Latarres, Lattices with an Arrow

Mohammad Ardeshir
Department of Mathematics
Sharif University of Technology
P.O. Box 11365-9415
Tehran, Iran
e-mail: mardeshir@sharif.edu

Wim Ruitenburg
Department of Mathematics, Statistics and Computer Science
Marquette University
P.O. Box 1881
Milwaukee, WI 53201, USA
e-mail: wimr@mscs.mu.edu

Abstract

A latarre is a lattice with an arrow. Its axiomatization looks natural. Latarres have a nontrivial theory which permits many constructions of latarres. Latarres appear as an end result of a series of generalizations of better known structures. These include Boolean algebras and Heyting algebras. Latarres need not have a distributive lattice.

1 Introduction

We introduce latarres as a natural class of universal algebra structures of lattices with an arrow. A reader may question our qualification of 'natural', maybe not so much because our definitions are unnatural, but rather that there may be other notions of 'lattice with arrow' which have a fair claim to be called natural. Regardless, we hope that the context of this paper is sufficient to convince that our qualification can be justified.

Latarres are the end result of a series of generalizations. Our process follows from earlier mathematical results obtained about Boolean algebras, Heyting algebras, Visser algebras (see [2], [3], and [8]), and what we call CJ algebras, after Celani and Jansana (weakly Heyting algebras in [5]). With this paper we have no pretense to offer a complete compilation of these results. Rather, we present a sufficiently extensive theory about latarres to show that they are a good class to consider. Much in this paper consists of modest generalizations of well-known material.

2 What is a Latarre?

A latarre is a LATtice with an ARRow. Before giving our formal definitions, let us look at the essentials of its language, and the essential 'natural' defining properties. The essential parts of its language consist of three binary operators $(\sqcap, \sqcup, \rightarrow)$. With restriction to $(\sqcap, \sqcup)$ a latarre is a lattice with meet $\sqcap$ and join $\sqcup$. For the arrow we have the additional schemas

$x \rightarrow y = (x \sqcup y) \rightarrow y$.
$x \rightarrow y = x \rightarrow (x \sqcap y)$.
$y \trianglelefteq z$ implies $x \rightarrow y \trianglelefteq x \rightarrow z$.
$y \trianglelefteq z$ implies $z \rightarrow x \trianglelefteq y \rightarrow x$.
$(x \rightarrow y) \sqcap (y \rightarrow z) \trianglelefteq x \rightarrow z$.

where $\trianglelefteq$ is the usual order definable by $x \trianglelefteq y$ exactly when $x \sqcap y = x$. None of these schemas is original; even the collection as a whole we expect is known, at least in the special case of distributive lattices.

Latarres form a universal algebra class. Below is an axiomatization by a collection of universal equations. For practical reasons we extend our essential list to $(\sqcap, \sqcup, \multimap, \varepsilon)$ by adding a constant ε to the three binary operators mentioned above. A *latarre* is a structure satisfying the universal algebra schemas of a lattice with meet $\sqcap$ and join $\sqcup$, plus

N1. $x \multimap y = (x \sqcup y) \multimap y$.

N2. $x \multimap y = x \multimap (x \sqcap y)$.

N3. $x \multimap (x \sqcap y \sqcap z) \trianglelefteq x \multimap (x \sqcap y)$.

N4. $y \multimap (y \sqcap z) \trianglelefteq (x \sqcap y) \multimap (x \sqcap y \sqcap z)$.

N5. $(x \multimap (x \sqcap y)) \sqcap ((x \sqcap y) \multimap (x \sqcap y \sqcap z)) \trianglelefteq x \multimap (x \sqcap y \sqcap z)$.

N6. $\varepsilon \multimap \varepsilon = \varepsilon$.

Element ε is an important convenience, but no more. With the proof of Proposition 2.1.4 we show that the additions of ε and its schema N6 are conservative over the subsystem without them. Additionally, given a subsystem without ε, we can add this element in only one way to get a latarre as defined above. That is, ε with N6 is uniquely definable over the subsystem.

Let us briefly ignore ε and schema N6. Then the remaining schemas N1 through N5 easily follow from the 'natural' schemas near the beginning of this Section. The following Proposition includes the reverse direction.

Proposition 2.1. *Latarres satisfy schemas*

1. $y \trianglelefteq z$ *implies* $x \multimap y \trianglelefteq x \multimap z$.
2. $y \trianglelefteq z$ *implies* $z \multimap x \trianglelefteq y \multimap x$.
3. $(x \multimap y) \sqcap (y \multimap z) \trianglelefteq x \multimap z$.
4. $x \multimap y \trianglelefteq z \multimap z$.

Proof. Item 1: Suppose $y \trianglelefteq z$. With N2 and N3 we have $x \multimap y = x \multimap (x \sqcap y) = x \multimap (x \sqcap y \sqcap z) \trianglelefteq x \multimap (x \sqcap z) = x \multimap z$.

Item 2: Suppose $y \trianglelefteq z$. With N2 and N4 we have $z \multimap x = z \multimap (x \sqcap z) \trianglelefteq (y \sqcap z) \multimap (x \sqcap y \sqcap z) = y \multimap (x \sqcap y) = y \multimap x$.

Item 3: Apply N2, N4, N5, N3, and N2 to get

$$\begin{aligned}&(x \multimap y) \sqcap (y \multimap z) = (x \multimap (x \sqcap y)) \sqcap (y \multimap (y \sqcap z)) \trianglelefteq \\ &(x \multimap (x \sqcap y)) \sqcap ((x \sqcap y) \multimap (x \sqcap y \sqcap z)) \trianglelefteq \\ &x \multimap (x \sqcap y \sqcap z) \trianglelefteq x \multimap (x \sqcap z) = x \multimap z.\end{aligned}$$

Item 4: With N2 and N1 we get schema $(x \sqcap y) \multimap (x \sqcap y) = (x \sqcap y) \multimap y = ((x \sqcap y) \sqcup y) \multimap y = y \multimap y$. So by symmetry we have schema $x \multimap x = (x \sqcap y) \multimap (x \sqcap y) = y \multimap y$. The value of $z \multimap z$ is constant and independent of z. With N2 and N4 we have schema $x \multimap y = x \multimap (x \sqcap y) \trianglelefteq (x \sqcap y) \multimap (x \sqcap y) = z \multimap z$. ☐

The proof of Proposition 2.1 does not use N6, and $z \multimap z$ is constant and the largest value possible for $x \multimap y$. So with N6 we only assign name ε to this constant $z \multimap z$ of Proposition 2.1.4. So we have

Corollary 2.2. *Latarres satisfy schemas*

1. $x \multimap y \trianglelefteq \varepsilon$.

2. $x \rightarrow x = \varepsilon$.

3. $x \trianglelefteq y$ *implies* $x \rightarrow y = \varepsilon$.

4. $x \rightarrow y = \varepsilon$ *implies* $z \rightarrow x \trianglelefteq z \rightarrow y$ *and* $y \rightarrow z \trianglelefteq x \rightarrow z$.

Proof. Item 3: $x \trianglelefteq y$ implies $x \rightarrow y = x \rightarrow (x \sqcap y) = x \rightarrow x = \varepsilon$.

Item 4: $x \rightarrow y = \varepsilon$ implies $z \rightarrow x = (z \rightarrow x) \sqcap (x \rightarrow y) \trianglelefteq z \rightarrow y$ and $y \rightarrow z = (x \rightarrow y) \sqcap (y \rightarrow z) \trianglelefteq x \rightarrow z$. □

In Section 3 we start with trivial examples of latarres that include ones that are neither distributive nor have a largest element. So ε need not be top. The following are further examples of schemas that have shown useful.

Proposition 2.3. *Latarres satisfy schemas*

1. $x \rightarrow (y \sqcap z) = (x \rightarrow y) \sqcap (x \rightarrow z)$.

2. $(y \sqcup z) \rightarrow x = (y \rightarrow x) \sqcap (z \rightarrow x)$.

3. $z \rightarrow x \trianglelefteq (x \rightarrow y) \sqcap (y \rightarrow z)$ *implies* $(z \rightarrow x) = (z \rightarrow y) \sqcap (y \rightarrow x)$. *In particular,* $z \trianglerighteq y \trianglerighteq x$ *implies* $(z \rightarrow x) = (z \rightarrow y) \sqcap (y \rightarrow x)$.

4. $(x \rightarrow y) \sqcap (y \rightarrow z) = (x \sqcup y) \rightarrow (y \sqcap z)$.

5. $y \rightarrow z = \varepsilon$ *implies* $(x \sqcup y) \rightarrow z = x \rightarrow (x \sqcap z) = x \rightarrow z$.

6. $z \rightarrow x = \varepsilon$ *implies* $z \rightarrow (x \sqcap y) = (z \sqcup y) \rightarrow y = z \rightarrow y$.

7. $y \rightarrow z \trianglelefteq (x \sqcap y) \rightarrow (x \sqcap z)$.

8. $(y \rightarrow x) \sqcap (y \rightarrow z) = (y \rightarrow x) \sqcap ((x \sqcap y) \rightarrow (x \sqcap z))$.

Proof. Item 1: We have

$$(x \rightarrow (x \sqcap y)) \sqcap (x \rightarrow (x \sqcap z)) = x \rightarrow (x \sqcap y \sqcap z),$$

where direction $\trianglelefteq$ follows with N4 and N5, and direction $\trianglerighteq$ follows with two applications of N3. With N2 this schema is equivalent to

$$(x \rightarrow y) \sqcap (x \rightarrow z) = x \rightarrow (y \sqcap z).$$

Item 2: By Proposition 2.1.2 we have $(y \sqcup z) \rightarrow x \trianglelefteq (y \rightarrow x) \sqcap (z \rightarrow x)$. Conversely, with Propositions 2.1.1 and 2.1.3 and with N1 we have $(y \rightarrow x) \sqcap (z \rightarrow x) \trianglelefteq (y \rightarrow (x \sqcup z)) \sqcap (z \rightarrow x) = ((x \sqcup y \sqcup z) \rightarrow (x \sqcup z)) \sqcap ((x \sqcup z) \rightarrow x) \trianglelefteq ((x \sqcup y \sqcup z) \rightarrow x) = (y \sqcup z) \rightarrow x$.

Item 3: We always have $(z \rightarrow y) \sqcap (y \rightarrow x) \trianglelefteq z \rightarrow x$. In the other direction, $z \rightarrow x \trianglelefteq (x \rightarrow y) \sqcap (y \rightarrow z)$ implies $(z \rightarrow x) = (z \rightarrow x) \sqcap (x \rightarrow y) \trianglelefteq z \rightarrow y$ and $(z \rightarrow x) = (y \rightarrow z) \sqcap (z \rightarrow x) \trianglelefteq y \rightarrow x$.

Item 4: $(x \rightarrow y) \sqcap (y \rightarrow z) = ((x \sqcup y) \rightarrow y) \sqcap (y \rightarrow (y \sqcap z))$. Apply item 3.

Item 5: $(x \sqcup y) \rightarrow z = (x \rightarrow z) \sqcap (y \rightarrow z) = x \rightarrow z = x \rightarrow (x \sqcap z)$.

Item 6: $z \rightarrow (x \sqcap y) = (z \rightarrow x) \sqcap (z \rightarrow y) = z \rightarrow y = (z \sqcup y) \rightarrow y$.

Item 7: We have $y \rightarrow z = y \rightarrow (y \sqcap z) \trianglelefteq (x \sqcap y) \rightarrow (x \sqcap y \sqcap z) = (x \sqcap y) \rightarrow (x \sqcap z)$.

Item 8: With item 7 it suffices to show direction $\trianglerighteq$. We have $(y \rightarrow x) \sqcap ((x \sqcap y) \rightarrow (x \sqcap z)) = (y \rightarrow (x \sqcap y)) \sqcap ((x \sqcap y) \rightarrow (x \sqcap z)) \trianglelefteq y \rightarrow (x \sqcap z) = (y \rightarrow x) \sqcap (y \rightarrow z)$. □

Besides schemas we also have useful relations between schemas:

Proposition 2.4. *Let a, b, and c be elements of a latarre $\mathfrak{A}$. Then*

1. $c \sqcap \varepsilon = c \sqcap (b \rightarrow a)$ *if and only if* $\mathfrak{A}$ *satisfies schema* $c \sqcap ((a \sqcap b) \rightarrow (a \sqcap x)) = c \sqcap (b \rightarrow x)$.

2. $\mathfrak{A}$ *satisfies schema* $a \sqcap \varepsilon \trianglelefteq z \twoheadrightarrow a$ *if and only if* $\mathfrak{A}$ *satisfies schema* $a \sqcap ((a \sqcap x) \twoheadrightarrow (a \sqcap y)) = a \sqcap (x \twoheadrightarrow y)$.

Proof. Item 1: From right to left, substitute a for x. From left to right with Proposition 2.3.8, $c \sqcap ((a \sqcap b) \twoheadrightarrow (a \sqcap x)) = c \sqcap \varepsilon \sqcap ((a \sqcap b) \twoheadrightarrow (a \sqcap x)) = c \sqcap (b \twoheadrightarrow a) \sqcap ((a \sqcap b) \twoheadrightarrow (a \sqcap x)) = c \sqcap (b \twoheadrightarrow a) \sqcap (b \twoheadrightarrow x) = c \sqcap \varepsilon \sqcap (b \twoheadrightarrow x) = c \sqcap (b \twoheadrightarrow x)$.

Item 2: Clearly $a \sqcap \varepsilon \trianglelefteq z \twoheadrightarrow a$ if and only if $a \sqcap \varepsilon = a \sqcap (z \twoheadrightarrow a)$. Apply item 1. □

We inductively define $\nabla^n x$ for all n by $\nabla^0 x = x$ and $\nabla^{n+1} x = \varepsilon \twoheadrightarrow \nabla^n x$.

Proposition 2.5. *Latarres satisfy schemas*

1. $\nabla^n(x \sqcap y) = \nabla^n x \sqcap \nabla^n y$.
2. $x \sqcap y \twoheadrightarrow z = \varepsilon$ *implies* $y \twoheadrightarrow x \trianglelefteq y \twoheadrightarrow z$.
   So $x \trianglelefteq y \twoheadrightarrow x$ *plus* $x \sqcap y \twoheadrightarrow z = \varepsilon$ *implies* $x \trianglelefteq y \twoheadrightarrow z$.
3. $x \trianglelefteq y \twoheadrightarrow z$ *implies* $x \sqcap (w \twoheadrightarrow y) \trianglelefteq w \twoheadrightarrow z$.
4. $y \twoheadrightarrow \varepsilon = \varepsilon$ *implies* $\nabla x \sqcap ((x \sqcap y) \twoheadrightarrow (x \sqcap z)) = \nabla x \sqcap (y \twoheadrightarrow z)$.
5. $y \twoheadrightarrow \varepsilon = \varepsilon$ *plus* $x \sqcap y \twoheadrightarrow z = \varepsilon$ *implies* $\nabla x \trianglelefteq y \twoheadrightarrow z$.

Proof. Item 1: By induction on n, using Proposition 2.3.1.
Item 2: $y \twoheadrightarrow x = (y \twoheadrightarrow (x \sqcap y)) \sqcap ((x \sqcap y) \twoheadrightarrow z) \trianglelefteq y \twoheadrightarrow z$.
Item 3 follows immediately with Proposition 2.1.3.
Item 4: Use that $\varepsilon \twoheadrightarrow x = (y \twoheadrightarrow \varepsilon) \sqcap (\varepsilon \twoheadrightarrow x) \trianglelefteq y \twoheadrightarrow x$, and Proposition 2.3.8.
Item 5: Use that $\varepsilon \twoheadrightarrow x = (y \twoheadrightarrow \varepsilon) \sqcap (\varepsilon \twoheadrightarrow x) \trianglelefteq y \twoheadrightarrow x$, and item 2. □

3 Examples of Latarres

Next we consider some simple examples of latarres and ways to construct more. We do not aim for maximum generality.

One collection of trivial latarres is the following. Start with any lattice $\mathfrak{M}$ and any element m of $\mathfrak{M}$. Set $x \twoheadrightarrow y = m$ for all elements x and y of $\mathfrak{M}$. This defines a 'trivial' latarre with $\varepsilon = m$ and $\mathfrak{M}$ as underlying lattice.

Definition 3.1. Some of the examples below invite new definitions. Examples: A latarre is called *unitary* if the lattice has a top 1 and $\varepsilon = 1$. A latarre is called *arrow persistent* if it satisfies schema $x \sqcap \varepsilon \trianglelefteq y \twoheadrightarrow x$. A latarre is called *Heyting* if it satisfies schema $x = \nabla x$. A latarre is called *Boolean* if it satisfies schema $(x \twoheadrightarrow (x \sqcap y)) \twoheadrightarrow (x \sqcap y) = x$. Obviously sublatarres of Boolean latarres are again Boolean, sublatarres of Heyting latarres are again Heyting, sublatarres of arrow persistent latarres are again arrow persistent, and sublatarres of unitary latarres are again unitary. All latarres satisfy schema $\nabla x \trianglelefteq \varepsilon$, so a latarre is unitary arrow persistent exactly when it satisfies schema $x \trianglelefteq \nabla x$. So Heyting latarres are unitary arrow persistent. When we set $x = y$ in the defining schema of Boolean latarres, we get $(x \twoheadrightarrow x) \twoheadrightarrow x = \nabla x = x$. So Boolean latarres are Heyting.

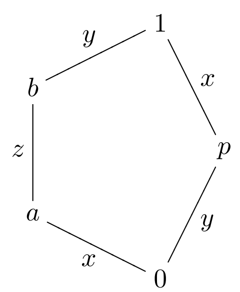
A latarre is called *complete* if its underlying lattice is a complete lattice. A latarre is called *almost-complete* if for each subset S which contains an element, $\bigsqcup S$ exists or, equivalently, if for each subset S with a lower bound, $\bigsqcap S$ exists. So complete implies almost-complete. A well-known class of complete latarres is the following. A complete lattice is called a *frame* (or a complete Heyting algebra or a locale) when it satisfies $m \sqcap \bigsqcup S = \bigsqcup\{m \sqcap s \ : \ s \in S\}$, for all sets of elements $\{m\} \cup S$. The lattice $\mathcal{O}(X)$ of open sets of a topological space forms a frame. On a frame $\mathfrak{M}$ we can define an arrow $x \twoheadrightarrow y = \bigsqcup\{z \ : \ x \sqcap z \trianglelefteq y\}$. The resulting structure $(\mathfrak{M}, \twoheadrightarrow, 1, 0)$ is a complete Heyting algebra. Each filter F on frame $\mathfrak{M}$ is the domain of an almost-complete Heyting latarre $(\mathfrak{F}, \twoheadrightarrow, 1)$.

A function $f : \mathfrak{A} \to \mathfrak{B}$ between latarres is called a latarre *homomorphism*, or simply a *morphism*, if f preserves the defining operations of $\sqcap$, $\sqcup$, $\rightarrowtriangle$, and ε. Latarres are an equational class, so its class of models is closed under submodels, products, and (homomorphic) images.

Example 3.2. We can define Boolean algebras in terms of basic operations $\sqcap$, $\sqcup$, $\rightarrowtriangle$, 0, and 1, with their usual properties. Complement is definable by $-x := x \rightarrowtriangle 0$. When we ignore 0 as special element and set $\varepsilon := 1$, we get a Boolean latarre, say $\mathfrak{B}$. Filters on $\mathfrak{B}$ are exactly the upward closed sublatarres of $\mathfrak{B}$.

Example 3.3. The claims about Boolean algebras have the expected straightforward generalization to Heyting algebras and Heyting latarres. When we ignore 0 and set $\varepsilon := 1$, we get a Heyting latarre, say $\mathfrak{C}$. Filters on $\mathfrak{C}$ are exactly the upward closed sublatarres of $\mathfrak{C}$.

Example 3.4. Define a unitary latarre on lattice N_5 as follows. In the diagram of N_5 below, labels x, y, and z mean that we set $1 \rightarrowtriangle b = y$, set $b \rightarrowtriangle a = z$, and so on. The letters x, y and z are values to be chosen freely from the domain $\{0, a, b, p, 1\}$ with the only restrictions that $x \trianglelefteq z$ and $y \trianglelefteq z$.



The properties of unitary latarres allow us to uniquely extend the arrow by $p \rightarrowtriangle p = 1$ and $1 \rightarrowtriangle a = (1 \rightarrowtriangle b) \sqcap (b \rightarrowtriangle a) = y \sqcap z = y$ and $a \rightarrowtriangle p = a \rightarrowtriangle a \sqcap p = a \rightarrowtriangle 0 = x$, and so on. Verifying the arrow axioms is tedious but straightforward.

Here are two results on how to construct new latarres from old.

Proposition 3.5. *Let $\mathfrak{A} = (\mathfrak{M}, \sqcup, \rightarrowtriangle, \varepsilon)$ be a latarre and $f : \mathfrak{M} \to \mathfrak{M}$ be a meet semilattice endomorphism. Define $\mathfrak{A}_f = (\mathfrak{M}, \sqcup, \rightarrowtriangle_f, f(\varepsilon))$ by $a \rightarrowtriangle_f b = f(a \rightarrowtriangle b)$. Then $\mathfrak{A}_f$ is a latarre.*

Proof. All arrow axioms are easily verified. For example,

$$\begin{aligned}
&(x \rightarrowtriangle_f (x \sqcap y)) \sqcap ((x \sqcap y) \rightarrowtriangle_f (x \sqcap y \sqcap z)) = \\
&f(x \rightarrowtriangle (x \sqcap y)) \sqcap f((x \sqcap y) \rightarrowtriangle (x \sqcap y \sqcap z)) = \\
&f((x \rightarrowtriangle (x \sqcap y)) \sqcap ((x \sqcap y) \rightarrowtriangle (x \sqcap y \sqcap z))) \trianglelefteq \\
&f(x \rightarrowtriangle (x \sqcap y \sqcap z)) = x \rightarrowtriangle_f (x \sqcap y \sqcap z).
\end{aligned}$$

□

Proposition 3.6. *Let $\mathfrak{A} = (\mathfrak{N}, \rightarrowtriangle, \varepsilon)$ be a latarre and $g : \mathfrak{N} \to \mathfrak{N}$ be a lattice endomorphism. Define $\mathfrak{A}^g = (\mathfrak{N}, \rightarrowtriangle^g, \varepsilon)$ by $a \rightarrowtriangle^g b = g(a) \rightarrowtriangle g(b)$. Then $\mathfrak{A}^g$ is a latarre.*

Proof. All arrow axioms are easily verified. For example, $(x \sqcup y) \rightarrowtriangle^g y == g(x \sqcup y) \rightarrowtriangle gy = (gx \sqcup gy) \rightarrowtriangle gy = gx \rightarrowtriangle gy = x \rightarrowtriangle^g y$. □

Example 3.7. Let $f : A \to A$ be a bijection on set A. Then f extends to a bijection $f : \mathcal{P}(A) \to \mathcal{P}(A)$ on the power set of A, defined by $f(X) = \{f(x) : x \in X\}$. Following Example 3.2, power set $\mathcal{P}(A)$ is the domain of a complete Boolean latarre $\mathfrak{B} = (\mathfrak{M}, \sqcup, \rightarrowtriangle, \varepsilon)$ with $\varepsilon = A$, and $\mathcal{P}(A)$ is also the domain of the corresponding meet semilattice $\mathfrak{M}$. Clearly f is a semilattice morphism on $\mathfrak{M}$. By Proposition 3.5 we get a new latarre $\mathfrak{B}_f$ from $\mathfrak{B}$ by redefining $X \rightarrowtriangle_f Y := f(X \rightarrowtriangle Y) = f(X^c \cup Y)$.

Example 3.8. Let $g : A \to A$ be a continuous function on a topological space $\mathcal{O}(A)$. Then inverse image map $h = g^{-1} : \mathcal{O}(A) \to \mathcal{O}(A)$ is a meet semilattice morphism on the meet semilattice part $\mathfrak{N}$ of the frame (or complete Heyting algebra, or locale) $\mathcal{O}(A)$. Following Example 3.3, $\mathfrak{N}$ is the meet semilattice part of the corresponding complete Heyting latarre $\mathfrak{C} = (\mathfrak{N}, \sqcup, \twoheadrightarrow, A)$. By Proposition 3.5 we get a new latarre $\mathfrak{C}_h$ from $\mathfrak{C}$ by redefining $\varepsilon_h = g^{-1}(\varepsilon)$ and $U \twoheadrightarrow_h V = h(U \twoheadrightarrow V) = g^{-1}(U \twoheadrightarrow V) = \bigcup\{g^{-1}(W) : W \cap U \subseteq V\}$. Map $h = g^{-1}$ is also a lattice morphism on $(\mathfrak{N}, \sqcup)$. So by Proposition 3.6 we get another new latarre $\mathfrak{C}^h$ from $\mathfrak{C}$ by redefining $U \twoheadrightarrow^h V = g^{-1}(U) \twoheadrightarrow g^{-1}(V) = \bigcup\{W : g(W \cap g^{-1}(U)) \subseteq V\}$.

Here are two other results on how to construct new latarres from old.

Proposition 3.9. *Let $f : \mathfrak{M} \to \mathfrak{N}$ be a lattice morphism, and $g : \mathfrak{N} \to \mathfrak{M}$ be map which preserves meet $\sqcap$. Let $\mathfrak{B} = (\mathfrak{N}, \twoheadrightarrow, \varepsilon)$ be a latarre. Define ε_m and $\twoheadrightarrow_m$ on $\mathfrak{M}$ by $\varepsilon_m = g(\varepsilon)$ and $x \twoheadrightarrow_m y = g(f(x) \twoheadrightarrow f(y))$. Then $\mathfrak{A} = (\mathfrak{M}, \twoheadrightarrow_m, \varepsilon_m)$ is a latarre.*

Proof. It suffices to check the following schemas.

Clearly $x \twoheadrightarrow_m x = g(f(x) \twoheadrightarrow f(x)) = g(\varepsilon) = \varepsilon_m$, and $x \twoheadrightarrow_m y \trianglelefteq \varepsilon_m$.

We have $(y \sqcup z) \twoheadrightarrow_m x = g(f(y \sqcup z) \twoheadrightarrow f(x)) = g((f(y) \sqcup f(z)) \twoheadrightarrow f(x)) = g((f(y) \twoheadrightarrow f(x)) \sqcap (f(z) \twoheadrightarrow f(x))) = g(f(y) \twoheadrightarrow f(x)) \sqcap g(f(z) \twoheadrightarrow f(x)) = (y \twoheadrightarrow_m x) \sqcap (z \twoheadrightarrow_m x)$.

We have $x \twoheadrightarrow_m (y \sqcap z) = g(f(x) \twoheadrightarrow f(y \sqcap z)) = g(f(x) \twoheadrightarrow (f(y) \sqcap f(z))) = g((f(x) \twoheadrightarrow f(y)) \sqcap (f(x) \twoheadrightarrow f(z))) = g(f(x) \twoheadrightarrow f(y)) \sqcap g(f(x) \twoheadrightarrow f(z)) = (x \twoheadrightarrow_m y) \sqcap (x \twoheadrightarrow_m z)$.

Finally, $(x \twoheadrightarrow_m y) \sqcap (y \twoheadrightarrow_m z) = g(f(x) \twoheadrightarrow f(y)) \sqcap g(f(y) \twoheadrightarrow f(z)) = g((f(x) \twoheadrightarrow f(y)) \sqcap (f(y) \twoheadrightarrow f(z))) \trianglelefteq g(f(x) \twoheadrightarrow f(z)) = x \twoheadrightarrow_m z$. □

Map $f : \mathfrak{A} \to \mathfrak{B}$ of Proposition 3.9 need not be a latarre morphism. By Proposition 3.5 we have a latarre $\mathfrak{B}_{fg} = (\mathfrak{N}, \twoheadrightarrow_{fg}, fg(\varepsilon))$ with $x \twoheadrightarrow_{fg} y = fg(x \twoheadrightarrow y)$. Map $f : \mathfrak{A} \to \mathfrak{B}_{fg}$ is a latarre morphism.

Suppose map $g : \mathfrak{N} \to \mathfrak{M}$ of Proposition 3.9 is a lattice morphism. Map $g : \mathfrak{B} \to \mathfrak{A}$ need not be a latarre morphism. By Proposition 3.6 we have a latarre $\mathfrak{B}^{fg} = (\mathfrak{N}, \twoheadrightarrow^{fg}, \varepsilon)$ with $x \twoheadrightarrow^{fg} y = fg(x) \twoheadrightarrow fg(y)$. Map $g : \mathfrak{B}^{fg} \to \mathfrak{A}$ is a latarre morphism.

Proposition 3.10. *Let $\mathfrak{A}_1 = (\mathfrak{M}, \varepsilon_1 \twoheadrightarrow_1)$ and $\mathfrak{A}_2 = (\mathfrak{M}, \varepsilon_2, \twoheadrightarrow_2)$ be latarres on the same lattice $\mathfrak{M}$. Define $\mathfrak{A} = (\mathfrak{M}, \varepsilon, \twoheadrightarrow)$ by $\varepsilon = \varepsilon_1 \sqcap \varepsilon_2$ and $x \twoheadrightarrow y = (x \twoheadrightarrow_1 y) \sqcap (x \twoheadrightarrow_2 y)$. Then $\mathfrak{A}$ is a latarre.*

Proof. All arrow properties are easy. For example, $x \twoheadrightarrow (y \sqcap z) = (x \twoheadrightarrow_1 (y \sqcap z)) \sqcap (x \twoheadrightarrow_2 (y \sqcap z)) = (x \twoheadrightarrow_1 y) \sqcap (x \twoheadrightarrow_1 z) \sqcap (x \twoheadrightarrow_2 y) \sqcap (x \twoheadrightarrow_2 z) = (x \twoheadrightarrow y) \sqcap (x \twoheadrightarrow z)$. □

Suppose lattice $\mathfrak{M}$ in Proposition 3.10 is complete, and $\{\mathfrak{A}_s : s \in S\}$ is a collection of latarres $\mathfrak{A}_s = (\mathfrak{M}, \twoheadrightarrow_s, \varepsilon_s)$. Then $\mathfrak{A}_S = \bigsqcap\{\mathfrak{A}_s : s \in S\} = (\mathfrak{M}, \twoheadrightarrow_S, \varepsilon_S)$ with $x \twoheadrightarrow_S y = \bigsqcap\{x \twoheadrightarrow_s y : s \in S\}$ and $\varepsilon_S = \bigsqcap\{\varepsilon_s : s \in S\}$ is a well-defined structure. An easy verification of the arrow properties shows that $\mathfrak{A}_S$ is a latarre.

Example 3.11. Here is an application of Proposition 3.9. Let $\mathfrak{B}_2 = (\mathfrak{N}_2, \twoheadrightarrow, 1)$ be the usual Boolean algebra with domain $\{0, 1\}$, but treated as a Boolean latarre which happens to have a least element. So $\mathfrak{N}_2$ is a 2-element linearly ordered lattice. Let F be a prime filter on a lattice $\mathfrak{M}$. The map $f : \mathfrak{M} \to \mathfrak{N}_2$ defined by $f(x) = 1$ exactly when $x \in F$, is a lattice morphism. Let $a \trianglelefteq b$ be elements of $\mathfrak{M}$. Define map $g : \mathfrak{N}_2 \to \mathfrak{M}$ by $g(1) = b$ and $g(0) = a$. Clearly g preserves meet $\sqcap$. On $\mathfrak{M}$ define $x \twoheadrightarrow_m y = g(f(x) \twoheadrightarrow f(y))$. Then $\mathfrak{A} = (\mathfrak{M}, \twoheadrightarrow_m, b)$ is a latarre. For all x and y in $\mathfrak{A}$ we have $x \twoheadrightarrow_m y = b$ or $x \twoheadrightarrow_m y = a$. If $x \in F$ and $y \notin F$, then $x \twoheadrightarrow_m y = a$. If $x \notin F$ or $y \in F$, then $x \twoheadrightarrow_m y = b$.

We can combine the construction above with Proposition 3.10. Given prime filters F and G on lattice $\mathfrak{M}$, and pairs $a \trianglelefteq b$ and $c \trianglelefteq d$ of elements of $\mathfrak{M}$, we apply the construction above twice to build, besides $\mathfrak{A} = (\mathfrak{M}, \twoheadrightarrow_m, b)$, another latarre $\mathfrak{B} = (\mathfrak{M}, \twoheadrightarrow_n, d)$. Proposition 3.10 allows us to form a new latarre $\mathfrak{C} = (\mathfrak{M}, \twoheadrightarrow, b \sqcap d)$ satisfying

$x \rightharpoonup y = (x \rightharpoonup_m y) \sqcap (x \rightharpoonup_n y)$. The domain M of $\mathfrak{M}$ is the disjoint union of the sets $e = F \cap G$, $p = F \setminus G$, $q = G \setminus F$, and $o = M \setminus (F \cup G)$. The value of $x \rightharpoonup y$ depends on which set the elements x or y belong to, as implied by the table

$\rightharpoonup$	e	p	q	o
e	$b \sqcap d$	$b \sqcap c$	$a \sqcap d$	$a \sqcap c$
p	$b \sqcap d$	$b \sqcap d$	$a \sqcap d$	$a \sqcap d$
q	$b \sqcap d$	$b \sqcap c$	$b \sqcap d$	$b \sqcap c$
o	$b \sqcap d$	$b \sqcap d$	$b \sqcap d$	$b \sqcap d$

Example 3.12. Let R be a commutative ring. Its collection of ideals is closed under intersections, so forms a complete lattice ordered by set inclusion, Let $\mathfrak{M}$ be the complete lattice of ideals, with $I \sqcap J = I \cap J$ for all ideals I and J. Lattice $\mathfrak{M}$ need not be distributive. An ideal I is called a *radical* ideal if $r^2 \in I$ implies $r \in I$, for all $r \in R$. The set $\sqrt{I} = \{r \in R \ : \ r^n \in I \text{ for some } n\}$ is the least radical ideal containing I. Given ideals I and J, the set $J : I = \{r \in R \ : \ rI \subseteq J\}$ is an ideal. We construct a unitary complete latarre $\mathfrak{A}$ on lattice $\mathfrak{M}$ as follows. Set $I \rightharpoonup J = \sqrt{J : I}$ and $\varepsilon = R$. It suffices to check the following schemas.

$I : I = R$, so $I \rightharpoonup I = R = \varepsilon$ is the largest ideal.

Clearly $(I \sqcup J) \rightharpoonup K \trianglelefteq (I \rightharpoonup K) \sqcap (J \rightharpoonup K)$. Conversely, suppose $r \in (I \rightharpoonup K) \sqcap (J \rightharpoonup K)$. So there is n with $r^n I \subseteq K$ and $r^n J \subseteq K$. Let $s \in I \sqcup J$. There are $i \in I$ and $j \in J$ with $s = i + j$. So $r^n s = r^n i + r^n j \in K$. Thus $r \in (I \sqcup J) \rightharpoonup K$.

Clearly $I \rightharpoonup (J \sqcap K) \trianglelefteq (I \rightharpoonup J) \sqcap (I \rightharpoonup K)$. Conversely, suppose $r \in (I \rightharpoonup J) \sqcap (I \rightharpoonup K)$. So there is n with $r^n I \subseteq J$ and $r^n I \subseteq K$. So $r^n I \subseteq J \cap K = J \sqcap K$. Thus $r \in I \rightharpoonup (J \sqcap K)$.

Finally, suppose $r \in (I \rightharpoonup J) \sqcap (J \rightharpoonup K)$. So there is n with $r^n I \subseteq J$ and $r^n J \subseteq K$. So $r^{2n} I \subseteq K$. Thus $r \in (I \rightharpoonup K)$.

So we have a latarre $\mathfrak{A} = (\mathfrak{M}, \rightharpoonup, R)$ of ideals of R with $I \sqcap (I \rightharpoonup J) = I \sqcap \sqrt{J}$.

Example 3.13. This example is motivated by the Kripke models and theory of Visser's Basic Propositional Logic, see [8] and [3]. Let $(K, \prec)$ be a set with relation $x \prec y$ satisfying the schemas of anti-symmetry $(x \prec y) \wedge (y \prec x) \rightarrow (x = y)$ and transitivity $(x \prec y) \wedge (y \prec z) \rightarrow (x \prec z)$. So each node may or may not be reflexive. On the lattice $\mathfrak{M}$ of the Alexandrov topology $\mathcal{O}(K)$ on the collection

$$\{u \subseteq K \ : \ \forall k, m \in K(u \ni k \prec m \ \rightarrow \ u \ni m)\}$$

of upward closed subsets of K we define

$$u \rightharpoonup v = \{k \in K \ : \ \forall m \in K((k \prec m) \wedge (m \in u) \ \rightarrow \ (m \in v))\}.$$

Then $\mathfrak{A} = (\mathfrak{M}, \rightharpoonup, K)$ is a latarre such that $(\mathfrak{M}, \rightharpoonup, K, \emptyset)$ is a Visser algebra as in [2] (called a Basic algebra in [3]). Here is another way to see this. We have the usual complete Heyting algebra $(\mathcal{O}(K), \rightharpoonup_i, K, \emptyset)$. Define operator $j : \mathcal{O}(K) \to \mathcal{O}(K)$ by $ju = \{k \in K \ : \ \forall m \in K((k \prec m) \to (m \in u))\}$. Then j preserves meets (is multiplicative), and $x \rightharpoonup y = j(x \rightharpoonup_i y)$. Apply Proposition 3.5. Note that $\nabla x = jx$.

Example 3.14. Let $\mathcal{O}(X)$ be a T_0 topological space. So we have a complete Heyting latarre $\mathfrak{A} = (\mathcal{O}(X), \rightharpoonup, X)$. Define operator $j : \mathcal{O}(X) \to \mathcal{O}(X)$ by

$$ju = \bigsqcup\{u \cup \{x\} \ : \ u \cup \{x\} \text{ is open}\}.$$

So ju extends u with all isolated elements of the complement of u. Operator j preserves meets (is multiplicative), so by Proposition 3.5 we can define $x \rightharpoonup_j y = j(x \rightharpoonup y)$ and get a new latarre $(\mathcal{O}(X), \rightharpoonup_j, X)$. Note that $\nabla_j x = X \rightharpoonup_j x = jx$. Even in the case of $\mathcal{O}(\mathbb{R})$, the usual topology on the reals, there are u with $j^{n+1} u \neq j^n u$ for all n.

Example 3.15. This example generalizes Example 3.14 from T_0 topological spaces to almost-complete frames. Let $\mathfrak{M} = (M, \sqcap, \sqcup)$ be an almost-complete lattice. We define v *covers or equals* u, written $u \trianglelefteq_1 v$, by

$$u \trianglelefteq_1 v \;\leftrightarrow\; (u \trianglelefteq v \wedge \forall t(u \trianglelefteq t \trianglelefteq v \;\rightarrow\; (u = t \vee t = v))).$$

If $\mathfrak{M}$ is the lattice of a T_0 space $\mathcal{O}(X)$, then $u \trianglelefteq_1 v$ exactly when there is $\xi \in X$ with $u \trianglelefteq v \trianglelefteq u \cup \{\xi\}$. Define operator $j : \mathfrak{M} \to \mathfrak{M}$ by

$$jx = \bigsqcup\{u \;:\; x \trianglelefteq_1 u\}.$$

Map j is well-defined, and $x \trianglelefteq jx$.

Now add that $\mathfrak{M}$ is modular. Claim: j is order preserving.
Proof of the claim: Suppose $x \trianglelefteq y$. To show: $jx \trianglelefteq jy$. Let $x \trianglelefteq_1 v$. It suffices to show that $y \trianglelefteq_1 y \sqcup v$. We have $x \trianglelefteq y \sqcap v \trianglelefteq v$, so $x = y \sqcap v$ or $y \sqcap v = v$. If $y \sqcap v = v$, then $y \sqcup v = y$ and we are done. Suppose $x = y \sqcap v$. By the classic modularity Theorem 6.1, interval sublattice $[y, y \sqcup v]$ is isomorphic to $[y \sqcap v, v] = [x, v]$. Thus $y \trianglelefteq_1 y \sqcup v$.

We need the auxiliary claim: $x \trianglelefteq_1 v$ implies $x \sqcap y \trianglelefteq_1 v \sqcap y$.
Proof of the auxiliary claim: By modularity, interval sublattice $[x \sqcap y, v \sqcap y]$ is isomorphic to $[x, x \sqcup (v \sqcap y)]$. Since $x \trianglelefteq x \sqcup (v \sqcap y) \trianglelefteq v$ we have $x \trianglelefteq_1 x \sqcup (v \sqcap y)$. Thus $x \sqcap y \trianglelefteq_1 v \sqcap y$.

Now add that $\mathfrak{M}$ is distributive. Claim: $x \trianglelefteq_1 u$ and $y \trianglelefteq_1 v$ implies $u \sqcap v \trianglelefteq j(x \sqcap y)$.
Proof of the claim: By the auxiliary claim above we have $x \sqcap y \trianglelefteq_1 u \sqcap y \trianglelefteq_1 u \sqcap v$ and $x \sqcap y \trianglelefteq_1 x \sqcap v \trianglelefteq_1 u \sqcap v$. So $u \sqcap y \trianglelefteq j(x \sqcap y)$ and $x \sqcap v \trianglelefteq j(x \sqcap y)$. If $x \sqcap y = u \sqcap y$ or $u \sqcap y = u \sqcap v$ or $x \sqcap y = x \sqcap v$ or $x \sqcap v = u \sqcap v$, then $x \sqcap y \trianglelefteq_1 u \sqcap v$, so $u \sqcap v \trianglelefteq j(x \sqcap y)$ and we are done. We have $x \trianglelefteq x \sqcup (u \sqcap y) \trianglelefteq u$, so $x = x \sqcup (u \sqcap y)$ or $x \sqcup (u \sqcap y) = u$. If $x = x \sqcup (u \sqcap y)$, then $u \sqcap y \trianglelefteq x$, so $u \sqcap y = x \sqcap y$ and we are done. So we may suppose that $x \sqcup (u \sqcap y) = u$ or, with modularity, that $u \sqcap (x \sqcup y) = u \trianglerighteq x$. Similarly we may suppose that $v \sqcap (x \sqcup y) = v \trianglerighteq y$. So $u \sqcup v = x \sqcup y$. With distributivity, $u \sqcap v \sqcap ((u \sqcap y) \sqcup (x \sqcap v)) = u \sqcap v \sqcap (x \sqcup y) = u \sqcap v$. Thus $u \sqcap v \trianglelefteq (u \sqcap y) \sqcup (x \sqcap v) \trianglelefteq j(x \sqcap y)$.

Finally add that $\mathfrak{M}$ is an almost-complete frame. Then $jx \sqcap jy = \bigsqcup\{u \;:\; x \trianglelefteq_1 u\} \sqcap \bigsqcup\{v \;:\; y \trianglelefteq_1 v\} = \bigsqcup\{u \sqcap v \;:\; x \trianglelefteq_1 u \;\wedge\; y \trianglelefteq_1 v\} \trianglelefteq j(x \sqcap y)$. Thus j preserves meets (is multiplicative).

On almost-complete frame $\mathfrak{M}$ we can define the usual Heyting latarre $\mathfrak{A} = (\mathfrak{M}, \twoheadrightarrow, 1)$. With Proposition 3.5 we get another latarre $\mathfrak{A}_j = (\mathfrak{M}, \twoheadrightarrow_j, 1)$ by defining $x \twoheadrightarrow_j y = j(x \twoheadrightarrow y)$. This generalizes Example 3.14 from T_0 topological spaces to almost-complete frames.

4 General Substitution Rules

In this Section we consider substitution rules that apply to all latarres. Later we consider further substitution rules that only apply in special cases.

With each latarre $\mathfrak{A}$ we associate a predicate logic language $\mathcal{L}(\mathfrak{A})$ in the expected way, with function symbols that correspond with the defining functions of $\mathfrak{A}$, and with for each element of $\mathfrak{A}$ a constant symbol. Whenever convenient, we use the functions of the model themselves as symbols in the language. In our approach constants are nullary functions, and constant symbols are nullary function symbols. For convenience we may write $t(x)$ even if term $t(x)$ has other variables besides x. Given a term $t(x)$ of $\mathcal{L}(\mathfrak{A})$, we define positivity and negativity of occurrences of x in $t(x)$ in the usual inductive way.

Proposition 4.1. *Let $t(x)$ be a term over a latarre $\mathfrak{A}$. If x is only positive in $t(x)$, then $x \trianglelefteq y$ implies $t(x) \trianglelefteq t(y)$. If x is only negative in $t(x)$, then $x \trianglelefteq y$ implies $t(y) \trianglelefteq t(x)$.*

Proof. Both claims are proved simultaneously by induction on the complexity of $t(x)$. The case for atoms is trivial. For the induction steps use the rules

$x \trianglelefteq y$ implies $z \circ x \trianglelefteq z \circ y$ for $\circ \in \{\sqcap, \sqcup, \rightarrowtriangle\}$,   and
$x \trianglelefteq y$ implies $y \rightarrowtriangle z \trianglelefteq x \rightarrowtriangle z$.

$\square$

Definition 4.2. An x occurs at *depth* $n \geq 0$ in term $t(x)$ if x occurs n levels deep inside implication subformulas of implication subformulas and so on. So x occurs at depth 2 in $(y \rightarrowtriangle (w \sqcap (x \sqcup v))) \rightarrowtriangle z$, and x occurs at depth n in $\nabla^n x$. The x occurs *informally* if depth $n = 0$, otherwise x occurs *formally.* Obviously informal occurrences are always positive, and negative occurrences are always formal.

Proposition 4.3. *Let $t(x)$ be a term over a latarre $\mathfrak{A}$ and $n \geq 0$ be such that x only occurs at depth n in $t(x)$. If x is only positive in $t(x)$, then $\mathfrak{A}$ satisfies schema $\nabla^n(x \rightarrowtriangle y) \trianglelefteq t(x) \rightarrowtriangle t(y)$. If x is only negative in $t(x)$, then $\mathfrak{A}$ satisfies schema $\nabla^n(x \rightarrowtriangle y) \trianglelefteq t(y) \rightarrowtriangle t(x)$.*

Proof. We may suppose that x occurs only once in $t(x)$. For example if $t(x) = u(x, x)$ for a term $u(z, w)$ with z and w negative, we use $\nabla^n(x \rightarrowtriangle y) \trianglelefteq (u(y, y) \rightarrowtriangle u(y, x)) \sqcap (u(y, x) \rightarrowtriangle u(x, x)) \trianglelefteq t(y) \rightarrowtriangle t(x)$. Given this supposition, we complete the proof by induction on n.

We complete the proof of case $n = 0$ by induction on the complexity of $t(x)$. In this case x is positive in $t(x)$. The cases of $t(x)$ equal to x or without x are trivial. The induction step on the complexity of $t(x)$: Suppose $t(x)$ equals $p \sqcap q(x)$. With induction we have $(p \sqcap q(x)) \rightarrowtriangle (p \sqcap q(y)) = (p \sqcap q(x)) \rightarrowtriangle q(y) \trianglerighteq q(x) \rightarrowtriangle q(y) \trianglerighteq x \rightarrowtriangle y$. Suppose $t(x)$ has form $p \sqcup q(x)$. Then $(p \sqcup q(x)) \rightarrowtriangle (p \sqcup q(y)) = q(x) \rightarrowtriangle (p \sqcup q(y)) \trianglerighteq q(x) \rightarrowtriangle q(y) \trianglerighteq x \rightarrowtriangle y$.

Induction step: Suppose the case holds for some value n, and suppose x occurs in $t(x)$ at depth $n+1$. There is a least subterm $u(x)$ of $t(x)$ in which x occurs at depth $n+1$. So $t(x) = v(u(x))$ where x is informal in $v(x)$. Subterm $u(x)$ is of the form $r \rightarrowtriangle s(x)$ or of the form $s(x) \rightarrowtriangle r$, with x at depth n in $s(x)$. We have four combinations of x occurring positive or negative in $u(x)$ and x occurring positive or negative in $s(x)$. Here is one of these four cases. Suppose x is positive in $u(x)$ and negative in $s(x)$. So $u(x)$ is of the form $s(x) \rightarrowtriangle r$. Then with induction $\nabla^{n+1}(x \rightarrowtriangle y) \trianglelefteq (s(x) \rightarrowtriangle r) \rightarrowtriangle (\nabla^n(x \rightarrowtriangle y) \sqcap (s(x) \rightarrowtriangle r)) \trianglelefteq (s(x) \rightarrowtriangle r) \rightarrowtriangle ((s(y) \rightarrowtriangle s(x)) \sqcap (s(x) \rightarrowtriangle r)) \trianglelefteq (s(x) \rightarrowtriangle r) \rightarrowtriangle (s(y) \rightarrowtriangle r) = u(x) \rightarrowtriangle u(y)$. By the already proven case for $n = 0$ we have $u(x) \rightarrowtriangle u(y) \trianglelefteq v(u(x)) \rightarrowtriangle v(u(y))$. So $\nabla^{n+1}(x \rightarrowtriangle y) \trianglelefteq t(x) \rightarrowtriangle t(y)$. The proofs of the other three cases of the four are similar. This completes the induction step.

So by induction the claim holds for all $n \geq 0$. $\square$

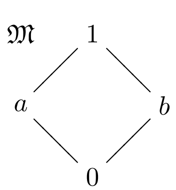
Some example special cases of Proposition 4.3 are: Let $t(x)$ be a term in which x is only positive. There is n such that all x occur at depth at most n in $t(x)$. So

$$\textstyle\prod_{i \leq n} \nabla^i(x \rightarrowtriangle y) \;\trianglelefteq\; t(x) \rightarrowtriangle t(y).$$

Another example. Given a term $t(x)$ in which x is only negative, let a and b be elements such that $a \rightarrowtriangle b \trianglelefteq \nabla(a \rightarrowtriangle b)$. Then $a \rightarrowtriangle b \trianglelefteq \nabla^n(a \rightarrowtriangle b)$ for all $n \geq 0$, and

$$a \rightarrowtriangle b \trianglelefteq t(b) \rightarrowtriangle t(a).$$

We do not always have that x positive in $t(x)$ implies $x \rightarrowtriangle y \trianglelefteq t(x) \rightarrowtriangle t(y)$. For otherwise with $t(x) = \nabla x$ it would imply $x \rightarrowtriangle y \trianglelefteq (\varepsilon \rightarrowtriangle x) \rightarrowtriangle (\varepsilon \rightarrowtriangle y)$, so in particular with $x = \varepsilon$ we would have $\nabla y \trianglelefteq \nabla^2 y$. Here is a counterexample to this last equation. Consider the Boolean lattice $\mathfrak{M}$.

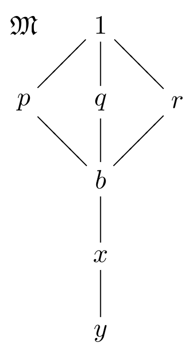


By Example 3.7 we have a (unique) unitary latarre on $\mathfrak{M}$ with $\varepsilon \rightarrowtriangle a = 1 \rightarrowtriangle a = b$ and $1 \rightarrowtriangle b = a$. So $\nabla b = a$ and $\nabla^2 b = b$.

Proposition 4.4. *Let $t(x)$ be a term built without join $\sqcup$ over a latarre $\mathfrak{A}$, and $n \geq 1$ be such that x only occurs at depth n in $t(x)$. If x is only positive in $t(x)$, then $\mathfrak{A}$ satisfies schema $\nabla^{n-1}(x \rightarrowtriangle y) \sqcap t(x) \trianglelefteq t(y)$. If x is only negative in $t(x)$, then $\mathfrak{A}$ satisfies schema $\nabla^{n-1}(x \rightarrowtriangle y) \sqcap t(y) \trianglelefteq t(x)$.*

Proof. We may suppose that x occurs at most once in $t(x)$. There is a least subterm $u(x)$ of $t(x)$ such that x is at depth n in $u(x)$ and $t(x)$ equals $v(u(x))$ for a term $v(x)$. So x is informal in $v(x)$. Term $u(x)$ has form $r \rightarrowtriangle s(x)$ or form $s(x) \rightarrowtriangle r$, with x at depth $n-1$ in $s(x)$. We have four combinations of x occurring positive or negative in $u(x)$ and x occurring positive or negative in $s(x)$. Here is one of these four cases. Suppose x is positive in $u(x)$ and positive in $s(x)$. So $u(x)$ has form $r \rightarrowtriangle s(x)$. With Proposition 4.3 we have $u(x) \sqcap \nabla^{n-1}(x \rightarrowtriangle y) \trianglelefteq u(x) \sqcap (s(x) \rightarrowtriangle s(y)) \trianglelefteq u(y)$. We complete the proof for all $t(x)$ by induction on the complexity of $v(x)$. Since x is informal in $v(x)$, the following observation suffices. If $q(x) \trianglelefteq q(y)$, then $p \sqcap q(x) \trianglelefteq p \sqcap q(y)$. The proofs of the other three cases of the four are similar. $\square$

In the proof of Proposition 4.4 an induction step for $\sqcup$ may fail unless extra conditions are employed as in Proposition 5.9. For $\nabla^{n-1}(x \rightarrowtriangle y) \sqcap q(x) \trianglelefteq q(y)$ need not imply $\nabla^{n-1}(x \rightarrowtriangle y) \sqcap (p \sqcup q(x)) \trianglelefteq p \sqcup q(y)$. Here is an example where equation $(x \rightarrowtriangle y) \sqcap (b \rightarrowtriangle x) \trianglelefteq b \rightarrowtriangle y$ holds and equation $(x \rightarrowtriangle y) \sqcap (p \sqcup (b \rightarrowtriangle x)) \trianglelefteq p \sqcup (b \rightarrowtriangle y)$ is false. Consider the modular lattice $\mathfrak{M}$.



By Proposition 6.4.9 and Theorem 6.8 we can construct a unitary latarre on $\mathfrak{M}$ with $b \rightarrowtriangle x = r$ and $x \rightarrowtriangle y = q$. So $b \rightarrowtriangle y = r \sqcap q = b$, and $(x \rightarrowtriangle y) \sqcap (p \sqcup (b \rightarrowtriangle x)) = q \sqcap (p \sqcup r) = q \sqcap 1 = q$, while $p \sqcup (b \rightarrowtriangle y) = p \sqcup b = p$.

Proposition 4.5. *Let $t(x)$ be a term over a latarre $\mathfrak{A}$ in which x occurs only at depths at least n in $t(x)$, for some $n \geq 1$. Let $a, b \in A$ be such that $\nabla^{n-1}(a \rightarrowtriangle b) = \varepsilon$. If x is only positive in $t(x)$, then $t(a) \trianglelefteq t(b)$. If x is only negative in $t(x)$, then $t(b) \trianglelefteq t(a)$.*

Proof. We may suppose that x occurs exactly once in $t(x)$. There is a least subterm $u(x)$ of $t(x)$ such that x is at depth n in $u(x)$ and $t(x)$ equals $v(u(x))$ for a term $v(x)$. Term $u(x)$ is of the form $r \rightarrowtriangle s(x)$ or of the form $s(x) \rightarrowtriangle r$ with x at depth $n-1$ in $s(x)$. We have eight combinations of x occurring positive or negative in $v(x)$, positive or negative in $u(x)$, and positive or negative in $s(x)$. Here is one of these eight cases. Suppose x is positive in $v(x)$, negative in $u(x)$, and positive in $s(x)$. So $u(x)$ has form $s(x) \rightarrowtriangle r$. By Proposition 4.3 we have $s(a) \rightarrowtriangle s(b) = \varepsilon$. So $u(b) = (s(a) \rightarrowtriangle s(b)) \sqcap (s(b) \rightarrowtriangle r) \trianglelefteq u(a)$. So by Proposition 4.1 we have $t(b) = v(u(b)) \trianglelefteq v(u(a)) = t(a)$. The proofs of the other seven cases of the eight are essentially the same. $\square$

Write $x \leftrightarrowtriangle y$ as short for $(x \rightarrowtriangle y) \sqcap (y \rightarrowtriangle x)$. If x is only formal in $t(x)$, then $a \leftrightarrowtriangle b = \varepsilon$ implies $t(a) = t(b)$. The following is essentially a special case of Proposition 4.5.

Proposition 4.6. *Let $t(x)$ be a term over a latarre $\mathfrak{A}$, and $a, b \in A$ are such that $a \twoheadrightarrow b = \varepsilon$. If x is only positive in $t(x)$, then $t(a) \twoheadrightarrow t(b) = \varepsilon$. If x is only negative in $t(x)$, then $t(b) \twoheadrightarrow t(a) = \varepsilon$.*

Proof. Given $a \twoheadrightarrow b = \varepsilon$, suppose x is positive in $t(x)$. Let $u(y, x)$ be term $t(y) \twoheadrightarrow t(x)$. So x is positive and formal in $u(y, x)$. By Proposition 4.5 we have $t(y) \twoheadrightarrow t(a) \trianglelefteq t(y) \twoheadrightarrow t(b)$. Substitution of a for y gives $\varepsilon = t(a) \twoheadrightarrow t(a) \trianglelefteq t(a) \twoheadrightarrow t(b)$. The proof for x negative in $t(x)$ is similar. □

Example 4.7. Proposition 4.6 allows for another technique by which to construct new latarres from old ones. Given a latarre $\mathfrak{A}$, define equivalence relation $x \sim y$ by $x \leftrightarrow y = \varepsilon$. We write $a^{(1)}$ or a' for the equivalence class of a, and $A^{(1)}$ or A' for the collection of equivalence classes. In fact relation $x \sim y$ a congruence. For on this collection A' we can define the following derived latarre. If $a \leftrightarrow b = \varepsilon$, then $t(a) \leftrightarrow t(b) = \varepsilon$ for all terms $t(x)$. So the following are well-defined on A': Define $x' \sqcap' y' = (x \sqcap y)'$ and $x' \sqcup' y' = (x \sqcup y)'$ and $x' \twoheadrightarrow' y' = (x \twoheadrightarrow y)'$. So with these operations, $\mathfrak{A}' = (A', \sqcap', \sqcup', \twoheadrightarrow', \varepsilon')$ is a latarre, and the map $x \mapsto x'$ is an onto latarre morphism from $\mathfrak{A}$ onto $\mathfrak{A}'$. Now $a' \trianglelefteq' b'$ exactly when $(a \sqcap b)' = a'$ exactly when $\varepsilon = (a \sqcap b) \leftrightarrow a = a \twoheadrightarrow b$. So $a' \trianglelefteq' b'$ exactly when $a \twoheadrightarrow b = \varepsilon$.

We can repeat this construction and form $\mathfrak{A}'' = \mathfrak{A}^{(2)}$ by defining $x' \sim y'$ on $\mathfrak{A}'$ by $(x \leftrightarrow y)' = x' \leftrightarrow' y' = \varepsilon'$ or, equivalently, by $(x \leftrightarrow y) \sim \varepsilon$, that is, $(x \leftrightarrow y) \leftrightarrow \varepsilon = \varepsilon$. That is, by $\nabla(x \leftrightarrow y) = \nabla(x \twoheadrightarrow y) \sqcap \nabla(y \twoheadrightarrow x) = \varepsilon$. Both by repeating the previous construction of $\mathfrak{A}'$ from $\mathfrak{A}$, or by a direct appeal to Proposition 4.6, do we see that we have a latarre and onto morphisms $\mathfrak{A} \to \mathfrak{A}' \to \mathfrak{A}''$ in the expected way. Continuing in this way, we get a chain

$$\mathfrak{A} = \mathfrak{A}^{(0)} \to \mathfrak{A}^{(1)} \to \mathfrak{A}^{(2)} \to \mathfrak{A}^{(3)} \to \dots$$

with for all $a, b \in A$ and $n \geq 1$ we have $a^{(n)} = b^{(n)}$ in $A^{(n)}$ exactly when $\nabla^{n-1}(a \leftrightarrow b) = \varepsilon$. A sketch of a proof of this last claim, by induction on n, runs as follows. Suppose in $\mathfrak{A}^{(n)}$ we have $x^{(n)} = y^{(n)}$ exactly when $\nabla^{n-1}(x \leftrightarrow y) = \varepsilon$. To construct $\mathfrak{A}^{(n+1)}$ we set $x^{(n)} \sim x^{(n)}$ iff $x^{(n)} \leftrightarrow^{(n)} y^{(n)} = \varepsilon^{(n)}$ iff $(x \leftrightarrow y)^{(n)} = \varepsilon^{(n)}$ iff $\nabla^{n-1}((x \leftrightarrow y) \leftrightarrow \varepsilon) = \varepsilon$ iff $\nabla^{n}(x \leftrightarrow y) = \varepsilon$.

5 Visser Latarres and Meet Substitution

In this Section we establish the close connection between weakly Visser latarres and (relative) meet substitution. But first we get some naming conventions settled.

Definition 5.1. The following is a non-exhaustive list of varieties of distributive latarres.

A *distributive latarre* is a latarre satisfying schema $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$.

A *CJ latarre* is a unitary distributive latarre.

A *Visser latarre* is a distributive latarre satisfying the schema $x \trianglelefteq \nabla x$ of unitary arrow persistence.

As defined in Example 3.1, a *Heyting latarre* is a latarre satisfying schema $x = \nabla x$. With Proposition 5.2 we show that Heyting latarres are distributive.

As defined in Example 3.1, a *Boolean latarre* is a latarre satisfying the schema $(x \twoheadrightarrow (x \sqcap y)) \twoheadrightarrow (x \sqcap y) = x$. In Example 3.1 we showed that Boolean latarres are Heyting.

Each of the varieties of latarres listed above is contained in the preceding one. A unitary arrow persistent latarre need not be distributive, since *all* lattices with top 1 turn into unitary arrow persistent latarres when we define $x \twoheadrightarrow y = 1$.

CJ latarres with a least element 0 are the same as CJ algebras as explained below, and Visser latarres with a least element 0 are the same as Visser algebras. With Proposition 5.2 below it is a straightforward standard task to establish that Heyting latarres with a least element 0 are the same as Heyting algebras, and Boolean latarres with a least element 0 are the same as Boolean algebras. We imagine that the names of Visser latarre, Heyting latarre, and Boolean latarre are sufficiently motivated by [2], and by the extensive literature on Heyting algebras and Boolean algebras. CJ latarres with a least element 0 are defined by Celani and Jansana in [5] as weakly Heyting algebras. In the context of names like Boolean latarre, Heyting latarre, and Visser latarre, we imagine that the choice of the name CJ latarre is a natural approximation of a continuation of this pattern.

We choose to say little about Boolean latarres or Heyting latarres since their properties are essentially as the well-known ones of the literature. We make an exception for the following fundamental result.

Proposition 5.2. *The following are equivalent for a latarre $\mathfrak{A}$.*

1. *$\mathfrak{A}$ is Heyting.*
2. *$\mathfrak{A}$ satisfies $x \sqcap y \trianglelefteq z$ if and only if $x \trianglelefteq y \rightarrow z$, for all x, y, and z.*

Item 2 implies distributivity.

Proof. Suppose item 1. So $\mathfrak{A}$ is unitary. With Proposition 2.5.3 we get that $x \trianglelefteq y \rightarrow z$ implies $x \sqcap y \trianglelefteq z$. With Proposition 2.5.2 we get that $x \sqcap y \trianglelefteq z$ implies $x \trianglelefteq y \rightarrow z$. So item 2 holds.

Suppose item 2. Setting $y = z$ implies that $\mathfrak{A}$ is unitary, so $\varepsilon = 1$. Setting $x = z$ and $y = 1$ gives schema $z \trianglelefteq \nabla z$. Setting $x = \nabla z$ and $y = 1$ gives schema $\nabla z \trianglelefteq z$. So $\mathfrak{A}$ is Heyting.

Finally, item 2 implies that $\mathfrak{A}$ as a poset category has for all elements a a pair of functors, left adjoint $x \mapsto a \sqcap x$ and right adjoint $z \mapsto a \rightarrow z$. The left adjoints preserve colimits, which implies distributivity. □

Generalizing from Example 3.1, an element a of a latarre is called *arrow persistent* if it satisfies schema $a \sqcap \varepsilon \trianglelefteq y \rightarrow a$. Element a is called *unitary arrow persistent* if it satisfies schema $a \trianglelefteq y \rightarrow a$. So a is unitary arrow persistent exactly when both $a \trianglelefteq \varepsilon$ and a is arrow persistent. A *weakly Visser latarre* is a distributive latarre satisfying the schema $x \sqcap \varepsilon \trianglelefteq y \rightarrow x$ of arrow persistence. So a Visser latarre is a unitary weakly Visser latarre.

Proposition 5.3. *The following are equivalent for an element a of a latarre.*

1. *a is arrow persistent.*
2. *$a \sqcap (a \rightarrow y) \trianglelefteq z \rightarrow y$, for all y and z.*
3. *$(a \sqcap y \rightarrow z) = \varepsilon$ implies $a \sqcap \varepsilon \trianglelefteq y \rightarrow z$, for all y and z.*
4. *$a \sqcap y \trianglelefteq z$ implies $a \sqcap \varepsilon \trianglelefteq y \rightarrow z$, for all y and z.*

Proof. Suppose item 1. Then $a \sqcap (a \rightarrow y) \trianglelefteq (z \rightarrow a) \sqcap (a \rightarrow y) \trianglelefteq z \rightarrow y$, so item 2 holds. Item 2 immediately implies item 1 by setting $y = a$. Suppose item 1. To prove item 3, suppose $a \sqcap y \rightarrow z = \varepsilon$. Then $a \sqcap \varepsilon \trianglelefteq y \rightarrow a = (y \rightarrow (a \sqcap y)) \sqcap (a \sqcap y \rightarrow z) \trianglelefteq y \rightarrow z$. Item 3 immediately implies item 4. Suppose item 4. Setting $z = a$ immediately implies item 1. □

Corollary 5.4. *The following are equivalent for an element a of a latarre.*

1. *a is unitary arrow persistent.*

2. *$a \trianglelefteq \varepsilon$, and $a \sqcap (a \rightarrow y) \trianglelefteq z \rightarrow y$, for all y and z.*

3. *$(a \sqcap y \rightarrow z) = \varepsilon$ implies $a \trianglelefteq y \rightarrow z$, for all y and z.*

4. *$a \sqcap y \trianglelefteq z$ implies $a \trianglelefteq y \rightarrow z$, for all y and z.*

Arrow persistence allows for some stronger substitution rules than in Section 4.

Proposition 5.5. *Let $t(x)$ be a term built without join $\sqcup$ over a latarre $\mathfrak{A}$, and $n \geq 1$ be such that x only occurs at depth at least n in $t(x)$. Let a and b be elements of $\mathfrak{A}$ such that $a \rightarrow b$ is arrow persistent. If x is only positive in $t(x)$, then $\mathfrak{A}$ satisfies $\nabla^{n-1}(a \rightarrow b) \sqcap t(a) \trianglelefteq t(b)$. If x is only negative in $t(x)$, then $\mathfrak{A}$ satisfies $\nabla^{n-1}(a \rightarrow b) \sqcap t(b) \trianglelefteq t(a)$.*

Proof. We have $a \rightarrow b = (a \rightarrow b) \sqcap \varepsilon \trianglelefteq \nabla(a \rightarrow b)$. So $\nabla^k(a \rightarrow b) \trianglelefteq \nabla^m(a \rightarrow b)$ for all $m \geq k \geq 0$. Apply Proposition 4.4. □

Proposition 5.6. *Let $t(x)$ be a term over a latarre $\mathfrak{A}$, and $n \geq 0$ be such that x only occurs at depth at least n in $t(x)$. Let a and b be elements of $\mathfrak{A}$ such that $a \rightarrow b$ is arrow persistent. If x is only positive in $t(x)$, then $\mathfrak{A}$ satisfies $\nabla^n(a \rightarrow b) \trianglelefteq t(a) \rightarrow t(b)$. If x is only negative in $t(x)$, then $\mathfrak{A}$ satisfies $\nabla^n(a \rightarrow b) \trianglelefteq t(b) \rightarrow t(a)$.*

Proof. We have $a \rightarrow b = (a \rightarrow b) \sqcap \varepsilon \trianglelefteq \nabla(a \rightarrow b)$. So $\nabla^k(a \rightarrow b) \trianglelefteq \nabla^m(a \rightarrow b)$ for all $m \geq k \geq 0$. Apply Proposition 4.3. □

As a preliminary to our definition of meet substitution, we consider the following latarre constructions. Given a latarre $\mathfrak{A}$ and element a, we construct a latarre on the subset $\{x \in A \,:\, x \trianglelefteq a\}$ as follows. Set

$$\begin{aligned} &\varepsilon_a = \varepsilon \sqcap a, \\ &x \sqcap_a y = x \sqcap y, \\ &x \rightarrow_a y = a \sqcap (x \rightarrow y), \quad \text{and} \\ &x \sqcup_a y = x \sqcup y. \end{aligned}$$

The resulting structure $\mathfrak{A}_a$ is easily seen to be a latarre. The following are clear or straightforward. If $a \trianglelefteq \varepsilon$, then $\mathfrak{A}_a$ is unitary. If $\mathfrak{A}$ is unitary, arrow persistent, Visser, Heyting, or Boolean, then so is $\mathfrak{A}_a$.

The function $\pi_a(x) = a \sqcap x$ is an idempotent map from $\mathfrak{A}$ onto $\mathfrak{A}_a$. In general π_a is not a latarre morphism. Below we establish precisely when π_a is a morphism.

Morphism properties of π_a can be expressed in terms of substitution. Given a term $t(x)$ and element a of latarre $\mathfrak{A}$, we say that $t(x)$ *admits meet substitution over* $(\mathfrak{A}, a)$ if $\mathfrak{A}$ satisfies schema

$$a \sqcap x = a \sqcap y \text{ implies } a \sqcap t(x) = a \sqcap t(y).$$

One easily verifies that this notion of substitution over $(\mathfrak{A}, a)$ is equivalent to schema

$$a \sqcap t(x) = a \sqcap t(a \sqcap x),$$

which is a universal equation. We write $T(\mathfrak{A}, a)$ for the collection of terms over $\mathfrak{A}$ that admit meet substitution over $(\mathfrak{A}, a)$. We define that $\mathfrak{A}$ *admits meet substitution* if $T(\mathfrak{A}, a)$ includes all terms for all $a \in A$.

Proposition 5.7. *Let a be an element of latarre $\mathfrak{A}$. Then the collection $T(\mathfrak{A}, a)$ contains all terms without x, the term x itself, and is closed under $\sqcap$ and under composition. Additionally:*

1. *$\mathfrak{A}$ satisfies schema $a \sqcap \varepsilon \trianglelefteq x \rightarrow a$ if and only if $T(\mathfrak{A}, a)$ is closed under $\rightarrow$.*

2. $\mathfrak{A}$ *satisfies schema* $a \sqcap (x \sqcup y) = (a \sqcap x) \sqcup (a \sqcap y)$ *if and only if* $T(\mathfrak{A}, a)$ *is closed under* $\sqcup$.

Proof. The cases for terms without x and term x itself are easy. Suppose $t(x), u(x) \in T(\mathfrak{A}, a)$ and $a \sqcap x = a \sqcap y$. Then $a \sqcap t(x) = a \sqcap t(y)$ implies $a \sqcap t(x) \sqcap u(x) = a \sqcap t(y) \sqcap u(x)$, and $a \sqcap u(x) = a \sqcap u(y)$ implies $a \sqcap t(y) \sqcap u(x) = a \sqcap t(y) \sqcap u(y)$. Thus $a \sqcap t(x) \sqcap u(x) = a \sqcap t(y) \sqcap u(y)$. As to closure under composition, $a \sqcap t(x) = a \sqcap t(y)$ and the universal validity of common substitution give $u(a \sqcap t(x)) = u(a \sqcap t(y))$ and so $a \sqcap u(t(x)) = a \sqcap u(a \sqcap t(x)) = a \sqcap u(a \sqcap t(y)) = a \sqcap u(t(y))$.

Additional item 1: From left to right follows from Proposition 2.4.2. Conversely, closure of $T(\mathfrak{A}, a)$ under $\twoheadrightarrow$ implies schema $a \sqcap (a \sqcap x \twoheadrightarrow a \sqcap y) = a \sqcap (x \twoheadrightarrow y)$. By Proposition 2.4.2 this implies schema $a \sqcap \varepsilon \trianglelefteq x \twoheadrightarrow a$. An alternate argument for the converse: Closure of $T(\mathfrak{A}, a)$ under $\twoheadrightarrow$ implies $a \sqcap \varepsilon = a \sqcap (a \sqcap x \twoheadrightarrow a) = a \sqcap (x \twoheadrightarrow a) \trianglelefteq x \twoheadrightarrow a$.

Additional item 2: The equivalence easily follows with schema $a \sqcap ((a \sqcap x) \sqcup (a \sqcap y)) = (a \sqcap x) \sqcup (a \sqcap y)$. □

As a Corollary we get:

Theorem 5.8. *The following are equivalent for a latarre* $\mathfrak{A}$.

1. $\mathfrak{A}$ *is weakly Visser.*

2. *For all elements* a *of* $\mathfrak{A}$ *the map* $\pi_a : \mathfrak{A} \to \mathfrak{A}_a$ *is a latarre morphism.*

3. $\mathfrak{A}$ *admits meet substitution.*

Theorem 5.10 below is an extension of Proposition 5.5 for weakly Visser latarres. For its proof we first present an extension of Proposition 4.4 for distributive latarres.

Proposition 5.9. *Let* $t(x)$ *be a term over a distributive latarre* $\mathfrak{A}$*, and* $n \geq 1$ *be such that* x *only occurs at depth* n *in* $t(x)$*. If* x *is only positive in* $t(x)$*, then* $\mathfrak{A}$ *satisfies schema* $\nabla^{n-1}(x \twoheadrightarrow y) \sqcap t(x) \trianglelefteq t(y)$*. If* x *is only negative in* $t(x)$*, then* $\mathfrak{A}$ *satisfies schema* $\nabla^{n-1}(x \twoheadrightarrow y) \sqcap t(y) \trianglelefteq t(x)$.

Proof. The proof is as for Proposition 4.4, but with the following modifications of its last few lines: Since x is informal in $v(x)$, the following observations suffice. If $q(x) \trianglelefteq q(y)$, then $p \sqcap q(x) \trianglelefteq p \sqcap q(y)$ and $(p \sqcup q(x)) \sqcap r = (p \sqcap r) \sqcup (q(x) \sqcap r) \trianglelefteq (p \sqcap r) \sqcup (q(y) \sqcap r) = (p \sqcup q(y)) \sqcap r$. The proofs of the other three cases of the four are similar. □

Theorem 5.10. *Let* $t(x)$ *be a term over a weakly Visser latarre* $\mathfrak{A}$*, and* $n \geq 1$ *be such that* x *only occurs at depth at least* n *in* $t(x)$*. If* x *is only positive in* $t(x)$*, then* $\mathfrak{A}$ *satisfies schema* $\nabla^{n-1}(x \twoheadrightarrow y) \sqcap t(x) \trianglelefteq t(y)$*. If* x *is only negative in* $t(x)$*, then* $\mathfrak{A}$ *satisfies* $\nabla^{n-1}(x \twoheadrightarrow y) \sqcap t(y) \trianglelefteq t(x)$.

Proof. Weakly Visser latarres are distributive, and all elements of the form $x \twoheadrightarrow y$ are arrow persistent. Apply Propositions 5.9 and 5.5. □

6 Modular Latarres

Many of the latarres that we consider are distributive. One motivation for this Section is to show that there are many latarres that are not distributive. In this Section we give precise criteria for constructing latarres on all modular lattices whose interval sublattices $[m, n]$ have finite height. Our methods are motivated by [1].

Recall that a lattice is *modular* if it satisfies schema

$x \trianglerighteq y$ implies $x \sqcap (y \sqcup z) = y \sqcup (x \sqcap z)$.

A *modular latarre* is a latarre whose underlying lattice is modular.

The following is a classic result about modular lattices.

Theorem 6.1. *Given a modular lattice with elements a and b, the order preserving map $\sigma(x) = x \sqcap a$ from sublattice $[b, a \sqcup b]$ to sublattice $[a \sqcap b, a]$ is a lattice isomorphism. Its inverse is order preserving map $\tau(y) = y \sqcup b$. (See Birkhoff's [4, page 13].)*

Definition 6.2. Let $\mathfrak{M}$ be a lattice. Define relation $\succeq$ on $S = S(\mathfrak{M}) = \{(c, a) \in \mathfrak{M} \times \mathfrak{M} \ : \ c \trianglerighteq a\}$ as follows. Set $(d, b) \succeq (c, a)$ exactly when both $b \sqcap c = a$ and $b \sqcup c = d$. So we have exactly all pairs of the form $(b \sqcup c, b) \succeq (c, b \sqcap c)$. Over the dual lattice $\mathfrak{M}^d$ the corresponding relation satisfies $(q, p) \succeq^d (b, a)$ exactly when $(a, b) \succeq (p, q)$. Clearly $(d, b) \succeq (c, a)$ implies $d \trianglerighteq c$ and $b \trianglerighteq a$. If the lattice is modular, then $(d, b) \succeq (c, a)$ implies that $[b, d] \cong [a, c]$ as sublattices, by the isomorphism $\sigma(x) = x \sqcap c$ essentially as in Theorem 6.1. Over any latarre on lattice $\mathfrak{M}$ we have $(d, b) \succeq (c, a)$ implies $d \twoheadrightarrow b = (b \sqcup c) \twoheadrightarrow b = c \twoheadrightarrow b = c \twoheadrightarrow (b \sqcap c) = c \twoheadrightarrow a$.

Proposition 6.3. *Let $\mathfrak{M}$ be a lattice, and $S = S(\mathfrak{M}) = \{(c, a) \in \mathfrak{M} \times \mathfrak{M} \ : \ c \trianglerighteq a\}$.*

1. *Structure $(S, \succeq)$ is a partial order.*
2. *$(q, p) \succeq (b, a)$ if and only if $(q, b) \succeq (p, a)$.*
3. *$(c, b) \succeq (a, a)$ if and only if $c = b \trianglerighteq a$.*
   So by lattice duality we also have $(c, c) \succeq (b, a)$ if and only if $c \trianglerighteq b = a$.
4. *$(r, q) \succeq (c, b)$ plus $(q, p) \succeq (b, a)$ implies $(r, p) \succeq (c, a)$.*
5. *If $\mathfrak{M}$ is distributive, then partial order $\succeq$ admits amalgamation.*
   By duality, partial order $\preceq$ also admits amalgamation.
6. *A latarre $\mathfrak{A} = (\mathfrak{M}, \twoheadrightarrow, \varepsilon)$ is arrow persistent if and only if for all $a, b, c \in A$ we have $(b \sqcup c, b) \succeq (c, a)$ implies $b \sqcap \varepsilon \trianglelefteq c \twoheadrightarrow a$.*
7. *A unitary latarre $\mathfrak{A} = (\mathfrak{M}, \twoheadrightarrow, 1)$ is Heyting if and only if for all $a, b, c \in A$ with $c \trianglerighteq a$ we have $(b \sqcup c, b) \succeq (c, a)$ exactly when $a \trianglelefteq b \trianglelefteq c \twoheadrightarrow a$.*
8. *Let $\mathfrak{A} = (\mathfrak{M}, \twoheadrightarrow, 1)$ be a Heyting latarre, and $a, c \in A$ with $c \trianglerighteq a$. Then $(q, p) = (((c \twoheadrightarrow a) \sqcup c), c \twoheadrightarrow a)$ is the largest pair such that $(q, p) \succeq (c, a)$.*

Proof. Item 1: Reflexivity. If $c \trianglerighteq a$, then $c \sqcup a = c$ and $c \sqcap a = a$, so $(c, a) \succeq (c, a)$. Antisymmetry. Suppose $(d, b) \succeq (c, a) \succeq (d, b)$. Then $b \sqcap c = a$ plus $a \sqcap d = b$ implies $b \trianglerighteq a \trianglerighteq b$. And $b \sqcup c = d$ plus $a \sqcup d = c$ implies $c \trianglelefteq d \trianglelefteq c$. So $(d, b) = (c, a)$. Transitivity. Suppose $(c_3, a_3) \succeq (c_2, a_2) \succeq (c_1, a_1)$. Then $a_3 \sqcap c_1 = a_3 \sqcap c_2 \sqcap c_1 = a_2 \sqcap c_1 = a_1$, and $a_3 \sqcup c_1 = a_3 \sqcup a_2 \sqcup c_1 = a_3 \sqcup c_2 = c_3$. Thus $(c_3, a_3) \succeq (c_1, a_1)$.

Item 2: Both equations are equivalent to $p \sqcap b = a$ plus $p \sqcup b = q$.

Item 3: From right to left is immediate from the definitions. Conversely, $b \sqcap a = a$ implies $b \trianglerighteq a$, so $c = b \sqcup a = b$.

Item 4: Suppose $(r, q) \succeq (c, b)$ and $(q, p) \succeq (b, a)$. Then $c \sqcup p = c \sqcup b \sqcup p = c \sqcup q = r$ and $c \sqcap p = c \sqcap q \sqcap p = b \sqcap p = a$.

Item 5: Suppose $(c \sqcup x, x) \succeq (c, a)$ and $(c \sqcup y, y) \succeq (c, a)$. By symmetry it suffices to show that $(c \sqcup x \sqcup y, x \sqcup y) \succeq (c \sqcup x, x)$. Obviously $(x \sqcup y) \sqcup (c \sqcup x) = c \sqcup x \sqcup y$. And $(x \sqcup y) \sqcap (c \sqcup x) = ((x \sqcup y) \sqcap c) \sqcup x = a \sqcup a \sqcup x = x$ (where the next to last equation needs distributivity, modularity is not sufficient).

Item 6: Suppose $\mathfrak{A}$ is arrow persistent and $(b \sqcup c, b) \succeq (c, a)$. So $b \sqcap c = a$. By Proposition 5.3.4 this implies $b \sqcap \varepsilon \trianglelefteq c \twoheadrightarrow a$. Conversely, suppose $(b \sqcup c, b) \succeq (c, a)$ implies $b \sqcap \varepsilon \trianglelefteq c \twoheadrightarrow a$, for all a, b, c. If $c \trianglerighteq a$, then $(c, a) \succeq (c, a)$, so $a \sqcap \varepsilon \trianglelefteq c \twoheadrightarrow a$. Thus $\mathfrak{A}$ is arrow persistent.

Item 7: Suppose $\mathfrak{A}$ is Heyting. By item 6 we have that $(b \sqcup c, b) \succeq (c, a)$ implies $a \trianglelefteq b \trianglelefteq c \twoheadrightarrow a$. Suppose $c \trianglerighteq a$ and $a \trianglelefteq b \trianglelefteq c \twoheadrightarrow a$. Then $a = a \sqcap c \trianglelefteq b \sqcap c \trianglelefteq (c \twoheadrightarrow a) \sqcap c = a \sqcap c = a$. So $a = b \sqcap c$ and thus $(b \sqcup c, b) \succeq (c, a)$. Conversely, suppose $(b \sqcup c, b) \succeq (c, a)$ exactly when $a \trianglelefteq b \trianglelefteq c \twoheadrightarrow a$, for all a, b, c with $c \trianglerighteq a$. Equivalently, $b \sqcap c = a$ exactly when $a \trianglelefteq b \trianglelefteq c \twoheadrightarrow a$, for all a, b, c with $c \trianglerighteq a$. Set $c = 1$ and $b = a$. Then $a \trianglelefteq \nabla a$. Set $c = 1$ and $b = \nabla a$. Then $\nabla a = a$. Thus $\mathfrak{A}$ is Heyting.

Item 8: Combine items 5 and 7. □

Given a lattice $\mathfrak{M}$ and elements $a \trianglelefteq b$ of $\mathfrak{M}$, we say that sublattice interval $[a, b] = \{m \ : \ a \trianglelefteq m \trianglelefteq b\}$ is of *finite length* if there is $n < \omega$ such that all linearly ordered subsets of $[a, b]$ are of size at most $n + 1$. The least such n is called the *length* of $[a, b]$. If $a = b$, then the length of $[a, b] = [a, a]$ equals 0. We call b a *cover* of a exactly when the length of $[a, b]$ equals 1. Being a cover is equivalent to $b \triangleright a$ plus for all $r \trianglerighteq a$ we have $r \trianglerighteq b$ or $b \sqcap r = a$. We call b a *strong cover* of a exactly when $b \triangleright a$ plus for all $r \trianglerighteq a$ we have $r \trianglerighteq b$ or $r = a$. Elements can have at most one strong cover. An element a is called *meet irreducible* if for all $x, y \in M$, if $x \sqcap y = a$, then $x = a$ or $y = a$. A top element is clearly meet irreducible, and is therefore called *trivially* meet irreducible.

Proposition 6.4. *Let $\mathfrak{M}$ be a lattice. Let $S = S(\mathfrak{M}) = \{(c, a) \in \mathfrak{M} \times \mathfrak{M} \ : \ c \trianglerighteq a\}$. The following are equivalent for all $q \trianglerighteq p$.*

1. *(q, p) is maximal in $(S, \succeq)$.*
2. *$q \sqcap x = p$ implies $x = p$, for all x.*

So if $\mathfrak{M}$ has a top 1, then all $(1, p)$ are maximal in $(S, \succeq)$.
Additionally we have the following implications for $p \in M$.

3. *If $r \trianglerighteq q \trianglerighteq p$ and (q, p) is maximal in $(S, \succeq)$ then (r, p) is maximal in $(S, \succeq)$.*
4. *If p has a strong cover, then p is meet irreducible.*
5. *p is meet irreducible if and only if (q, p) is maximal in $(S, \succeq)$, for all $q \triangleright p$.*
6. *If (q, p) is maximal in $(S, \succeq)$ and q is a cover of p, then q is a strong cover of p.*

Let q be a cover of $p \in M$. Then the following are equivalent.

7. *q is a strong cover of p.*
8. *p is meet irreducible.*
9. *(q, p) is maximal in $(S, \succeq)$.*

Proof. Suppose that item 1 holds, and $q \sqcap x = p$. Then $(q \sqcup x, x) \succeq (q, p)$ so, by maximality, $x = p$. Conversely, suppose item 2 and $(y, x) \succeq (q, p)$. Then $q \sqcap x = p$, so $x = p$ and $y = p \sqcup q = q$.

Item 3: Suppose (q, p) is maximal and $r \trianglerighteq q$. If $x \sqcap r = p$, then $x \sqcap q = p$, so by maximality $x = p$. By the equivalence of items 2 and 1, (r, p) is maximal.

Item 4: Let q be a strong cover of p and $x \sqcap y = p$. Then $x \trianglerighteq q$ or $x = p$, and $y \trianglerighteq q$ or $y = p$. Thus $x = p$ or $y = p$.

Item 5: Suppose p is meet irreducible and $q \triangleright p$. Let x be such that $q \sqcap x = p$. Then $q = p$ or $x = p$, so $x = p$. By the equivalence of items 2 and 1, (q, p) is maximal. Conversely, suppose that (q, p) is maximal in $(S, \succeq)$, for all $q \triangleright p$. Let $x \sqcap y = p$ with $x \triangleright p$. Then $(x \sqcup y, y) \succeq (x, p)$. By maximality of (x, p) we have $y = p$. Thus p is meet irreducible.

Item 6: Let q be a cover of p and (q, p) be maximal. Let $x \trianglerighteq p$. Then $x \trianglerighteq q$ or $q \sqcap x = p$, so $x \trianglerighteq q$ or $x = p$.

The implications from item 7 to item 8 to item 9 to item 7 easily follow from item 4, 5, and 6 respectively. □

Maximality of (q,p) in $(S,\succeq)$ with $q \rhd p$ does not always imply meet irreducibility of p. For example consider the 4-element Boolean lattice on $\{0,a,b,1\}$ with $a \sqcap b = 0$ and $a \sqcup b = 1$. Then $(1,0)$ is maximal in $(S,\succeq)$, but 0 is not meet irreducible.

Meet irreducibility of p does not imply that p has a cover. For example all elements of the rationals $\mathbb{Q}$ as linearly ordered lattice are meet irreducible, but none have a cover.

Define $(c,a) \approx (d,b)$ as the equivalence relation on $S = S(\mathfrak{M}) = \{(c,a) \in \mathfrak{M} \times \mathfrak{M} \ : \ c \unrhd a\}$, generated by $\succeq$. So over latarres we have $(c,a) \approx (d,b)$ implies $c \multimap a = d \multimap b$.

Each properly descending finite chain $c_n \rhd c_{n-1} \rhd c_{n-2} \rhd \ldots \rhd c_0$ of a lattice produces a list of subintervals $[c_i, c_{i+1}]$. Two such chains $\langle c_i \rangle_{i \leq n}$ and $\langle d_i \rangle_{i \leq n}$ are called *projectively equivalent* if there is a permutation σ on $\{0,1,2,\ldots,n-1\}$ such that $(c_{i+1}, c_i) \approx (d_{\sigma(i)+1}, d_{\sigma(i)})$ for all i. Projective equivalence is an equivalence relation since the collection of permutations forms a group.

A lattice is called *semimodular* if for all $a \neq b$, if $a \sqcup b$ is a cover of both a and b, then both a and b are covers of $a \sqcap b$. So modular lattices are semimodular by Theorem 6.1. Substructures $[a,b]$ of a semimodular lattice are again semimodular lattices.

By Zorn's Lemma all chains of a lattice extend to maximal chains.

What follows are well-known Theorems from lattice theory, associated with the names Dedekind, Hölder, and Jordan. See Birkhoff's [4], or Jacobson's [6, chapter 8].

Theorem 6.5. *If all chains of a semimodular lattice are finite, then all its maximal chains are of equal length.*

Theorem 6.6. *All maximal finite chains of a modular lattice are projectively equivalent.*

Let $\mathfrak{M}$ be a modular lattice whose intervals $[a,b]$ are all of finite length. The following is a way to construct all latarres with $\mathfrak{M}$ as underlying lattice.

The restriction of equivalence relation $(c,a) \approx (d,b)$ on $S = S(\mathfrak{M})$ to subset $C = C(\mathfrak{M}) = \{(c,a) \in \mathfrak{M} \times \mathfrak{M} \ : \ c \text{ is a cover of } a\}$ creates a set of partitions $C_\approx = C_\approx(\mathfrak{M})$. We write $(c,a)_\approx$ for the equivalence class of (c,a) in $C_\approx$. Note that for modular lattices we have that $(q,p) \succeq (b,a)$ implies b is a cover of a exactly when q is a cover of p (see Theorem 6.1). Given the modular lattice $\mathfrak{M}$ above, let $\mathfrak{P}$ be a meet subsemilattice of $\mathfrak{M}$ with top ε, and let v_C be a function from $C_\approx$ to $\mathfrak{P}$. We show below that each such function v_C 'extends' to a unique latarre arrow on $\mathfrak{M}$ with all values $(x \multimap y) \in \mathfrak{P}$ and, conversely, each latarre arrow on $\mathfrak{M}$ 'restricts' to one such function v_C. Define binary function $x \multimap y$ on $\mathfrak{M}$ as follows.

- P1. If b is a cover of a, then set $b \multimap a \ := \ v_C((b,a)_\approx)$.
- P2. Suppose $b \rhd a$ and b is not a cover of a. There is a maximal chain $b = c_n \rhd c_{n-1} \rhd c_{n-2} \rhd \ldots \rhd c_0 = a$ of length $n \geq 2$. So c_{i+1} is a cover of c_i, for all i. Set $b \multimap a \ := \ \bigsqcap_{i<n}(c_{i+1} \multimap c_i)$. Since all maximal finite chains of $[a,b]$ are projectively equivalent by Theorem 6.6, this is a sound definition, independent of the choice of the maximal finite chain.
- P3. If $a \unlhd b$, then set $a \multimap b \ := \ \varepsilon$.
- P4. If a and q are not compatible, then set $a \multimap q \ := \ (a \sqcup q) \multimap q$.

The cases P1 through P4 are disjoint and include all possibilities of pairs of elements x and y. Each case is well-defined when the previous cases are well-defined. So $x \multimap y$ is a well-defined binary function on the modular lattice $\mathfrak{M}$. If $x \multimap y$ is a binary function on $\mathfrak{M}$ such that $(\mathfrak{M}, \multimap)$ is a latarre, then there are subsemilattice $\mathfrak{P}$ of $\mathfrak{M}$ with top ε and function v_C as described above such that $x \multimap y$ satisfies conditions P1 through P4. This is the easy direction. We show below that, conversely, if an arrow $x \multimap y$ satisfies P1 through P4, then $(\mathfrak{M}, \multimap)$ is a latarre with all values $(x \multimap y) \in \mathfrak{P}$.

Proposition 6.7. *Let $\mathfrak{M}$ be a modular lattice with all intervals $[a, b]$ of finite length. Let $\mathfrak{P}$ be a meet subsemilattice of $\mathfrak{M}$ with top ε, and let v_C be a function from $C_{\approx}(\mathfrak{M})$ to $\mathfrak{P}$. Let binary function $x \multimap y$ on $\mathfrak{M}$ be defined as in P1 through P4. Then for all a, b, and c we have*

1. $(a \sqcup b) \multimap b = a \multimap (a \sqcap b)$.
2. $a \multimap b = (a \sqcup b) \multimap b = a \multimap (a \sqcap b)$.
3. $c \trianglerighteq b \trianglerighteq a$ *implies* $c \multimap a = (c \multimap b) \sqcap (b \multimap a)$.
4. $(a \multimap b) \sqcap (b \multimap c) = (a \sqcup b) \multimap (b \sqcap c)$.
5. $x \trianglelefteq y$ *implies* $a \multimap x \trianglelefteq a \multimap y$ *and* $y \multimap b \trianglelefteq x \multimap b$.
6. $(a \multimap b) \sqcap (b \multimap c) \trianglelefteq a \multimap c$.
7. $(a \multimap b) \sqcap (a \multimap c) = a \multimap (b \sqcap c)$.
8. $(b \multimap a) \sqcap (c \multimap a) = (b \sqcup c) \multimap a$.
9. $b \triangleright a$ *implies* $b \multimap a = \bigsqcap\{q \multimap p \ : \ (q, p) \in C(\mathfrak{M})$ *and* $a \trianglelefteq p \trianglelefteq q \trianglelefteq b\}$.

Proof. Item 1: If $a \trianglelefteq b$, then both sides equal ε. Otherwise, the intervals $[b, a \sqcup b]$ and $[a \sqcap b, a]$ are isomorphic by map $\sigma(x) = a \sqcap x$ with inverse $\tau(y) = b \sqcup y$. Suppose p, q are such that $b \trianglelefteq p \trianglelefteq q \trianglelefteq a \sqcup b$. Then $p \sqcup (a \sqcap q) = (p \sqcup a) \sqcap q = q$, so $(q, p) \succeq (a \sqcap q, a \sqcap p) = (\sigma(q), \sigma(p))$. So σ sends each pair in an equivalence class of $\approx$ to a pair in the same equivalence class, and therefore sends maximal chains of covers to projectively equivalent maximal chains of covers. So with the definition of $\multimap$ we have $(a \sqcup b) \multimap b = a \multimap (a \sqcap b)$.

Item 2: If a and b are incompatible, then this is immediate from item 1 and the definition of $a \multimap b$. If $a \trianglelefteq b$ then $a \sqcup b = b$ and $a \sqcap b = a$ make all implications equal to ε. If $a \trianglerighteq b$ then $a \sqcup b = a$ and $a \sqcap b = b$, making all arrows 'syntactically' equal to $a \multimap b$.

Item 3: The cases where $a = b$ or $b = c$ are trivial. If $c \triangleright b \triangleright a$, then string maximal finite chains from $[a, b]$ and $[b, c]$ together and apply the definitions.

Item 4: $(a \multimap b) \sqcap (b \multimap c) = ((a \sqcup b) \multimap b) \sqcap (b \multimap (b \sqcap c))$. Apply item 3.

Item 5: We have $a \trianglerighteq a \sqcap y \trianglerighteq a \sqcap x$, so $a \multimap x = a \multimap (a \sqcap x) = (a \multimap (a \sqcap y)) \sqcap ((a \sqcap y) \multimap (a \sqcap x)) \trianglelefteq a \multimap (a \sqcap y) = a \multimap y$. We have $b \sqcup y \trianglerighteq b \sqcup x \trianglerighteq b$, so $y \multimap b = (b \sqcup y) \multimap b = ((b \sqcup y) \multimap (b \sqcup x)) \sqcap ((b \sqcup x) \multimap b) \trianglelefteq (b \sqcup x) \multimap b = x \multimap b$.

Item 6: By $(a \multimap b) \sqcap (b \multimap c) = (a \sqcup b) \multimap (b \sqcap c)$ and item 5.

Item 7: We have $a \multimap x = a \multimap (a \sqcap x)$. Apply this for x equal to b, to c, and to $b \sqcap c$ respectively. So we must prove that $(a \multimap (a \sqcap b)) \sqcap (a \multimap (a \sqcap c)) = a \multimap (a \sqcap b \sqcap c)$. In other words and easier to spell out, in proving the original requested equation we may suppose without loss of generality that $a \trianglerighteq b \sqcup c$. With that we apply item 3 and get $(a \multimap b) \sqcap (a \multimap c) = (a \multimap (b \sqcup c)) \sqcap ((b \sqcup c) \multimap b) \sqcap ((b \sqcup c) \multimap c) = (a \multimap (b \sqcup c)) \sqcap ((b \sqcup c) \multimap b) \sqcap (b \multimap (b \sqcap c)) = a \multimap (b \sqcap c)$.

Item 8: We have $x \multimap a = (a \sqcup x) \multimap a$. Apply this for x equal to b, to c, and to $b \sqcup c$ respectively. So we must prove that $((a \sqcup b) \multimap a) \sqcap ((a \sqcup c) \multimap a) = (a \sqcup b \sqcup c) \multimap a$. In other words and easier to spell out, in proving the original requested equation we may suppose without loss of generality that $a \trianglelefteq b \sqcap c$. With that we apply item 3 and get $(b \multimap a) \sqcap (c \multimap a) = (b \multimap (b \sqcap c)) \sqcap (c \multimap (b \sqcap c)) \sqcap ((b \sqcap c) \multimap a) = ((b \sqcup c) \multimap c) \sqcap (c \multimap (b \sqcap c)) \sqcap ((b \sqcap c) \multimap a) = (b \sqcup c) \multimap a$.

Item 9: With item 3 we have that $b \multimap a$ is a lower bound of the set

$$\{q \multimap p \ : \ (q, p) \in C(\mathfrak{M}) \text{ and } a \trianglelefteq p \trianglelefteq q \trianglelefteq b\}.$$

In the converse direction, each cover $q \trianglerighteq p$ in interval $[a, b]$ extends to a finite maximal chain. Apply the definition of $\multimap$. □

Theorem 6.8. *Let $\mathfrak{M}$ be a modular lattice with all intervals $[a, b]$ of finite length. Let $\mathfrak{P}$ be a meet subsemilattice of $\mathfrak{M}$ with top ε, and let v_C be a function from $C_{\approx}(\mathfrak{M})$ to $\mathfrak{P}$. Let binary function $x \twoheadrightarrow y$ on $\mathfrak{M}$ be defined as in P1 through P4. Then $(\mathfrak{M}, \twoheadrightarrow)$ is a modular latarre with $(a \twoheadrightarrow b) \in \mathfrak{P}$ for all $a, b \in \mathfrak{M}$.*

Proof. All latarre axioms follow from the definitions and Proposition 6.7. □

In [1], Alizadeh and Joharizadeh construct what we now call CJ latarres as in Definition 5.1, on finite distributive lattices. Finite distributive lattices allow them to employ that each equivalence class in $C_{\approx}(\mathfrak{M})$ has a unique largest element, see our Proposition 6.3.5. Consequently the part of their construction corresponding with our function v_C is simpler and more elegant.

7 Fixed Points and Löb

Ever since Visser's paper [8] we have a special interest in logic and algebraic structures with explicit fixed points. In the context of latarres we now give explicit fixed points a further look.

Over Visser latarres we know (see [8] or [3]) that all terms $t(x)$ have *explicit fixed point* $t(1)$, that is, $t(t(1)) = t(1)$, exactly when all terms of the form $t_a(x) = x \twoheadrightarrow a$ have explicit fixed point $t_a(1)$, that is, $t_a(t_a(1)) = t_a(1)$. Let us call an element a of a Visser latarre a *Löb* element if it satisfies equation $\nabla a \twoheadrightarrow a = t_a(t_a(1)) = t_a(1) = \nabla a$. The name Löb is chosen because the form of this equation corresponds with the key Löb equation of the axiomatization of provability modal logic.

Definition 7.1. We extend the notions of explicit fixed points $t(1)$ and of Löb elements to all weakly Visser latarres. One immediate problem is that weakly Visser latarres need not have a largest element 1. The following are straightforward generalizations of these concepts to all latarres. An equation $t(x) = u(x)$ has *ultimate solutions* over latarre $\mathfrak{A}$ if for all a there are $b \trianglerighteq a$ such that $t(b) = u(b)$. If $\mathfrak{A}$ has top 1, equation $t(x) = u(x)$ has ultimate solutions if and only if $t(1) = u(1)$. Note that a comparison $t(x) \trianglelefteq u(x)$ can also have ultimate solutions over $\mathfrak{A}$, since it corresponds with equation $t(x) \sqcap u(x) = t(x)$. We call a term $t(x)$ *ultimately fixed* or *U-fixed* over $\mathfrak{A}$ if equation $t(t(x)) = t(x)$ has ultimate solutions over $\mathfrak{A}$. We call an element a a *U-Löb* element over $\mathfrak{A}$ if $t_a(t_a(x)) = t_a(x)$ has ultimate solutions over $\mathfrak{A}$, where $t_a(x) = x \twoheadrightarrow a$. U-fixed and U-Löb are obvious generalizations of the notions of explicit fixed point $t(1)$ and of Löb element over unitary latarres. From now on we only use U-fixed and U-Löb and take the liberty to re-use the earlier expressions of explicit fixed point $t(1)$ and of Löb element when appropriate.

A latarre is *U-fixed* if all its terms are U-fixed. A latarre is *U-Löb* if all its element are U-Löb. Obviously U-fixed implies U-Löb. By [8] or [3], over Visser latarres we have the converse direction that U-Löb implies U-fixed. The logical forms of these U-definitions are unfortunately more complicated. Below we introduce simple equational definitions of what we call fixed and Löb, and show that these are equivalent over weakly Visser latarres to U-fixed and U-Löb.

Observe that $t(x) \trianglelefteq \varepsilon$ implies $t(t(\varepsilon) \sqcap \varepsilon) = t(t(\varepsilon))$. In particular $t_a(t_a(\varepsilon) \sqcap \varepsilon) = t_a(t_a(\varepsilon))$. This is partial motivation for the following definitions. A term $t(x)$ is called *fixed* over a latarre $\mathfrak{A}$ if $\mathfrak{A}$ satisfies schema $t(t(x) \sqcap x) = t(x)$. An element a is called *Löb* over a latarre $\mathfrak{A}$ if $\mathfrak{A}$ satisfies schema $t_a(t_a(x) \sqcap x) = t_a(x)$, where $t_a(x) = x \twoheadrightarrow a$. A latarre is *fixed* if all its terms are fixed. A latarre is *Löb* if all its elements are Löb. Obviously fixed implies Löb.

An easy example: If $t(x)$ satisfies schema $x \trianglelefteq t(x)$ over $\mathfrak{A}$, then term $t(x)$ is fixed over $\mathfrak{A}$. A more involved example is contained in the following Proposition.

Proposition 7.2. *Let $t(x)$ be a term over a latarre $\mathfrak{A}$.*

If x is only positive in $t(x)$, then $\mathfrak{A}$ satisfies schema $t(t(x) \sqcap x) \trianglelefteq t(x)$. If x is only negative in $t(x)$, then $\mathfrak{A}$ satisfies schema $t(x) \trianglelefteq t(t(x) \sqcap x)$.

If $\mathfrak{A}$ is weakly Visser, then $\mathfrak{A}$ satisfies schema $t(x) \trianglelefteq t(t(x) \sqcap x)$.

So if $\mathfrak{A}$ is weakly Visser and x is only positive in $t(x)$, then $\mathfrak{A}$ satisfies schema $t(t(x) \sqcap x) = t(x)$, that is, $t(x)$ is fixed over $\mathfrak{A}$.

Proof. The claims about positivity and negativity are immediate by Proposition 4.1. By Theorem 5.8 weakly Visser implies meet substitution, so $t(x) \sqcap t(t(x) \sqcap x) = t(x) \sqcap t(x) = t(x)$. □

Next come some technical Propositions which we need to identify the various notions of fixed and Löb.

Proposition 7.3. *Let $t(x)$ be a term over a weakly Visser latarre $\mathfrak{A}$. Then $\mathfrak{A}$ satisfies schema $t(t(x) \sqcap x) \sqcap (x \twoheadrightarrow t(x)) \sqcap (y \twoheadrightarrow t(x)) = t(x) \sqcap \varepsilon$.*

Proof. There is a term $u(y,z)$ with y only positive in $u(y,z)$ and z only negative in $u(y,z)$ such that $t(x)$ equals $u(x,x)$. So schema $u(t(x) \sqcap x, z) \trianglelefteq u(x,z)$ holds. Now z is also only formal in $u(y,z)$. By Theorem 5.10 with $n = 1$, we have that $\mathfrak{A}$ satisfies schema $(z \twoheadrightarrow w) \sqcap u(y,w) \trianglelefteq u(y,z)$. So

$$\begin{aligned}
&t(t(x) \sqcap x) \sqcap (x \twoheadrightarrow t(x)) \sqcap (y \twoheadrightarrow t(x)) \trianglelefteq \\
&u(x, t(x) \sqcap x) \sqcap (x \twoheadrightarrow t(x)) \sqcap (y \twoheadrightarrow t(x)) \trianglelefteq \\
&u(x, x \sqcap x) \sqcap (x \twoheadrightarrow t(x)) \sqcap (y \twoheadrightarrow t(x)) = \\
&t(x) \sqcap (x \twoheadrightarrow t(x)) \sqcap (y \twoheadrightarrow t(x)) \trianglelefteq t(x) \sqcap \varepsilon.
\end{aligned}$$

The converse direction holds by Proposition 7.2 and the schema $t(x) \sqcap \varepsilon \trianglelefteq z \twoheadrightarrow t(x)$. □

Proposition 7.3 immediately implies:

Proposition 7.4. *Let $t(x)$ be a term over a weakly Visser latarre $\mathfrak{A}$. Then $\mathfrak{A}$ satisfies $t(t(\varepsilon) \sqcap \varepsilon) \sqcap \nabla t(\varepsilon) = t(\varepsilon) \sqcap \varepsilon$. So if $\mathfrak{A}$ is Visser, then $\mathfrak{A}$ satisfies $t(t(1)) \sqcap \nabla t(1) = t(1)$.*

Proposition 7.5. *Let $t(x)$ be a term over a weakly Visser latarre $\mathfrak{A}$, and a be an element of $\mathfrak{A}$. Define $u(z) = z \twoheadrightarrow t(a)$. If there is $b \trianglerighteq a$ with $u(u(b) \sqcap b) = u(b)$, then $t(t(a) \sqcap a) \sqcap \varepsilon = t(a) \sqcap \varepsilon$.*

So if term $t(x)$ satisfies schema $t(x) \trianglelefteq \varepsilon$ over weakly Visser latarre $\mathfrak{A}$, and $t(a)$ is U-Löb over $\mathfrak{A}$ for all a, then $t(x)$ is fixed over $\mathfrak{A}$.

Proof. By Proposition 7.3 we have $t(t(a) \sqcap a) \sqcap (b \twoheadrightarrow t(a)) = t(a) \sqcap \varepsilon$. So with Proposition 5.3.4 we have $t(t(a) \sqcap a) \sqcap \varepsilon \trianglelefteq (b \twoheadrightarrow t(a)) \twoheadrightarrow (t(a) \sqcap \varepsilon) \trianglelefteq ((b \twoheadrightarrow t(a)) \sqcap b) \twoheadrightarrow t(a) = u(u(b) \sqcap b) = u(b) = b \twoheadrightarrow t(a)$. Thus $t(t(a) \sqcap a) \sqcap \varepsilon = t(t(a) \sqcap a) \sqcap (b \twoheadrightarrow t(a)) = t(a) \sqcap \varepsilon$. □

Proposition 7.5 implies that Visser latarres are fixed exactly when they are Löb. In this Section we broaden this result.

Proposition 7.6. *Let a be element of a latarre $\mathfrak{A}$. Then a is Löb over $\mathfrak{A}$ implies a is U-Löb over $\mathfrak{A}$. If $\mathfrak{A}$ is weakly Visser, then a is U-Löb over $\mathfrak{A}$ implies a is Löb over $\mathfrak{A}$.*

Proof. In the first case over a general latarre, a is Löb means schema $(((x \twoheadrightarrow a) \sqcap x) \twoheadrightarrow a) = x \twoheadrightarrow a$ holds. In this schema we can plug in any $x \trianglerighteq \varepsilon$, thereby establishing ultimate solutions for $((x \twoheadrightarrow a) \twoheadrightarrow a) = x \twoheadrightarrow a$. So a is U-Löb.

The case for a weakly Visser latarre: Let $t(x)$ be term $x \twoheadrightarrow a$. Suppose for all x there is $y \trianglerighteq x$ such that $t(t(y)) = t(y)$. We already know by Proposition 7.2 that weakly Visser implies schema $t(x) \trianglelefteq t(t(x) \sqcap x)$. Given x, it suffices to show $t(t(x) \sqcap x) \trianglelefteq t(x)$. There is $y \trianglerighteq x$ such that $t(t(y)) = t(y)$. Weakly Visser implies meet substitution, so $t(x) \sqcap x = t(x \sqcap y) \sqcap x = t(y) \sqcap x$. So

$$
\begin{aligned}
&t(t(x) \sqcap x) = t(t(y) \sqcap x) = (t(y) \sqcap x) \twoheadrightarrow a = \\
&(x \twoheadrightarrow ((t(y) \sqcap x) \twoheadrightarrow a)) \sqcap ((t(y) \sqcap x) \twoheadrightarrow a) = \\
&(x \twoheadrightarrow (((t(y) \sqcap x) \twoheadrightarrow a) \sqcap x)) \sqcap ((t(y) \sqcap x) \twoheadrightarrow a) = \\
&(x \twoheadrightarrow ((t(y) \twoheadrightarrow a) \sqcap x)) \sqcap ((t(y) \sqcap x) \twoheadrightarrow a) = \\
&(x \twoheadrightarrow (t(t(y)) \sqcap x)) \sqcap ((t(y) \sqcap x) \twoheadrightarrow a) = \\
&(x \twoheadrightarrow (t(y) \sqcap x)) \sqcap ((t(y) \sqcap x) \twoheadrightarrow a) \trianglelefteq x \twoheadrightarrow a = t(x).
\end{aligned}
$$

□

Proposition 7.6 is sound justification for re-using the name Löb in our new definition.

Proposition 7.7. *Let $t(x)$ be a term over a Visser latarre $\mathfrak{A}$. Then $t(x)$ is fixed over $\mathfrak{A}$ if and only if $t(x)$ has explicit fixpoint $t(1)$.*

Proof. Visser latarres are unitary, so we have $\varepsilon = 1$. If schema $t(t(x) \sqcap x) = t(x)$ holds, then set $x = 1$ to obtain $t(t(1)) = t(1)$. Conversely, suppose $t(t(1)) = t(1)$. We have $x = 1 \sqcap x$. So with meet substitution, $t(t(x) \sqcap x) \sqcap x = t(t(1)) \sqcap x = t(1) \sqcap x \trianglelefteq t(1)$. So $t(t(x) \sqcap x) \trianglelefteq x \twoheadrightarrow t(1) = x \twoheadrightarrow t(x)$. Application of Proposition 7.3 with $y = x$ gives $t(t(x) \sqcap x) = t(x)$. □

For latarres with top 1 a term $t(x)$ is U-fixed exactly when $t(1)$ is an explicit fixed point. So Proposition 7.7 for Visser latarres is a justification for using the name fixed in our new definition.

Next we consider what happens when latarres are fixed or Löb. The following Proposition is of interest on its own.

Proposition 7.8. *Let latarre $\mathfrak{A}$ be such that for all terms $t(x)$ in which x occurs only once, $\mathfrak{A}$ satisfies schema $t(t(x) \sqcap x) = t(x)$. Then the schema holds for all terms $t(x)$, that is, $\mathfrak{A}$ is fixed.*

Proof. We prove that $t(x)$ is fixed, by induction on the number n of occurrences of x in term $t(x)$. The cases for $n \leq 1$ are trivial or are given. Induction step: Suppose the case holds for terms with at most n occurrences of x. Let $t(x)$ equal term $u(x, x)$, where x occurs once in $u(x, y)$, and y occurs n times in $u(x, y)$. So we have schemas, first, by induction, $u(x, y) = u(x, u(x, y) \sqcap y)$ and, second, $u(z, u(x, y) \sqcap y) = u(u(z, u(x, y) \sqcap y) \sqcap z, u(x, y) \sqcap y)$. In the second schema, set z equal to x, and apply the first schema twice to get $u(x, y) = u(x, u(x, y) \sqcap y) = u(u(x, y) \sqcap x, u(x, y) \sqcap y)$. Finally set y equal to x to get $t(x) = t(t(x) \sqcap x)$. □

Proposition 7.9. *A fixed latarre is U-fixed.*

Proof. Let $t(x)$ be a term over a fixed latarre. So we have schema $t(t(x) \sqcap x) = t(x)$. For U-fixed it suffices to find a such that $t(b) \trianglelefteq b$ for all $b \trianglerighteq a$. There is a term $u(y_1, y_2, \ldots, y_n)$ built from x and the elements of A using at most $\sqcap$ and $\sqcup$, and arrow formula terms (that is, terms of the form $v \twoheadrightarrow w$) $r_1, r_2, \ldots, r_n$ such that term $t(x)$ equals term $u(r_1, r_2, \ldots, r_n)$. Set $v(x)$ equal to term $u(\varepsilon, \varepsilon, \ldots, \varepsilon)$. So schema $t(x) \trianglelefteq v(x)$ holds, and x is at most positive in $v(x)$. Let a be an upper bound of all elements of A that occur in term $v(x)$. Then for all $b \trianglerighteq a$ we have $t(b) \trianglelefteq v(b) \trianglelefteq b$, where this very last $\trianglelefteq$ follows easily by induction on the complexity of term $v(x)$. □

Proposition 7.10. *A U-fixed latarre is weakly Visser and U-Löb.*

Proof. Let $\mathfrak{A}$ be a U-fixed latarre. Then $\mathfrak{A}$ is obviously U-Löb. Next we show weakly Visser. Let $u_a(x)$ be term $a \sqcap (x \twoheadrightarrow a)$. Then $u_a(u_a(x)) = a \sqcap ((a \sqcap (x \twoheadrightarrow a)) \twoheadrightarrow a) = a \sqcap \varepsilon$. So ultimate solutions of $u_a(u_a(x)) = u_a(x)$ imply that a is weakly arrow persistent. Distributivity: Let $v_{abc}(x)$ be term $a \sqcap ((x \sqcap b) \sqcup (x \sqcap c))$. If $x \trianglerighteq b \sqcup c$, then $v_{abc}(x) = a \sqcap (b \sqcup c)$ and $v_{abc}(v_{abc}(x)) = a \sqcap ((a \sqcap b) \sqcup (a \sqcap c)) = (a \sqcap b) \sqcup (a \sqcap c)$. So ultimate solutions of $v_{abc}(v_{abc}(x)) = v_{abc}(x)$ imply distributivity. □

Theorem 7.11. *The following are equivalent for a latarre $\mathfrak{A}$.*

1. *$\mathfrak{A}$ is U-fixed.*
2. *$\mathfrak{A}$ is a weakly Visser and U-Löb.*
3. *$\mathfrak{A}$ is a weakly Visser and Löb.*
4. *$\mathfrak{A}$ is fixed.*

Proof. Item 1 implies item 2 by Proposition 7.10.

Item 2 implies item 3 by Proposition 7.6.

Suppose item 3. To prove: Item 4. By Proposition 7.8 it suffices to show schema $t(t(x)\sqcap x) = t(x)$ for terms $t(x)$ in which x occurs once. If x is positive in $t(x)$, then we are done by Proposition 7.2. Otherwise, suppose x is negative in $t(x)$. So x is formal in $t(x)$. By Proposition 7.2 (or Proposition 4.1), it suffices to show that $t(t(x) \sqcap x) \trianglelefteq t(x)$. Let $u(x)$ be the largest arrow subterm of $t(x)$ which contains x. Without loss of generality we may suppose that there is a term $v(y)$ built from constants and variables but only one single occurrence of y using at most $\sqcap$ and $\sqcup$, such that we have schema $t(x) = v(u(x))$, and x is negative in $u(x)$. We have schema $u(x) \trianglelefteq \varepsilon$. So by Propositions 7.5 and 7.6 we have schema $u(u(x) \sqcap x) = u(x)$. To extend the fixedness of $u(x)$ to fixedness of $t(x)$, it suffices to show that the collection of terms $w(x)$ in which x is negative and which satisfy schema $w(w(x) \sqcap x) = w(x)$, is closed under taking $\sqcap$ and $\sqcup$ with constants. Let $r(x) = w(x) \sqcap a$ with $w(w(x) \sqcap x) = w(x)$. Then $r(r(x) \sqcap x) = w(r(x) \sqcap x) \sqcap a = w(w(x) \sqcap a \sqcap x) \sqcap a = w(w(x) \sqcap x) \sqcap a = w(x) \sqcap a = r(x)$. Let $s(x) = w(x) \sqcup a$ with $w(w(x) \sqcap x) = w(x)$. Then $s(s(x) \sqcap x) = w(s(x) \sqcap x) \sqcup a = w((w(x) \sqcup a) \sqcap x) \sqcup a \trianglelefteq w(w(x)\sqcap x)\sqcup a = w(x)\sqcup a = s(x)$. So the collection is closed as wished, thus $t(x)$ is also fixed.

Item 4 implies item 1 by Proposition 7.9. □

Some of the proofs of the Propositions and Theorem so far in this Section imply generalizations. We end this Section with one of these.

Proposition 7.12. *The following are equivalent for a latarre $\mathfrak{A}$.*

1. *$\mathfrak{A}$ is weakly Visser, and all elements of the form $a \rightarrow b$ are Löb.*
2. *$\mathfrak{A}$ is fixed.*

Proof. With Theorem 7.11 it suffices to show that item 1 implies item 2.

So suppose item 1. The proof is almost identical to the proof of 7.11.4 from 7.11.3. Follow that proof to the sentence: We have schema $u(x) \trianglelefteq \varepsilon$. Then observe that by supposition, $u(a)$ is Löb for all a. Then continue the earlier proof: So by Propositions 7.5 and 7.6 we have schema $u(u(x) \sqcap x) = u(x)$. Then continue to the end of the earlier proof. □

An element that can be written in the form $a \rightarrow b$ is called an arrow element, or an arrow element of the 1^{st} kind. Given an arrow element t of the n^{th} kind, we call an element $a \rightarrow t$ an arrow element of the $(n+1)^{\text{th}}$ kind. An arrow element of the n^{th} kind is also an arrow element of the m^{th} kind, for all $n \geq m \geq 1$.

Theorem 7.13. *The following are equivalent for a latarre $\mathfrak{A}$.*

1. *$\mathfrak{A}$ is weakly Visser, and there is $n \geq 1$ such that all arrow elements of the n^{th} kind are Löb.*
2. *$\mathfrak{A}$ is fixed.*

Proof. Obviously item 2 implies item 1, see Proposition 7.12.

Suppose item 1. To prove: item 2. We complete the proof by induction on n. The case for $n = 1$ holds by Proposition 7.12. Assume the equivalence holds for n, and all arrow elements of the $(n+1)^{\text{th}}$ kind are Löb. It suffices to show that all arrow elements of the n^{th} kind are Löb. Let c be an arrow element of the n^{th} kind. Let $t(x)$ be term $x \rightarrow c$, and a be an arbitrary element. Then $t(a)$ is an arrow element of the $(n+1)^{\text{th}}$ kind. By assumption term $t(a)$ is Löb. Since a is arbitrary, by Propositions 7.5 and 7.6 term $t(x)$ is fixed, that is, c is Löb. □

References

[1] Majid Alizadeh and Nima Joharizadeh *Counting weak Heyting algebras on finite distributive lattices*, Logic Journal of the Interest Group in Pure and Applied Logics **23**, no. 2 (2015), 247–258.

[2] Majid Alizadeh, Mohammad Ardeshir, and Wim Ruitenburg. *Boolean Algebras in Visser Algebras*, Notre Dame Journal of Formal Logic **57** (2016), 141–150.

[3] Mohammad Ardeshir and Wim Ruitenburg. *Basic propositional calculus I*, Mathematical Logic Quarterly **44** (1998), 317–343.

[4] Garrett Birkhoff. **Lattice Theory**, American Mathematical Society Colloquium Publications Vol. 25, Third edition, third printing, American Mathematical Society, Providence RI, 1979.

[5] Sergio Celani and Ramon Jansana. *Bounded distributive lattices with strict implication*, Mathematical Logic Quarterly **51** (2005), 219–246.

[6] Nathan Jacobson. **Basic Algebra I**, W.H. Freeman and Company, 1974.

[7] Jorge Picado and Aleš Pultr. **Frames and Locales, Topology without points**, Frontiers in Mathematics, Birkhäuser, 2012.

[8] Albert Visser. *A propositional logic with explicit fixed points*, Studia Logica **40** (1981), 155–175.