The Unsettled Story of the Proof Interpretation

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1 Constructive Mathematics

There are, from a historical perspective, three major schools of constructive mathematics: Brouwer, Markov, and Bishop. We add one other source of constructive mathematics and logic for which the term 'school' may not apply. Category theory in general, and topos theory in particular, have a peculiar fruitful relation with constructive mathematics and intuitionistic logic.

Heyting's intuitionistic predicate logic represents very general regularities observed in language used in constructive mathematical proofs. The intended meanings of the logical constants are clarified through Heyting's proof interpretation. A re-evaluation of proof interpretation and predicate logic leads to the new constructive Basic logic properly contained in intuitionistic logic. The proposition logical part of Basic logic is due to Albert Visser.

2 Mathematical Logic as Applied Mathematics

Heyting studied axiomatic theories of geometry and algebra, where the use of very general hypotheticals is natural. Heyting (1978) writes that

> Logic can be considered in different ways. As a study of regularities in language it is an experimental science which, like any such science, needs mathematical notions; therefore it belongs to applied mathematics.

Did a classical mathematician in 1927 have with predicate logic a 'good' formal language? Yes. Did a classical mathematician in 1927 have a complete set of axioms and rules? Yes by Gödel's Completeness Theorem of 1930. These two questions about classical logic can also be asked in the intuitionistic case.

In 1927 the Dutch 'Mathematical Society' posted a prize question about a formalization of Brouwer's intuitionistic mathematics, including the problem of formalizing an intuitionistic predicate logic. To this Heyting wrote the winning essay. An earlier partial version is due to A.N. Kolmogorov (1925).

Heyting chose a formal language with a collection of logical operators equivalent to $\top$, $\bot$, $A \wedge B$, $A \vee B$, $A \to B$, $\neg A$, $\forall x A$, and $\exists x A$. The axioms and rules are a proper subset of the axioms and rules known for classical logic. Is Heyting's language a 'good' language? No new unquestioned logical constants have shown up. So yes.

Did Heyting have a complete set of axioms and rules? Heyting (1956, 1971) states:

> It must be remembered that no formal system can be proved to represent adequately an intuitionistic theory. There always remains a residue of ambiguity in the interpretation of the signs, and it can never be proved with mathematical rigour that the system of axioms really embraces every valid method of proof.

and

> ...one is never sure that the formal system represents fully any domain of mathematical thought; at any moment the discovering of new methods of reasoning may force us to extend the formal system.

What evidence justifies us to conclude that a formal sentence should not be derivable from the formal system? Brouwer offered so-called weak counterexamples against principles like Excluded Middle $A \vee \neg A$. The evidence for non-provability is different in nature from provability.

An illustration of Kripke models as weak Brouwerian counterexamples. A version based on the halting problem. Consider a two-node reflexive transitive Kripke model with nodes $\alpha \prec \beta$. Imagine a sequence $t_1, t_2, t_3, \ldots$ with $t_i \in \{\alpha, \beta\}$ for all i, and $t_i \preceq t_j$ for all $i \leq j$, and with $t_1 = \alpha$. The constructivist need not know whether there is some i for which $t_i = \beta$. As long as $t_i = \alpha$, the constructvist knows structure $\mathfrak{A}_\alpha$, with the potential of discovering at a later integer $j > i$ that $t_j = \beta$ and the constructivist knows structure $\mathfrak{A}_\beta$. The logical statements which that constructivist accepts are the ones forced at node α.

One may read in the existence of 'classical' Kripke model counterexamples to the intuitionistic provability of statements that such statements are not constructive tautologies. Insufficient, yes. Evidence of non-provability, also yes. If one accepts such models as evidence, then Heyting's system is complete.

Intuitionistic predicate logic has been accepted by all major schools of constructivism. A **Third Question**: Are the axioms and rules of intuitionistic predicate logic constructive?

3 Proof Interpretations for Intuitionistic Logic

Heyting (1933) writes (my translation)

> I went through the axioms and theorems of principia mathematica, and made a system of independent axioms from the ones found acceptable. Because of the relative completeness of the one in principia is, in my opinion, the completeness of my system assured in the best possible way.

Heyting (1978) writes:

> Logic can be considered in different ways. As a study of regularities in language it is an experimental science which, like any such science, needs mathematical notions; therefore it belongs to applied mathematics. If we consider logic not from the linguistic point of view but turn our attention to the intended meaning, then logic expresses very general mathematical theorems about sets and their subsets.

Heyting's 1934 proof interpretation of the logical constants, the Brouwer-Heyting-Kolmogorov BHK interpretation, this version is by Troelstra and van Dalen (1988).

H1. A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.

H2. A proof of $A \vee B$ is given by presenting either a proof of A or a proof of B (plus the stipulation that we want to regard the proof presented as evidence for $A \vee B$).

H3. A proof of $A \to B$ is a construction which permits us to transform any proof of A into a proof of B.

H4. Absurdity $\perp$ (contradiction) has no proof; a proof of $\neg A$ is a construction which transforms any hypothetical proof of A into a proof of a contradiction.

H5. A proof of $\forall x A(x)$ is a construction which transforms a proof of $d \in D$ (D the intended range of x) into a proof of $A(d)$.

H6. A proof of $\exists x A(x)$ is given by providing $d \in D$, and a proof of $A(d)$.

The proof interpretation has been challenged from multiple sides, mostly to refine or clarify, not as a challenge to intuitionistic predicate logic.

The Brouwer-Heyting-Kreisel explanation, a somewhat different proof interpretation. The most significant differences are the required 'insights' (Troelstra 1977)

H3'. A proof of $A \to B$ consists of a construction c which transforms any proof of A into a proof of B (*together* with the insight that c has the property: d proves A $\Rightarrow$ cd proves B).

H5'. ... we can explain a proof of $\forall x A x$ as a construction c which on application to any $d \in D$ yields a proof $c(d)$ of Ad, together with the insight that c has this property. ...

Kushner (2006) writes about (Bishop and) Markov:

> ... [Bishop] could not avoid the key problem of any system of constructive mathematics, namely, the problem of clarifying implication. Markov spent the last years of his life struggling to develop a large "stepwise" semantic system in order to achieve, above all, a satisfactory theory of implication.

Bishop (1967) writes about the interpretation of implication (emphasis added):

> Statements formed with this connective, for example, statements of the type ((P implies Q) implies R), have a less immediate meaning than the statements from which they are formed, although in actual practice this does not *seem* to lead to difficulties in interpretation.

Michael Dummett (2000) about the interpretation:

> The principal reason for suspecting these explanations of incoherence is their apparently highly impredicative character; if we know which constructions are proofs of the atomic statements of any first-order theory, then the explanations of the logical constants, taken together, determine which constructions are proofs of any of the statements of that theory; yet the explanations require us, in determining whether or not a construction is a proof of a conditional or of a negation, to consider its effect when applied to an arbitrary proof of the antecedent or of the negated statement, so that we must, in some sense, be able to survey or grasp some totality of constructions which will include all possible proofs of a given statement.

4 Axiomatics and a New Constructive Logic

Our method to find constructive predicate logic is based on an axiomatic approach. The accumulation of proofs is closed under certain canonical rules, which are sufficient to uniquely determine constructive predicate logic. The axioms and rules need not be a complete set for a theory of constructions and proofs. We do not require the existence of a collection of all proofs. Heyting (1931) writes (my translation):

> A proof for a proposition is a mathematical construction, which itself again can be considered mathematically.

We write (A, p, B) for a proof p with assumption A and conclusion B. We write $A \vdash B$, with intended meaning B is derivable from A, if a proof (A, p, B) exists. This approach is in line with a suggestion of Gödel (1938), and distinct from Kreisel (1962) and (p, B). The word 'assumption' replaces 'hypothetical' of earlier interpretations. In our axiomatic approach we claim properties that assumptions imply without a need to further specify the meaning of 'assumption'.

For each formula A we have a trivial proof (A, p, A). So we also have logical axiom schema

$$A \vdash A$$

If (A, p, B) and (B, q, C) are proofs, then so is $(A, q \cdot p, C)$, also written as (A, qp, C). So we have logical rule

$$\frac{A \vdash B \quad B \vdash C}{A \vdash C}$$

4.1 Propositional Logic

Negation $\neg A$ is defined by $A \to \bot$, and bi-implication $A \leftrightarrow B$ is defined by $(A \to B) \wedge (B \to A)$.

There are trivial proofs $(A \wedge B, p_1, A)$ and $(A \wedge B, p_2, B)$. So, with composition, a proof $(C, q, A \wedge B)$ implies that we have proofs $(C, p_1 q, A)$ and $(C, p_2 q, B)$. If we have proofs (C, p, A) and (C, q, B), then there is a proof which we name $(C, \langle p, q \rangle, A \wedge B)$. So we have rules

$$\frac{C \vdash A \wedge B}{C \vdash A \quad C \vdash B} \quad \text{and} \quad \frac{C \vdash A \quad C \vdash B}{C \vdash A \wedge B}$$

For every A a trivial proof $(A, p.\top)$, and axiom

$$A \vdash \top$$

There are trivial proofs $(A, s_1, A \vee B)$ and $(B, s_2, A \vee B)$. So, with composition, a proof $(A \vee B, p, C)$ implies that we have proofs (A, ps_1, C) and (B, ps_2, C). If we have proofs (A, p, C) and (B, q, C), then there is a proof which we name $(A \vee B, [p, q], C)$. So we have rules

$$\frac{A \vee B \vdash C}{A \vdash C \quad B \vdash C} \quad \text{and} \quad \frac{A \vdash C \quad B \vdash C}{A \vee B \vdash C}$$

For every A a trivial proof $(\bot, p.A)$, and axiom

$$\bot \vdash A$$

We axiomatize that we have proofs $(A\wedge(B\vee C), d, (A\wedge B)\vee(A\wedge C))$. When a constructivist assumes $A\wedge(B\vee C)$, then this constructivist also assumes $A \wedge ((A \wedge B) \vee (A \wedge C))$. So we have distributivity axiom

$$A \wedge (B \vee C) \;\vdash\; (A \wedge B) \vee (A \wedge C)$$

Add a further clarification.

If we have a proof $(A \wedge B, p, C)$, then we have a proof $(A, p_A, B \rightarrow C)$, where p_A takes proof p, replaces its assumptions $A \wedge B$ by assumption B and derives $A \wedge B$ using the assumption A of p_A. Finally append that the result is a proof for conclusion $B \rightarrow C$. So we have rule

$$\frac{A \wedge B \;\vdash\; C}{A \;\vdash\; B \rightarrow C}$$

Assume $A \rightarrow B$ and $B \rightarrow C$. So we assume proofs (A, x, B) and (B, y, C) without specifying x and y any further. Bishop (1967) writes:

> Mathematics takes another leap, from the entity which is constructed in fact to the entity whose construction is hypothetical. To some extent hypothetical entities are present from the start: whenever we assert that every positive integer has a certain property, in essence we are considering a positive integer whose construction is hypothetical.

In this same sense x and y are hypothetical. From the assumed x and y we construct the proof (A, yx, C). So we have rule

$$(A \to B) \wedge (B \to C) \ \vdash \ A \to C$$

Assume $A \to B$ and $A \to C$. So we assume proofs (A, x, B) and (A, y, C). So we have proof $(A, \langle x, y\rangle, B \wedge C)$, and we have rule

$$(A \to B) \wedge (A \to C) \ \vdash \ A \to (B \wedge C)$$

Assume $B \to A$ and $C \to A$. So we assume proofs (B, x, A) and (C, y, A). So we have proof $(B \vee C, [x, y], A)$, and we have rule

$$(B \to A) \wedge (C \to A) \ \vdash \ (B \vee C) \to A$$

This complete system axiomatizes the Basic Propositional Logic of Albert Visser.

To get Intuitionistic Propositional Logic it suffices to add schema $\top \to A \ \vdash \ A$. Heyting (1978) writes that

> . . . logic expresses very general mathematical theorems about sets and their subsets.

The principle $\top \to A \ \vdash \ A$ is not one of these very general mathematical theorems. A proof $((\top \to A), p, A)$ is asked to turn a 'hypothetical' assumed proof $(\top, x, A)$ into an actual proof p with conclusion A. Dummett (2000) writes:

> As mathematics advances, we become able to conceive of new operations and to recognize them and others as effectively transforming proofs of B into proofs of C; and so the meaning of $B \to C$ would change, if a grasp of it

> required us to circumscribe such operations in thought. Moreover, an operation which would transform any proof of $B \to C$ available to us now into a proof of D might not so transform proofs of $B \to C$ which became available to us with the advance of mathematics: and so what would now count as a valid proof of $(B \to C) \to D$ would no longer count as one.

A proof x may only become available after mathematics has advanced with newly conceived operations. Limited versions of modus ponens still hold in Basic Logic, like

$$\frac{C \vdash A \qquad \vdash A \to B}{C \vdash B}$$

A completeness theorem for Basic Propositional Logic with transitive Kripke models, so not necessarily reflexive as for Intuitionistic Propositional Logic. Insufficient yes. Evidence of non-provability, also yes.

Similar to Dummett, Mark van Atten (2018) writes:

> Intuitionists consider the notion of proof to be open-ended, not only epistemically (at no moment do we know all possible proofs) but ontologically, and hence they deny that there is such a thing as the totality of all intuitionistic proofs [. . . Brouwer's PhD . . .]. There is only a *growing* universe of mathematical objects and proofs.

Over Basic logic we have the equivalence of $\neg\neg A$ and $\neg\neg\neg\neg A$. Brouwer writes in 1924 that even $\neg A$ and $\neg\neg\neg A$ are equivalent. His key step is that A implies $\neg\neg A$. In particular a constructive proof $((\top \to \bot), p, \bot)$ which turns a hypothetical proof of inconsistency into an actual proof of a contradiction. This argument is circular.

4.2 Predicate Logic

In this Section we write π, σ or τ for proof objects, and ξ, η or ζ for proof variables.

Variables x range over descriptions of elements that are intended to belong to a domain of discourse. The word 'description' replaces 'construction'. Bishop (1967) writes 'description' for sets defined in a possibly incomplete way. We write $\mathrm{E}\,x$ or $\mathrm{E}(x)$ for the propositional statement that the element described by x belongs to the domain of discourse.

Write $(A, \pi\mathbf{x}, B)$ for a proof $\pi\mathbf{x}$ with assumption A and conclusion B, where list $\mathbf{x}$ includes all free variables. We write $A \vdash_{\mathbf{x}} B$ if a proof $(A, \pi\mathbf{x}, B)$ exists.

We have a constant symbol ℓ for the empty description, with proof $(\mathrm{E}\,\ell, \pi, \bot)$, and axiom

$$\mathrm{E}\,\ell \vdash \bot$$

We have rules

$$\frac{A \vdash_{\mathbf{x}y} B}{A \vdash_{\mathbf{x}} B} \quad y \notin \mathrm{FV}(A, B), \quad \text{and} \quad \frac{A \vdash_{\mathbf{x}} B}{A \vdash_{\mathbf{x}y} B}$$

We have substitution rule

$$\frac{A\mathbf{x} \ \vdash_{\mathbf{x}} \ B\mathbf{x}}{A\mathbf{y} \ \vdash_{\mathbf{y}} \ B\mathbf{y}} \quad \text{no variables of } \mathbf{y} \text{ become bound}$$

and substitution rule

$$\frac{A\mathbf{x}y \ \vdash_{\mathbf{x}y} \ B\mathbf{x}y}{A\mathbf{x}c \ \vdash_{\mathbf{x}} \ B\mathbf{x}c}$$

Many rules of predicate logic are the same as in the proposition logical case, for example

$$A \vdash_{\mathbf{x}} A$$

and

$$\frac{A \vdash_{\mathbf{x}} B \quad B \vdash_{\mathbf{y}} C}{A \vdash_{\mathbf{xy}} C}$$

The same for $\wedge$ and $\vee$. Farther below implication $\to$ is combined with universal quantification.

We have abstract axiom schemas $(P\mathbf{x}, \pi\mathbf{x}, \mathrm{E}\,\mathbf{x})$ for

$$P\mathbf{x} \ \vdash_{\mathbf{x}} \ \mathrm{E}\,\mathbf{x}$$

There are trivial proofs $(A \wedge \mathrm{E}\,x, \sigma x\mathbf{y}, \exists x A)$. Suppose we have a proof $(\exists x A, \pi\mathbf{z}, B)$. By composition we have a proof $(A \wedge \mathrm{E}\,x, (\pi\sigma)x\mathbf{yz}, B)$. In the other direction, suppose we have a proof $(A \wedge \mathrm{E}\,x, \pi x\mathbf{y}, B)$, where x is not free in B. Then there is a proof which we name $(\exists x A, [\pi]\mathbf{y}, B)$. So we have rules

$$\frac{A \wedge \mathrm{E}\,x \ \vdash_{x\mathbf{y}} \ B}{\exists x A \ \vdash_{\mathbf{y}} \ B} \quad x \notin \mathrm{FV}(B), \qquad \frac{\exists x A \ \vdash_{\mathbf{z}} \ B}{A \wedge \mathrm{E}\,x \ \vdash_{x\mathbf{z}} \ B}$$

We axiomatize that we have proofs $(A \wedge \exists x B, \delta \mathbf{y}, \exists x(A \wedge B))$, where x is not free in A. So we have existential distributivity axiom

$$A \wedge \exists x B \ \vdash_{\mathbf{y}} \ \exists x(A \wedge B) \qquad x \text{ not free in } A$$

Add a further clarification.

We combine implication with universal quantification. Motivation: Suppose we have a proof $(A \wedge \mathrm{E}\, x, \pi x \mathbf{y}, B)$ with x not free in A. Then we have $(A, \sigma \mathbf{y}, \forall x(\mathrm{E}\, x \to B))$. Formula $\forall x(\mathrm{E}\, x \to B)$ is equivalent to $\forall x(\top \to B)$. Just as it is constructively acceptable to conclude $A \vdash_{\mathbf{y}} \forall x(\top \to C)$ from $A \wedge \mathrm{E}\, x \vdash_{x\mathbf{y}} C$ (x not free in A), it is constructively acceptable to conclude $A \vdash_{\mathbf{y}} \forall x(B \to C)$ from $A \wedge B \wedge \mathrm{E}\, x \vdash_{x\mathbf{y}} C$ (x not free in A). Nested quantifications like $\forall x \forall y(A \to B)$ are no longer available. So we introduce for each pair of formulas A and B universal implication formulas $\forall \mathbf{x}(A \to B)$. List $\mathbf{x}$ is allowed to have length 0. We write $A \to B$ as short for $\forall(A \to B)$.

We have rules

$$\frac{C \ \vdash_{y\mathbf{z}} \ \forall \mathbf{x} y(A \to B)}{C \ \vdash_{y\mathbf{z}} \ \forall \mathbf{x}((A \wedge \mathrm{E}\, y) \to B)}$$

and

$$\frac{A \wedge B \wedge \mathrm{E}\, \mathbf{x} \ \vdash_{\mathbf{x}\mathbf{y}} \ C}{A \ \vdash_{\mathbf{y}} \ \forall \mathbf{x}(B \to C)} \qquad \text{none of } \mathbf{x} \text{ free in } A$$

In particular $A \ \vdash_{\mathbf{y}} \ B \to C$ follows from $A \wedge B \ \vdash_{\mathbf{y}} \ C$.

Earlier axioms for implication are replaced by

$$\forall \mathbf{x}(A \to B) \wedge \forall \mathbf{x}(B \to C) \ \vdash_{\mathbf{y}} \ \forall \mathbf{x}(A \to C)$$

$$\forall \mathbf{x}(A \to B) \wedge \forall \mathbf{x}(A \to C) \vdash_{\mathbf{y}} \forall \mathbf{x}(A \to (B \wedge C))$$
$$\forall \mathbf{x}(B \to A) \wedge \forall \mathbf{x}(C \to A) \vdash_{\mathbf{y}} \forall \mathbf{x}((B \vee C) \to A)$$

For the existential quantifier $\exists$ we need a further such formalized version. We have axiom

$$\forall \mathbf{x} y(A \to B) \vdash_{\mathbf{z}} \forall \mathbf{x}(\exists y A \to B) \qquad y \notin \mathrm{FV}(B)$$

This completes our axiomatization of Basic Predicate Logic.

We have a completeness theorem for Basic Predicate Calculus with transitive Kripke models. Transitive Kripke models provide weak counterexamples to constructive provability of predicate logical statements. Insufficient, yes. Evidence of non-provability, also yes.