) \trianglelefteq x \multimap (x \sqcap y) \text{ and} \\
&\qquad z \multimap (x \sqcap y) \trianglelefteq y \multimap (x \sqcap y). \\
&z \multimap (x \sqcap y) \trianglelefteq (x \multimap (x \sqcap y)) \sqcap (y \multimap (x \sqcap y)). \\
&x \sqcup^* y \trianglelefteq (z \multimap (x \sqcap y)) \multimap (x \sqcap y) = z.
\end{aligned}$$

So $x \sqcup^* y$ is the least upper bound of x and y. Finally, Heyting semilatarres have

$$(x \sqcup^* y) \multimap z = (x \multimap z) \sqcap (y \multimap z).$$

Thus $\mathfrak{A}^*$ is a Boolean latarre. $\square$

U is a *complete ideal* if $U \subseteq A$ is a downward closed subset such that for all subsets $F \subseteq U$, if $\bigsqcup F$ exists, then $\bigsqcup F \in U$.

Proposition 0.13. *Let $\mathfrak{A}$ be a Heyting semilatarre. Let $\mathfrak{H} = \mathfrak{H}(\mathfrak{A})$ be the substructure of $\mathfrak{D}(\mathfrak{A})$ of complete ideals (only $\sqcup$ changes). Then $\mathfrak{H}$ is a Heyting latarre and a frame, with $\multimap$ equal the standard arrow. Map $\delta(a) = \langle\, a]$ is a semilatarre embedding which preserves all colimits, so is a latarre embedding if $\mathfrak{A}$ is a Heyting latarre. If $\mathfrak{A}$ is complete as lattice, then δ is a latarre isomorphism.*

Given a latarre $\mathfrak{A}$ and element a, we construct a latarre $\mathfrak{A}_a$ on domain $\langle\, a]$ as follows. Set

$$\begin{aligned}
&\varepsilon_a = \varepsilon \sqcap a \\
&x \multimap_a y = a \sqcap (x \multimap y) \\
&x \sqcap_a y = x \sqcap y \\
&x \sqcup_a y = x \sqcup y
\end{aligned}$$

Function $\pi_a(x) = a \sqcap x$ is an idempotent map from $\mathfrak{A}$ onto $\mathfrak{A}_a$.

Define $\mathfrak{A}$ *admits meet substitution* if for all terms $t(x)$ and $a \in A$ we have $\mathfrak{A} \models \forall xy(a \sqcap x = a \sqcap y \to a \sqcap t(x) = a \sqcap t(y))$. This is equivalent to $\mathfrak{A} \models \forall x(a \sqcap t(x) = a \sqcap t(a \sqcap x))$, which is a universal equation.

An element a of a latarre $\mathfrak{A}$ is called *weakly persistent* over $\mathfrak{A}$ if $\mathfrak{A}$ satisfies schema $a \sqcap \varepsilon \trianglelefteq (x \multimap a)$. A latarre $\mathfrak{A}$ is called *weakly Visser* if it is distributive, and if it satisfies schema

$$x \sqcap \varepsilon \;\trianglelefteq\; y \multimap x$$

Proposition 0.14. *A latarre $\mathfrak{A}$ is weakly Visser exactly when $\mathfrak{A}$ admits meet substitution.*

So if a latarre is unitary, then it admits meet substitution exactly when it is a Visser latarre.

Over Visser latarres we know that each term $t(x)$ has *explicit fixpoint* $t(1)$, that is, $t(t(1)) = t(1)$, exactly when for all elements a term $t_a(x) = x \multimap a$ has fixpoint $t_a(1)$, that is, $t_a(t_a(1)) = t_a(1)$.

A term $t(x)$ is called *fixed* over a (unitary) latarre $\mathfrak{A}$ if $\mathfrak{A}$ satisfies schema $t(t(x) \sqcap x) = t(x)$. An element a is called *Löb* over if $\mathfrak{A}$ satisfies schema $t_a(t_a(x) \sqcap x) = t_a(x)$. A latarre is *fixed* if all its terms are fixed. A latarre is *Löb* if all its elements are Löb. Obviously fixed implies Löb.

Proposition 0.15. *Let $t(x)$ be a term over a Visser latarre $\mathfrak{A}$. Then $t(x)$ is fixed over $\mathfrak{A}$ if and only if $t(x)$ has explicit fixpoint $t(1)$. So element a is Löb if and only if $t_a(t_a(1)) = t_a(1)$.*

Proposition 0.16. *The following are equivalent for a (unitary) latarre $\mathfrak{A}$.*

1. *$\mathfrak{A}$ has explicit fixpoints*
2. *$\mathfrak{A}$ is a weakly Visser and $t_a(t_a(1)) = t_a(1)$ for all a*
3. *$\mathfrak{A}$ is a weakly Visser and Löb*

4. $\mathfrak{A}$ *is fixed*