Intuitionistic Logic versus Constructive Logic

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This is joint work with Mohammad Ardeshir.

We develop a new predicate logic for constructive mathematics simultaneously with a matching new proof interpretation.

1 Constructive Mathematics

We give a brief 'historical' overview of the major schools of constructive mathematics. Many quotes from the literature imply that the insights that result in our reinterpretation, have a significant precursor in earlier insights.

Among constructivists it is broadly understood that constructive mathematics has precedence over constructive logic.

1.1 Brouwer

The first fully constructive mathematics and philosophy started with L.E.J. Brouwer's 1907 PhD thesis.

Brouwer condemned a logical foundation of mathematics independent of a priori human mental concepts. For otherwise one builds a linguistic structure definitely distinct from mathematics proper.

Brouwer's PhD student A. Heyting wrote his 1925 PhD thesis on an intuitionistic treatment of axiomatic projective geometry.

Brouwer's intuitionism is not consistent with classical mathematics.

1.2 Markov

In 2006 Boris A. Kushner wrote

> After World War II Markov's interests turned to axiomatic set theory, mathematical logic, and the foundations of mathematics. He founded the Russian school of constructive mathematics in the late 1940s and early 1950s.

A.A. Markov's main work is strongly influenced by the theory of and philosophy about the general recursive functions of the 1930s.

Some Markov constructivists contemplate broader possibilities. In 2016 Vladik Kreinovich offers some Main Challenges, including:

- The need to extend constructive mathematics to more complex mathematical objects.
- To be useful for data processing, algorithms must be able to handle possibly non-constructive data.

Markov's and Brouwer's do not include one another.

Markov's constructivism is not consistent with classical mathematics.

1.3 Bishop

In 1967 Errett Bishop writes:

> Our program is simple: to give numerical meaning to as much as possible of classical abstract analysis. Our motivation is the well-known scandal, exposed by Brouwer (and others) in great detail, that classical mathematics is deficient in numerical meaning.

Bishop's constructivism is consistent with classical mathematics, and appears contained in Brouwer's and Markov's.

1.4 Topos Theory

A constructive 'school' need not be founded by a single person who expounds a constructive point of view.

In the early 1960s appeared early theorems about categories which, as P.T. Johnstone writes in 1977,

> ...paved the way for a truly autonomous development of category theory as a foundation for mathematics.

The development of concern to us began with F.W. Lawvere's 1964 paper *An elementary theory of the category of sets*. Once Lawvere turned his attention to Grothendieck toposes as generalized set theories, a new elementary theory of the category of sets evolved called topos theory. What makes topos theory also a constructive mathematics is that its so-called internal logic is intuitionistic logic.

Internal topos mathematics is consistent with classical mathematics, but does not contain Bishop's constructivism even when a natural number object is added. Internal topos mathematics allows for liberal set constructions which are not part of Brouwer's intuitionism.

2 Mathematical Logic as Applied Mathematics

It seems that 'early' constructivists only have interest in constructive logic as a secondary matter.

Heyting used axiomatic theories of geometry and algebra, where the use of hypotheticals (if we have a structure satisfying the following axioms; then the following holds) is immediate.

Heyting wrote in 1978 that

> Logic can be considered in different ways. As a study of regularities in language it is an experimental science which, like any such science, needs mathematical notions; therefore it belongs to applied mathematics.

His realization that mathematical logic can be seen as applied mathematics dates back to the 1930s.

First, did a classical mathematician in 1927 know whether with predicate logic one had picked a 'good' formal language? One may reply yes since with the right collection of atomic formulas there is strong expressive power. Second, did a classical mathematician in 1927 have a complete set of axioms and rules? This second question was answered in the affirmative by Gödel's Completeness Theorem of 1930.

In 1927 the Dutch 'Mathematical Society' posted a problem question about a formalization of Brouwer's intuitionistic mathematics. To this Heyting wrote an essay for which he was awarded the prize the following year. An expanded version of the essay was published in 1930.

Heyting chose a formal language with a collection of logical operators equivalent to $\top$, $\bot$, $A \wedge B$, $A \vee B$, $A \to B$, $\neg A$, $\forall x A$, and $\exists x A$. First, this is a 'good' language because of its expressive power. We should consider the possible existence of logical constants beyond what Heyting included, which allow for further relevant distinctions.

Second, did Heyting have a complete set of axioms and rules? Brouwer offered so-called weak counterexamples. One may read in the existence of ‘classical’ Kripke model counterexamples to the intuitionistic provability of statements that such statements are plausibly not constructive tautologies. Insufficient, yes. Evidence of non-provability, also yes. So if one is willing to accept such models as evidence, then Heyting’s set of axioms and rules is complete.

Intuitionistic predicate logic has been accepted by all major schools of constructivism that we listed. However, there is a Third Question that needs an answer: Are the axioms and rules of Heyting’s intuitionistic predicate logic themselves constructively acceptable?

3 Proof Interpretations for Intuitionistic Logic

Let us expand the earlier 1978 quote of Heyting:

> Logic can be considered in different ways. As a study of regularities in language it is an experimental science which, like any such science, needs mathematical notions; therefore it belongs to applied mathematics. If we consider logic not from the linguistic point of view but turn our attention to the intended meaning, then logic expresses very general mathematical theorems about sets and their subsets.

How did Heyting address the intended meaning of the logical constants? This should settle the Third Question, and justify the axioms and rules of intuitionistic logic. Heyting in a 1933 letter writes (my translation)

> I went through the axioms and theorems of Principia mathematica, and made a system of independent axioms from the ones found acceptable. Because of the relative completeness of the one in Principia is, in my opinion, the completeness of my system assured in the best possible way.

Heyting's broadly recognized proof interpretation of the logical constants appear in 1934. The following description is from Troelstra and van Dalen in 1988.

H1. A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.

H2. A proof of $A \vee B$ is given by presenting either a proof of A or a proof of B (plus the stipulation that we want to regard the proof presented as evidence for $A \vee B$).

H3. A proof of $A \to B$ is a construction which permits us to transform any proof of A into a proof of B.

H4. Absurdity $\perp$ (contradiction) has no proof; a proof of $\neg A$ is a construction which transforms any hypothetical proof of A into a proof of a contradiction.

H5. A proof of $\forall x A(x)$ is a construction which transforms a proof of $d \in D$ (D the intended range of x) into a proof of $A(d)$.

H6. A proof of $\exists x A(x)$ is given by providing $d \in D$, and a proof of $A(d)$.

This interpretation is now known as the Brouwer-Heyting-Kolmogorov BHK interpretation.

The proof interpretation as stated is informal, and uses primitive terms like 'proof', 'construction', and 'hypothetical'. The proof interpretation has been challenged, mostly in the form of quests to refine or clarify Heyting's version, and always in support of Heyting's intuitionistic predicate logic.

Troelstra in 1977 presents a somewhat different proof interpretation. The most significant differences with the proof interpretation above are the required 'insights' in

H3'. A proof of $A \to B$ consists of a construction c which transforms any proof of A into a proof of B (*together* with the insight that c has the property: d proves A $\Rightarrow$ cd proves B).

H5'. ... we can explain a proof of $\forall x A x$ as a construction c which on application to any $d \in D$ yields a proof $c(d)$ of Ad, together with the insight that c has this property. ...

The origin of these 'insights' can be traced to G. Kreisel (1962–5), which Troelstra in 1981 describes as "Kreisel's attempts at a general theory of constructions and proofs". This variation on Heyting's proof interpretation is called the Brouwer-Heyting-Kreisel explanation.

The clarification of implication was a key problem for constructivists. Kushner writes in 2006 about (Bishop and) Markov:

> ... [Bishop] could not avoid the key problem of any system of constructive mathematics, namely, the problem of clarifying implication. Markov spent the last years of his life struggling to develop a large "stepwise" semantic system in order to achieve, above all, a satisfactory theory of implication.

Bishop writes in 1967 about the proof interpretation of implication (emphasis added):

> Statements formed with this connective, for example, statements of the type ((P implies Q) implies R), have a less immediate meaning than the statements from which they are formed, although in actual practice this does not *seem* to lead to difficulties in interpretation.

For the foundations of constructive mathematics along the lines of topos theory, the proof interpretation appears of marginal importance.

4 Axiomatics and a New Constructive Logic

For constructivists the proof interpretation is a major factor in justifying the rules of intuitionistic predicate logic. From Section 3 we see that the interpretation is highly impredicative. In 2000 Michael Dummett writes about the proof interpretation:

> The principal reason for suspecting these explanations of incoherence is their apparently highly impredicative character; if we know which constructions are proofs of the atomic statements of any first-order theory, then the explanations of the logical constants, taken together, determine which constructions are proofs of any of the statements of that theory; yet the explanations require us, in determining whether or not a construction is a proof of a conditional or of a negation, to consider its effect when applied to an arbitrary proof of the antecedent or of the negated statement, so that we must, in some sense, be able to survey or grasp some totality of constructions which will include all possible proofs of a given statement.

We develop a new proof interpretation and predicate logic simultaneously. We imagine an idealized constructivist who never makes mistakes, has perfect memory, and has unlimited 'time'. Our axioms are such that if the proof objects are considered from a constructive point of view, then the axioms about these proof objects are valid. We follow Heyting, who wrote in 1931 (my translation):

> A proof for a proposition is a mathematical construction, which itself again can be considered mathematically.

The objects of our axiomatic approach are formal constructions of statements and proofs, where the proofs are interpretable as constructive proofs.

Our proof objects p include an assumption as well as a conclusion. We also write (A, p, B) for a proof p with assumption A and conclusion B. We write $A \vdash B$, with intended meaning B is derivable from A, if a proof (A, p, B) exists for some p. Assumptions replace 'hypotheticals' of earlier proof interpretations.

We accept that if a constructivist assumes A, then A is accepted trivially. This is a clarification of the intended meaning of 'assumption'. For each formula A we have a trivial proof (A, p, A). So we also have logical axiom schema

$$A \vdash A$$

We have the following straightforward composition clause for proofs. If (A, p, B) and (B, q, C) are proofs, then so is (A, qp, C), where qp stands for the composition proof, and which we can construct in a uniform way in terms of p and q. So we have logical rule

$$\frac{A \vdash B \quad B \vdash C}{A \vdash C}$$

4.1 Propositional Logic

Negation $\neg A$ is defined by $A \to \bot$ and bi-implication $A \leftrightarrow B$ is defined by $(A \to B) \wedge (B \to A)$.

For each pair of formulas A and B we have a conjunction formula $A \wedge B$ with $\wedge$ the usual intended meaning of 'and'. There are trivial proofs $(A \wedge B, p_1, A)$ and $(A \wedge B, p_2, B)$. These come with the intended meaning of assuming a conjunction $A \wedge B$. Consequently, with composition, a proof $(C, q, A \wedge B)$ implies that we have proofs $(C, p_1 q, A)$ and $(C, p_2 q, B)$. In the other direction, if we have proofs (C, p, A) and (C, q, B), then there is a proof which we name $(C, (p, q), A \wedge B)$, and which we can construct in a uniform way in terms of p and q. So we have rules

$$\frac{C \vdash A \wedge B}{C \vdash A \quad C \vdash B} \quad \text{and} \quad \frac{C \vdash A \quad C \vdash B}{C \vdash A \wedge B}$$

We include a symbol $\top$ with intended meaning 'true', with for every A a trivial essentially vacuous proof $(A, p.\top)$, and axiom

$$A \vdash \top$$

For each pair of formulas A and B we have a disjunction formula $A \vee B$ with $\vee$ the usual intended meaning of 'or'. There are trivial proofs $(A, s_1, A \vee B)$ and $(B, s_2, A \vee B)$. Consequently, with composition, a proof $(A \vee B, p, C)$ implies that we have proofs (A, ps_1, C) and (B, ps_2, C). In the other direction, if we have proofs (A, p, C) and (B, q, C), then there is a proof which we name $(A \vee B, [p, q], C)$, and which we can construct in a uniform way in terms of p and q. This clarifies what it means to assume a disjunction $A \vee B$. So we have rules

$$\frac{A \vee B \vdash C}{A \vdash C \quad B \vdash C} \quad \text{and} \quad \frac{A \vdash C \quad B \vdash C}{A \vee B \vdash C}$$

We include a symbol $\perp$ with intended meaning 'false', with for every A a trivial proof $(\perp, p.A)$, and axiom

$$\perp \vdash A$$

With the axioms and rules so far we can show $(A \wedge B) \vee (A \wedge C) \vdash A \wedge (B \vee C)$. We axiomatize that we have proofs $(A \wedge (B \vee C), q, (A \wedge B) \vee (A \wedge C))$. This is a clarification of the meaning of 'assumption', in this case where the assumption has form $A \wedge (B \vee C)$. When a constructivist assumes $A \wedge (B \vee C)$, then this constructivist essentially also assumes $A \wedge ((A \wedge B) \vee (A \wedge C))$. So we have axiom

$$A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$$

For each pair of formulas A and B we have an implication formula $A \to B$ with $\to$ the intended meaning of 'implies'. Formula $A \to B$ has to reflect the meaning of $A \vdash B$ within the bounds of what is constructively acceptable. We write $\neg A$ as abbreviation for $A \to \bot$. If we have a proof $(A \wedge B, p, C)$, then we have a proof $(A, p_A, B \to C)$, where p_A takes proof p, replaces its assumptions $A \wedge B$ by assumption B and derives $A \wedge B$ using the assumption A of p_A. Finally append that the result is a proof for conclusion $B \to C$. So we have rule

$$\frac{A \wedge B \vdash C}{A \vdash B \to C}$$

Assume $A \to B$ and $B \to C$. So we assume proofs (A, x, B) and (B, y, C) without specifying x and y any further. As Bishop wrote in 1967:

> Mathematics takes another leap, from the entity which is constructed in fact to the entity whose construction is hypothetical. To some extent hypothetical entities are present from the start: whenever we assert that every positive integer has a certain property, in essence we are considering a positive integer whose construction is hypothetical.

In this same sense x and y are hypothetical. From the assumed x and y we construct the assumed proof (A, yx, C). So we have rule

$$(A \to B) \wedge (B \to C) \vdash (A \to C)$$

Assume $A \to B$ and $A \to C$. So we assume proofs (A, x, B) and (A, y, C). So we have hypothetical proof $(A, (x, y), B \wedge C)$, and we have rule

$$(A \to B) \wedge (A \to C) \vdash (A \to (B \wedge C))$$

Assume $B \to A$ and $C \to A$. So we assume proofs (B, x, A) and (C, y, A). So we have hypothetical proof $(B \vee C, [x, y], A)$, and we have rule

$$(B \to A) \wedge (C \to A) \vdash ((B \vee C) \to A)$$

This completes our axiomatization with restriction to the language of propositional logic.

Our system axiomatizes the so-called Basic Propositional Logic of Albert Visser of 1981, which is a proper subsystem of Intuitionistic Propositional Logic.

Heyting wrote in 1978,

> ... logic expresses very general mathematical theorems about sets and their subsets.

The principle $(\top \to A) \vdash A$ is not one of these very general mathematical theorems. The reason is that a proof $((\top \to A), p, A)$ is expected to turn a 'hypothetical' assumed proof $(\top, x, A)$ into an actual proof p with conclusion A. Such a very general constructive theorem is an impossibility. Our objection is in line with what Dummett wrote in 2000:

> As mathematics advances, we become able to conceive of new operations and to recognize them and others as effectively transforming proofs of B into proofs of C; and so the meaning of $B \to C$ would change, if a grasp of it required us to circumscribe such operations in thought. Moreover, an operation which would transform any proof of $B \to C$ available to us now into a proof of D might not so transform proofs of $B \to C$ which became available to us with the advance of mathematics: and so what would now count as a valid proof of $(B \to C) \to D$ would no longer count as one.

We do not agree with Dummett's follow-up "These fears are groundless".

Do we have a complete set of axioms and rules for constructive propositional logic? We have a completeness theorem for Basic Propositional Logic with transitive Kripke models, that is, Kripke models where the world relation is transitive but not necessarily reflexive as in the case for Intuitionistic Propositional Logic. Using transitive Kripke models as weak counterexamples to constructive provability of propositional statements has the same limited value as using reflexive transitive Kripke models has as a tool to make weak counterexamples in the intuitionistic case. Insufficient yes, but evidence of non-provability, also yes. If one is willing to accept transitive Kripke models as evidence, then Basic Propositional Logic is complete.

The end.
More on predicate logic in a later talk.