Latarres on Complete Lattices

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1 Introduction

Latarres are the end result of a series of generalizations. Our process follows from earlier mathematical results obtained about Boolean algebras, Heyting algebras, Visser algebras (see [1], [2], and [4]), and what we call CJ algebras, after Celani and Jansana (weakly Heyting algebras in [3]).

2 What is a Latarre?

2.1 Informal Definition

A latarre is a LATtice with an ARRow. The essential parts of its language consist of three binary operators $(\sqcap, \sqcup, \rightarrowtriangle)$. With restriction to $(\sqcap, \sqcup)$ a latarre is a lattice with meet $\sqcap$ and join $\sqcup$. For the arrow we have the additional 'natural' schemas

$$
\begin{aligned}
&x \rightarrowtriangle y = (x \sqcup y) \rightarrowtriangle y.\\
&x \rightarrowtriangle y = x \rightarrowtriangle (x \sqcap y).\\
&y \trianglelefteq z \text{ implies } x \rightarrowtriangle y \trianglelefteq x \rightarrowtriangle z.\\
&y \trianglelefteq z \text{ implies } z \rightarrowtriangle x \trianglelefteq y \rightarrowtriangle x.\\
&(x \rightarrowtriangle y) \sqcap (y \rightarrowtriangle z) \trianglelefteq x \rightarrowtriangle z.
\end{aligned}
$$

where $\trianglelefteq$ is the usual order definable by $x \trianglelefteq y$ exactly when $x \sqcap y = x$.

2.2 Formal Definition

For practical reasons we extend our language to $(\sqcap, \sqcup, \multimap, \varepsilon)$ by adding a constant ε to the three binary operators mentioned above. A *latarre* is a structure satisfying the universal algebra schemas of a lattice with meet $\sqcap$ and join $\sqcup$, plus

N1. $x \multimap y = (x \sqcup y) \multimap y$.

N2. $x \multimap y = x \multimap (x \sqcap y)$.

N3. $x \multimap (x \sqcap y \sqcap z) \trianglelefteq x \multimap (x \sqcap y)$.

N4. $y \multimap (y \sqcap z) \trianglelefteq (x \sqcap y) \multimap (x \sqcap y \sqcap z)$.

N5. $(x \multimap (x \sqcap y)) \sqcap ((x \sqcap y) \multimap (x \sqcap y \sqcap z)) \trianglelefteq x \multimap (x \sqcap y \sqcap z)$.

N6. $\varepsilon \multimap \varepsilon = \varepsilon$.

Element ε is an important convenience, that is, ε with N6 is uniquely definable over the subsystem without N6.

Proposition 2.1. *Latarres satisfy schemas*

1. $y \trianglelefteq z$ *implies* $x \rightarrow y \trianglelefteq x \rightarrow z$.
2. $y \trianglelefteq z$ *implies* $z \rightarrow x \trianglelefteq y \rightarrow x$.
3. $(x \rightarrow y) \sqcap (y \rightarrow z) \trianglelefteq x \rightarrow z$.
4. $x \rightarrow y \trianglelefteq z \rightarrow z$.
5. $x \rightarrow y \trianglelefteq \varepsilon$.
6. $x \rightarrow x = \varepsilon$.
7. $x \trianglelefteq y$ *implies* $x \rightarrow y = \varepsilon$.
8. $x \rightarrow y = \varepsilon$ *implies* $z \rightarrow x \trianglelefteq z \rightarrow y$ *and* $y \rightarrow z \trianglelefteq x \rightarrow z$.

Section 3 has trivial examples of latarres which are neither distributive nor have a largest element. So ε need not be top.

Proposition 2.2. *Latarres satisfy schemas*

1. $x \rightarrow (y \sqcap z) = (x \rightarrow y) \sqcap (x \rightarrow z)$.

2. $(y \sqcup z) \rightarrow x = (y \rightarrow x) \sqcap (z \rightarrow x)$.

3. $z \rightarrow x \trianglelefteq (x \rightarrow y) \sqcap (y \rightarrow z)$ *implies* $(z \rightarrow x) = (z \rightarrow y) \sqcap (y \rightarrow x)$. *In particular,* $z \trianglerighteq y \trianglerighteq x$ *implies* $(z \rightarrow x) = (z \rightarrow y) \sqcap (y \rightarrow x)$.

4. $(x \rightarrow y) \sqcap (y \rightarrow z) = (x \sqcup y) \rightarrow (y \sqcap z)$.

5. $y \rightarrow z = \varepsilon$ *implies* $(x \sqcup y) \rightarrow z = x \rightarrow (x \sqcap z) = x \rightarrow z$.

6. $z \rightarrow x = \varepsilon$ *implies* $z \rightarrow (x \sqcap y) = (z \sqcup y) \rightarrow y = z \rightarrow y$.

7. $y \rightarrow z \trianglelefteq (x \sqcap y) \rightarrow (x \sqcap z)$.

8. $(y \rightarrow x) \sqcap (y \rightarrow z) = (y \rightarrow x) \sqcap ((x \sqcap y) \rightarrow (x \sqcap z))$.

Let a, b, *and* c *be elements of a latarre* $\mathfrak{A}$. *Then*

1. $c \sqcap \varepsilon = c \sqcap (b \rightarrow a)$ *if and only if* $\mathfrak{A}$ *satisfies schema* $c \sqcap ((a \sqcap b) \rightarrow (a \sqcap x)) = c \sqcap (b \rightarrow x)$.

2. $\mathfrak{A}$ *satisfies schema* $a \sqcap \varepsilon \trianglelefteq z \rightarrow a$ *if and only if* $\mathfrak{A}$ *satisfies schema* $a \sqcap ((a \sqcap x) \rightarrow (a \sqcap y)) = a \sqcap (x \rightarrow y)$.

We inductively define $\nabla^n x$ for all n by $\nabla^0 x = x$ and $\nabla^{n+1} x = \varepsilon \rightarrow\!\!\!\triangleright \nabla^n x$.

Proposition 2.3. *Latarres satisfy schemas*

1. $\nabla^n (x \sqcap y) = \nabla^n x \sqcap \nabla^n y$.
2. $x \sqcap y \rightarrow\!\!\!\triangleright z = \varepsilon$ *implies* $y \rightarrow\!\!\!\triangleright x \trianglelefteq y \rightarrow\!\!\!\triangleright z$.
   So $x \trianglelefteq y \rightarrow\!\!\!\triangleright x$ *plus* $x \sqcap y \rightarrow\!\!\!\triangleright z = \varepsilon$ *implies* $x \trianglelefteq y \rightarrow\!\!\!\triangleright z$.
3. $x \trianglelefteq y \rightarrow\!\!\!\triangleright z$ *implies* $x \sqcap (w \rightarrow\!\!\!\triangleright y) \trianglelefteq w \rightarrow\!\!\!\triangleright z$.
4. $y \rightarrow\!\!\!\triangleright \varepsilon = \varepsilon$ *implies* $\nabla x \sqcap ((x \sqcap y) \rightarrow\!\!\!\triangleright (x \sqcap z)) = \nabla x \sqcap (y \rightarrow\!\!\!\triangleright z)$.
5. $y \rightarrow\!\!\!\triangleright \varepsilon = \varepsilon$ *plus* $x \sqcap y \rightarrow\!\!\!\triangleright z = \varepsilon$ *implies* $\nabla x \trianglelefteq y \rightarrow\!\!\!\triangleright z$.

3 Examples of Latarres

One collection of trivial latarres is the following. Start with any lattice $\mathfrak{M}$ and any element m of $\mathfrak{M}$. Set $x \rightarrowtriangle y = m$ for all elements x and y of $\mathfrak{M}$. This defines a 'trivial' latarre with $\varepsilon = m$ and $\mathfrak{M}$ as underlying lattice.

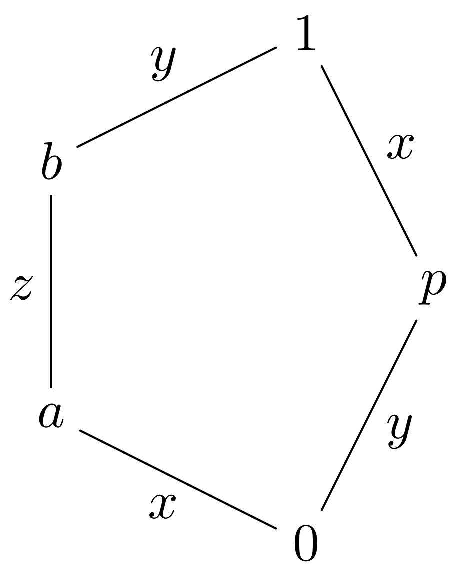
A latarre is called *unitary* if the lattice has a top 1 and $\varepsilon = 1$. A latarre is called *arrow persistent* if it satisfies schema $x \sqcap \varepsilon \trianglelefteq y \rightarrowtriangle x$. A latarre is called *Heyting* if it satisfies schema $x = \nabla x$. A latarre is called *Boolean* if it satisfies schema $(x \rightarrowtriangle (x \sqcap y)) \rightarrowtriangle (x \sqcap y) = x$. A latarre is unitary arrow persistent exactly when it satisfies schema $x \trianglelefteq \nabla x$. So Heyting latarres are unitary arrow persistent. Boolean latarres are Heyting.

A latarre is called *almost-complete* if for each subset S which contains an element, $\bigsqcup S$ exists or, equivalently, if for each subset S with a lower bound, $\bigsqcap S$ exists. So complete implies almost-complete. A *frame* (or a complete Heyting algebra or a locale) satisfies $m \sqcap \bigsqcup S = \bigsqcup \{m \sqcap s \ : \ s \in S\}$, for all sets of elements $\{m\} \cup S$. On a frame $\mathfrak{M}$ we can define an arrow $x \rightarrowtriangle y = \bigsqcup \{z \ : \ x \sqcap z \trianglelefteq y\}$. The resulting structure $(\mathfrak{M}, \rightarrowtriangle, 1, 0)$ is a frame. Each filter F on frame $\mathfrak{M}$ is the domain of an almost-complete Heyting latarre $(\mathfrak{F}, \rightarrowtriangle, 1)$.

Filters on a Boolean algebra $\mathfrak{B}$ are exactly the upward closed (Boolean) sublatarres of $\mathfrak{B}$.

Filters on a Heyting algebra $\mathfrak{C}$ are exactly the upward closed (Heyting) sublatarres of $\mathfrak{C}$.

Define a unitary latarre on lattice N_5 as follows. In the diagram of N_5 below, labels x, y, and z mean that we set $1 \rightarrow b = y$, set $b \rightarrow a = z$, and so on. The letters x, y and z are values to be chosen freely from the domain $\{0, a, b, p, 1\}$ with the only restrictions that $x \trianglelefteq z$ and $y \trianglelefteq z$.



The properties of unitary latarres allow us to uniquely extend the arrow by $p \rightarrow p = 1$ and $1 \rightarrow a = (1 \rightarrow b) \sqcap (b \rightarrow a) = y \sqcap z = y$ and $a \rightarrow p = a \rightarrow a \sqcap p = a \rightarrow 0 = x$, and so on.

A function $f : \mathfrak{A} \to \mathfrak{B}$ between latarres is called a latarre *(homo)morphism* if f preserves the defining operations of $\sqcap$, $\sqcup$, $\multimap$, and ε. Latarres are closed under submodels, products, and (homomorphic) images.

Proposition 3.1. *Let $\mathfrak{A} = (\mathfrak{M}, \sqcup, \multimap, \varepsilon)$ be a latarre and $f : \mathfrak{M} \to \mathfrak{M}$ be a meet semilattice endomorphism. Define $\mathfrak{A}_f = (\mathfrak{M}, \sqcup, \multimap_f, f(\varepsilon))$ by $a \multimap_f b = f(a \multimap b)$. Then $\mathfrak{A}_f$ is a latarre.*

Proposition 3.2. *Let $\mathfrak{A} = (\mathfrak{N}, \multimap, \varepsilon)$ be a latarre and $g : \mathfrak{N} \to \mathfrak{N}$ be a lattice endomorphism. Define $\mathfrak{A}^g = (\mathfrak{N}, \multimap^g, \varepsilon)$ by $a \multimap^g b = g(a) \multimap g(b)$. Then $\mathfrak{A}^g$ is a latarre.*

Let $g : A \to A$ be a continuous function on a topological space $\mathcal{O}(A)$. Inverse image map $h = g^{-1} : \mathcal{O}(A) \to \mathcal{O}(A)$ is a meet semilattice morphism on the meet semilattice part $\mathfrak{N}$ of the frame $\mathcal{O}(A)$. $\mathfrak{N}$ is the meet semilattice part of the corresponding complete Heyting latarre $\mathfrak{C} = (\mathfrak{N}, \sqcup, \multimap, A)$. By Proposition 3.1 we get a new latarre $\mathfrak{C}_h$ from $\mathfrak{C}$ by redefining $\varepsilon_h = g^{-1}(\varepsilon)$ and $U \multimap_h V = h(U \multimap V) = g^{-1}(U \multimap V) = \bigcup\{g^{-1}(W) \ : \ W \cap U \subseteq V\}$. Map $h = g^{-1}$ is also a lattice morphism on $(\mathfrak{N}, \sqcup)$. So by Proposition 3.2 we get another new latarre $\mathfrak{C}^h$ from $\mathfrak{C}$ by redefining $U \multimap^h V = g^{-1}(U) \multimap g^{-1}(V) = \bigcup\{W \ : \ g(W \cap g^{-1}(U)) \subseteq V\}$.

Proposition 3.3. *Let $f : \mathfrak{M} \to \mathfrak{N}$ be a lattice morphism, and $g : \mathfrak{N} \to \mathfrak{M}$ be map which preserves meet $\sqcap$. Let $\mathfrak{B} = (\mathfrak{N}, \rightarrowtriangle, \varepsilon)$ be a latarre. Define ε_m and $\rightarrowtriangle_m$ on $\mathfrak{M}$ by $\varepsilon_m = g(\varepsilon)$ and $x \rightarrowtriangle_m y = g(f(x) \rightarrowtriangle f(y))$. Then $\mathfrak{A} = (\mathfrak{M}, \rightarrowtriangle_m, \varepsilon_m)$ is a latarre.*

Map $f : \mathfrak{A} \to \mathfrak{B}$ of Proposition 3.3 need not be a latarre morphism. By Proposition 3.1 we have a latarre $\mathfrak{B}_{fg} = (\mathfrak{N}, \rightarrowtriangle_{fg}, fg(\varepsilon))$ with $x \rightarrowtriangle_{fg} y = fg(x \rightarrowtriangle y)$. Map $f : \mathfrak{A} \to \mathfrak{B}_{fg}$ is a latarre morphism.

Suppose map $g : \mathfrak{N} \to \mathfrak{M}$ of Proposition 3.3 is a lattice morphism. Map $g : \mathfrak{B} \to \mathfrak{A}$ need not be a latarre morphism. By Proposition 3.2 we have a latarre $\mathfrak{B}^{fg} = (\mathfrak{N}, \rightarrowtriangle^{fg}, \varepsilon)$ with $x \rightarrowtriangle^{fg} y = fg(x) \rightarrowtriangle fg(y)$. Map $g : \mathfrak{B}^{fg} \to \mathfrak{A}$ is a latarre morphism.

Proposition 3.4. *Let $\mathfrak{A}_1 = (\mathfrak{M}, \varepsilon_1 \rightarrowtriangle_1)$ and $\mathfrak{A}_2 = (\mathfrak{M}, \varepsilon_2, \rightarrowtriangle_2)$ be latarres on the same lattice $\mathfrak{M}$. Define $\mathfrak{A} = (\mathfrak{M}, \varepsilon, \rightarrowtriangle)$ by $\varepsilon = \varepsilon_1 \sqcap \varepsilon_2$ and $x \rightarrowtriangle y = (x \rightarrowtriangle_1 y) \sqcap (x \rightarrowtriangle_2 y)$. Then $\mathfrak{A}$ is a latarre.*

Proposition 3.4 can be strengthened for complete lattices, where we get a new arrow that looks like $x \rightarrowtriangle_S y = \bigsqcap\{x \rightarrowtriangle_s y \ : \ s \in S\}$, and $\varepsilon_S = \bigsqcap\{\varepsilon_s \ : \ s \in S\}$.

Let R be a commutative ring. Its collection of ideals forms a complete lattice ordered by set inclusion. Let $\mathfrak{M}$ be the complete lattice of ideals, with $I \sqcap J = I \cap J$ for all ideals I and J. Lattice $\mathfrak{M}$ need not be distributive. The set $\sqrt{I} = \{r \in R : r^n \in I \text{ for some } n\}$ is the least radical ideal containing I. Given ideals I and J, the set $J : I = \{r \in R : rI \subseteq J\}$ is an ideal. We construct a unitary complete latarre $\mathfrak{A}$ on lattice $\mathfrak{M}$ as follows. Set $I \multimap J = \sqrt{J : I}$ and $\varepsilon = R$. We have a unitary latarre $\mathfrak{A} = (\mathfrak{M}, \multimap, R)$ with $I \sqcap (I \multimap J) = I \sqcap \sqrt{J}$.

Let $\mathcal{O}(X)$ be a T_0 topological space. So we have a latarre $\mathfrak{A} = (\mathcal{O}(X), \multimap, X)$. Define operator $j : \mathcal{O}(X) \to \mathcal{O}(X)$ by

$$ju = \bigsqcup\{u \cup \{x\} \ : \ u \cup \{x\} \text{ is open}\}.$$

Define $x \multimap_j y = j(x \multimap y)$ and get a new latarre $(\mathcal{O}(X), \multimap_j, X)$, where $\nabla_j x = X \multimap_j x = jx$. Even in the case of $\mathcal{O}(\mathbb{R})$ there are u with $j^{n+1}u \neq j^n u$ for all n.

The above example generalizes to almost-complete frames. Let $\mathfrak{M} = (M, \sqcap, \sqcup)$ be an almost-complete lattice. Define v *covers or equals* u, written $u \trianglelefteq_1 v$, by
$u \trianglelefteq_1 v \ \leftrightarrow \ (u \trianglelefteq v \wedge \forall t(u \trianglelefteq t \trianglelefteq v \ \rightarrow \ (u = t \vee t = v)))$. Over a T_0 space $\mathcal{O}(X)$ this means $u \trianglelefteq_1 v$ exactly when there is $\xi \in X$ with $u \trianglelefteq v \trianglelefteq u \cup \{\xi\}$. Define operator $j : \mathfrak{M} \to \mathfrak{M}$ by

$$jx = \bigsqcup\{u \ : \ x \trianglelefteq_1 u\}.$$

4 General Substitution Rules

With each latarre $\mathfrak{A}$ we associate a predicate logic language $\mathcal{L}(\mathfrak{A})$. We may write $t(x)$ even if term $t(x)$ has other variables besides x. Given a term $t(x)$ of $\mathcal{L}(\mathfrak{A})$, we define positivity and negativity of occurrences of x in $t(x)$ in the usual inductive way.

Proposition 4.1. *Let $t(x)$ be a term over a latarre $\mathfrak{A}$. If x is only positive in $t(x)$, then $x \trianglelefteq y$ implies $t(x) \trianglelefteq t(y)$. If x is only negative in $t(x)$, then $x \trianglelefteq y$ implies $t(y) \trianglelefteq t(x)$.*

An x occurs at *depth* $n \geq 0$ in term $t(x)$ if x occurs n levels deep inside implication subformulas of implication subformulas and so on. So x occurs at depth 2 in $(y \rightarrow (w \sqcap (x \sqcup v))) \rightarrow z$, and x occurs at depth n in $\nabla^n x$. The x occurs *informally* if depth $n = 0$, otherwise x occurs *formally*.

Proposition 4.2. *Let $t(x)$ be a term over a latarre $\mathfrak{A}$ and $n \geq 0$ be such that x only occurs at depth n in $t(x)$. If x is only positive in $t(x)$, then $\mathfrak{A}$ satisfies schema $\nabla^n(x \rightarrow y) \trianglelefteq t(x) \rightarrow t(y)$. If x is only negative in $t(x)$, then $\mathfrak{A}$ satisfies schema $\nabla^n(x \rightarrow y) \trianglelefteq t(y) \rightarrow t(x)$.*

Proposition 4.3. *Let $t(x)$ be a term built without join $\sqcup$ over a latarre $\mathfrak{A}$, and $n \geq 1$ be such that x only occurs at depth n in $t(x)$. If x is only positive in $t(x)$, then $\mathfrak{A}$ satisfies schema $\nabla^{n-1}(x \multimap y) \sqcap t(x) \trianglelefteq t(y)$. If x is only negative in $t(x)$, then $\mathfrak{A}$ satisfies schema $\nabla^{n-1}(x \multimap y) \sqcap t(y) \trianglelefteq t(x)$.*

In Proposition 4.3 the exclusion of $\sqcup$ is essential.

Proposition 4.4. *Let $t(x)$ be a term over a latarre $\mathfrak{A}$ in which x occurs only at depths at least n in $t(x)$, for some $n \geq 1$. Let $a, b \in A$ be such that $\nabla^{n-1}(a \multimap b) = \varepsilon$. If x is only positive in $t(x)$, then $t(a) \trianglelefteq t(b)$. If x is only negative in $t(x)$, then $t(b) \trianglelefteq t(a)$.*

Write $x \leftrightarrow y$ as short for $(x \multimap y) \sqcap (y \multimap x)$. If x is only formal in $t(x)$, then $a \leftrightarrow b = \varepsilon$ implies $t(a) = t(b)$.

Proposition 4.5. *Let $t(x)$ be a term over a latarre $\mathfrak{A}$, and $a, b \in A$ are such that $a \multimap b = \varepsilon$. If x is only positive in $t(x)$, then $t(a) \multimap t(b) = \varepsilon$. If x is only negative in $t(x)$, then $t(b) \multimap t(a) = \varepsilon$.*

Given a latarre $\mathfrak{A}$, define equivalence relation $x \sim y$ by $x \leftrightarrow y = \varepsilon$. We write $a^{(1)}$ or a' for the equivalence class of a, and $A^{(1)}$ or A' for the collection of equivalence classes. Relation $x \sim y$ a congruence. On A' define the following latarre. If $a \leftrightarrow b = \varepsilon$, then $t(a) \leftrightarrow t(b) = \varepsilon$ for all terms $t(x)$. The following are well-defined on A': Define $x' \sqcap' y' = (x \sqcap y)'$ and $x' \sqcup' y' = (x \sqcup y)'$ and $x' \rightarrow' y' = (x \rightarrow y)'$. With these, $\mathfrak{A}' = (A', \sqcap', \sqcup', \rightarrow', \varepsilon')$ is a latarre. The map $x \mapsto x'$ is an onto latarre morphism from $\mathfrak{A}$ onto $\mathfrak{A}'$.

Repeat this construction and form $\mathfrak{A}'' = \mathfrak{A}^{(2)}$ by defining $x' \sim y'$ on $\mathfrak{A}'$ by $(x \leftrightarrow y)' = x' \leftrightarrow' y' = \varepsilon'$ or, equivalently, by $(x \leftrightarrow y) \sim \varepsilon$, that is, $(x \leftrightarrow y) \leftrightarrow \varepsilon = \varepsilon$. Continuing in this way, we get a chain

$$\mathfrak{A} = \mathfrak{A}^{(0)} \to \mathfrak{A}^{(1)} \to \mathfrak{A}^{(2)} \to \mathfrak{A}^{(3)} \to \dots$$

with for all $a, b \in A$ and $n \geq 1$ we have $a^{(n)} = b^{(n)}$ in $A^{(n)}$ exactly when $\nabla^{n-1}(a \leftrightarrow b) = \varepsilon$.

5 Visser Latarres and Substitution

We establish a close connection between weakly Visser latarres and (relative) meet substitution.

An element a of a latarre is called *arrow persistent* if it satisfies schema $a \sqcap \varepsilon \trianglelefteq y \rightarrow a$. Element a is called *unitary arrow persistent* if it satisfies schema $a \trianglelefteq y \rightarrow a$. A *weakly Visser latarre* is a distributive latarre satisfying the schema $x \sqcap \varepsilon \trianglelefteq y \rightarrow x$ of arrow persistence. A *Visser latarre* is a unitary weakly Visser latarre.

So a Visser latarre is a distributive latarre satisfying the schema $x \trianglelefteq \nabla x$ of unitary arrow persistence. Hence Heyting latarres are Visser latarres.

Proposition 5.1. *The following are equivalent for an element a of a latarre.*

1. *a is arrow persistent.*
2. *$a \sqcap (a \rightarrow y) \trianglelefteq z \rightarrow y$, for all y and z.*
3. *$(a \sqcap y \rightarrow z) = \varepsilon$ implies $a \sqcap \varepsilon \trianglelefteq y \rightarrow z$, for all y and z.*
4. *$a \sqcap y \trianglelefteq z$ implies $a \sqcap \varepsilon \trianglelefteq y \rightarrow z$, for all y and z.*

Given a latarre $\mathfrak{A}$ and element a, we construct a latarre on the subset $\{x \in A \,:\, x \trianglelefteq a\}$ as follows. Set

$$\begin{aligned}
&\varepsilon_a = \varepsilon \sqcap a, \\
&x \sqcap_a y = x \sqcap y, \\
&x \twoheadrightarrow_a y = a \sqcap (x \twoheadrightarrow y), \quad \text{and} \\
&x \sqcup_a y = x \sqcup y.
\end{aligned}$$

The resulting structure $\mathfrak{A}_a$ is a latarre. If $a \trianglelefteq \varepsilon$, then $\mathfrak{A}_a$ is unitary. If $\mathfrak{A}$ is unitary, arrow persistent, Visser, Heyting, or Boolean, then so is $\mathfrak{A}_a$.

The function $\pi_a(x) = a \sqcap x$ is an idempotent map from $\mathfrak{A}$ onto $\mathfrak{A}_a$. In general π_a is not a latarre morphism. Below we establish precisely when π_a is a morphism.

Given a term $t(x)$ and element a of latarre $\mathfrak{A}$, we say that $t(x)$ *admits meet substitution over* $(\mathfrak{A}, a)$ if $\mathfrak{A}$ satisfies schema

$$a \sqcap x = a \sqcap y \text{ implies } a \sqcap t(x) = a \sqcap t(y).$$

Equivalently, $(\mathfrak{A}, a)$ satisfies schema

$$a \sqcap t(x) = a \sqcap t(a \sqcap x).$$

We write $T(\mathfrak{A}, a)$ for the collection of terms over $\mathfrak{A}$ that admit meet substitution over $(\mathfrak{A}, a)$. We define $\mathfrak{A}$ *admits meet substitution* if $T(\mathfrak{A}, a)$ includes all terms for all $a \in A$.

Proposition 5.2. *Let a be an element of latarre $\mathfrak{A}$. Then the collection $T(\mathfrak{A}, a)$ contains all terms without x, the term x itself, and is closed under $\sqcap$ and under composition. Additionally:*

1. *$\mathfrak{A}$ satisfies schema $a \sqcap \varepsilon \trianglelefteq x \rightarrow a$ if and only if $T(\mathfrak{A}, a)$ is closed under $\rightarrow$.*

2. *$\mathfrak{A}$ satisfies schema $a \sqcap (x \sqcup y) = (a \sqcap x) \sqcup (a \sqcap y)$ if and only if $T(\mathfrak{A}, a)$ is closed under $\sqcup$.*

As a Corollary we get:

Theorem 5.3. *The following are equivalent for a latarre $\mathfrak{A}$.*

1. *$\mathfrak{A}$ is weakly Visser.*

2. *For all elements a of $\mathfrak{A}$ the map $\pi_a : \mathfrak{A} \to \mathfrak{A}_a$ is a latarre morphism.*

3. *$\mathfrak{A}$ admits meet substitution.*

6 Fixed Points and Löb

Over Visser latarres all terms $t(x)$ have *explicit fixed point* $t(1)$, that is, $t(t(1)) = t(1)$, exactly when all terms of the form $t_a(x) = x \multimap a$ have explicit fixed point $t_a(1)$, that is, $t_a(t_a(1)) = t_a(1)$.

An equation $t(x) = u(x)$ has *ultimate solutions* over latarre $\mathfrak{A}$ if for all a there are $b \trianglerighteq a$ such that $t(b) = u(b)$. We call a term $t(x)$ *ultimately fixed* or *U-fixed* over $\mathfrak{A}$ if equation $t(t(x)) = t(x)$ has ultimate solutions over $\mathfrak{A}$. We call an element a a *U-Löb* element over $\mathfrak{A}$ if $t_a(t_a(x)) = t_a(x)$ has ultimate solutions over $\mathfrak{A}$, where $t_a(x) = x \multimap a$.

A latarre is *U-fixed* if all its terms are U-fixed. A latarre is *U-Löb* if all its element are U-Löb. U-fixed implies U-Löb. Visser latarres add the converse direction that U-Löb implies U-fixed.

A term $t(x)$ is called *fixed* over a latarre $\mathfrak{A}$ if $\mathfrak{A}$ satisfies schema $t(t(x) \sqcap x) = t(x)$. An element a is called *Löb* over a latarre $\mathfrak{A}$ if $\mathfrak{A}$ satisfies schema $t_a(t_a(x) \sqcap x) = t_a(x)$, where $t_a(x) = x \multimap a$. A latarre is *fixed* if all its terms are fixed. A latarre is *Löb* if all its elements are Löb. Fixed implies Löb.

Obviously schema $x \trianglelefteq t(x)$ over $\mathfrak{A}$ implies that $t(x)$ is fixed over $\mathfrak{A}$.

Theorem 6.1. *The following are equivalent for a latarre $\mathfrak{A}$.*

1. *$\mathfrak{A}$ is U-fixed.*

2. *$\mathfrak{A}$ is a weakly Visser and U-Löb.*

3. *$\mathfrak{A}$ is a weakly Visser and Löb.*

4. *$\mathfrak{A}$ is fixed.*

An element that can be written in the form $a \rightharpoonup b$ is called an arrow element, or an arrow element of the 1st kind. Given an arrow element t of the n^{th} kind, we call an element $a \rightharpoonup t$ an arrow element of the $(n+1)^{\text{th}}$ kind.

Theorem 6.2. *The following are equivalent for a latarre $\mathfrak{A}$.*

1. *$\mathfrak{A}$ is weakly Visser, and there is $n \geq 1$ such that all arrow elements of the n^{th} kind are Löb.*

2. *$\mathfrak{A}$ is fixed.*

7 Almost-Complete Latarres

Another source of interest involves almost-complete lattices with an operator. In early cases this mostly involved frames $\mathfrak{M}$ with a map $j : \mathfrak{M} \to \mathfrak{M}$ satisfying the schemas

$$x \trianglelefteq jx \text{ (increasing)} \quad \text{and}$$
$$j(x \sqcap y) = jx \sqcap jy \text{ (multiplicative).}$$

An operator satisfying these conditions we dub a *nub*. With this terminology, j is a *nucleus* if j is an idempotent nub, that is, if j satisfies the extra schema

$$jjx = jx \text{ (idempotent).}$$

These definitions apply to all latarres. The following is a special case for current purposes.

Proposition 7.1. *Let j be a nub on a Visser latarre $\mathfrak{M}$. Define $x \rightarrow_j y = j(x \rightarrow y)$. Then $\mathfrak{M}$ with new arrow $\rightarrow_j$ forms a Visser latarre.*

Theorem 7.2. *Let j be a nub on an almost-complete frame $\mathfrak{M}$. Then there is a map w on $\mathfrak{M}$ satisfying*

- *w is a nucleus.*
- *w is least fixed point operator for j, that is, we have schemas $jwx = wx$, and $jy = y \trianglerighteq x$ implies $y \trianglerighteq wx$.*

Let $e : \mathfrak{N} \to \mathfrak{M}$ be the equalizer of j and $id : \mathfrak{M} \to \mathfrak{M}$. Then

- *$\mathfrak{N}$ is an almost-complete frame (and the image of w).*
- *$w = e\pi$ for a unique $\pi : \mathfrak{M} \to \mathfrak{N}$.*

For each $n \in \mathfrak{N}$ the inverse image structure $\pi^{-1}(\{n\}) = \mathfrak{M}_n$ is Visser and Löb, that is, is a fixed (point) latarre.

The lattice of $\pi^{-1}(\{n\}) = \mathfrak{M}_n$ is an almost-complete frame, but usually not a frame.

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