Basic Logic, K4, and Persistence

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Abstract

We characterize the first-order formulas with one free variable that are preserved under bisimulation and persistence or strong persistence over the class of Kripke models with transitive frames and unary persistent predicates.

1 Introduction

The characterization of syntactically definable sets of first-order formulas by model theoretic notions is well-established. The current line of work follows an approach started by Johan van Benthem in [8]. The case mentioned in the Abstract is well-known for the class of Kripke models over preordered frames, that is, frames where the world relation is both transitive and reflexive [10, pp. 318–320]: Let *Krip* be the classical first-order theory of preordered sets and persistent unary relations over a language with equality, a binary predicate $\leq$ for order, and countably infinitely many unary predicates for persistence. So *Krip* is axiomatizable by the reflexivity and transitivity axioms for $\leq$, and the axioms $Px \wedge x \leq y \rightarrow Py$ for all unary predicates P. For each (intuitionistic) propositional formula B there is a natural formula $I(B, x)$ over the language of *Krip* such that $\mathbf{K} \models I(B, k)$ if and only if $k \Vdash_{\mathbf{K}} B$, for all Kripke models $\mathbf{K}$ of *Krip* and $k \in K$, the domain of $\mathbf{K}$. Then a formula $A(x)$ over the language of *Krip* is persistent and preserved under bisimulations, if and only if there exists a propositional formula B such that $\mathit{Krip} \vdash A(x) \leftrightarrow I(B, x)$.

Let $\prec$ be the binary predicate for order in the language of transitive frames, and *Krit* be the theory of transitive frames and countably infinitely many persistent unary relations. Significantly, there are at least two natural ways to extend persistence to the larger class of models of *Krit*.

The first way defines $A(x)$ to be persistent if $\mathit{Krit} \vdash A(x) \wedge x \prec y \rightarrow A(y)$. At first one may expect that $A(x)$ over the language of *Krit* is persistent and preserved under bisimulations exactly when there exists a propositional formula B such that $\mathit{Krit} \vdash A(x) \leftrightarrow I(B, x)$, where in this case B is thought of as a formula over Basic Propositional Calculus BPC. But it is shown in [7] that this is very untrue. The authors of [7] extend the language of BPC essentially to a modal logic over Intuitionistic Propositional Calculus IPC, in which the implication of BPC is definable. This intuitionistic modal theory U exactly entails BPC, modulo a translation, over a natural sublanguage. All formulas B over U are such that the corresponding formulas $I''(B, x)$ over the language of *Krit* are persistent and preserved under bisimulations, and from it we can find the necessary B such that $\mathit{Krit} \vdash A(x) \leftrightarrow I''(B, x)$.

The second way defines $A(x)$ strongly persistent if $\mathit{Krit} \vdash A(x) \leftrightarrow \forall y(x \prec y \rightarrow A(y))$. For each propositional formula B of the language of BPC, let $\Xi(B)$ be the formula $((\top \rightarrow B) \rightarrow B) \rightarrow (\top \rightarrow B)$. Then a formula $A(x)$ over the language of *Krit* is strongly persistent and preserved under bisimulations, if and only if there exists a propositional formula B such that $\mathit{Krit} \vdash A(x) \leftrightarrow I(\Xi(B), x)$.

Both characterizations can be extended to all theories $\Gamma \supseteq \mathit{Krit}$.

2 Basic Logic

Basic Propositional Calculus BPC was first introduced by Albert Visser in [9]. His paper also gives a completeness theorem for the class of transitive Kripke models. See also [1]. We don't need to know Basic Logic, or even see an axiomatization of BPC, to understand any of the later sections below. This brief section is intended as motivation for the study of BPC.

BPC was introduced as a proper subsystem of Intuitionistic Propositional Calculus IPC satisfying the informal equations

$$\frac{CPC}{S5} = \frac{IPC}{S4} = \frac{BPC}{K4},$$

where CPC is Classical Propositional Calculus; S5, S4, and K4 are the familiar modal logics. The weakness of the system BPC allows for an extension FPC which extends the list of informal equations above with the quotient

$$\frac{FPC}{GL},$$

where GL is the well-known provability modal logic. Visser showed that FPC satisfies the Explicit Fixed Point Theorem: For each propositional formula $A[p]$, FPC $\vdash A[\top] \leftrightarrow A[A[\top]]$.

An alternate motivation for the study of Basic Logic, first expressed by this author in [5], is based on a search for a different 'constructive' interpretation of the connectives. A satisfactory description is beyond the scope of this paper. Interested readers are invited to look in [5] or [6] for further reading.

3 Strong Persistence

When setting up the case for strong persistence, we make extensive use of [10]: For the purposes of this paper we may assume that the language $\mathcal{L}$ of BPC is built with countably many propositional variables. As language $\mathcal{L}_{Krit}$ for the theory of transitive Kripke models we have $=$ for equality, $\prec$ for the order, and countably many unary predicate symbols. The theory *Krit* extends the classical first-order logic with equality with the nonlogical axioms of

- $(x \prec y) \wedge (y \prec z) \rightarrow (x \prec z)$ (transitivity); and
- $Px \wedge x \prec y \rightarrow Py$ for all unary predicate symbols P (persistence).

We will write $x \preceq y$ as short for $x = y \vee x \prec y$ whenever it is convenient. The expressions $\succ$ and $\succeq$ are the duals of $\prec$ and $\preceq$.

Define an embedding I of $\mathcal{L}$ into $\mathcal{L}_{Krit}$ inductively by

- $I(p, x)$ equals Px (we assume some bijection $p \mapsto P$);
- $I(\bot, x)$ equals $\bot$;
- $I(\top, x)$ equals $\top$;
- $I(A \wedge B, x)$ equals $I(A, x) \wedge I(B, x)$;
- $I(A \vee B, x)$ equals $I(A, x) \vee I(B, x)$; and
- $I(A \rightarrow B, x)$ equals $\forall y(x \prec y \wedge I(A, y) \rightarrow I(B, y))$ modulo a possible renaming of bound variables.

Transitive Kripke models and models of *Krit* are essentially the same. Clearly we have $k \Vdash_{\mathbf{K}} A$ if and only if $\mathbf{K} \models I(A, k)$, for all propositional formulas A, all models $\mathbf{K}$, and elements $k \in K$, the domain of $\mathbf{K}$. By the completeness theorem for BPC we have $\vdash_{\mathrm{BPC}} A$ if and only if $\mathbf{K} \models I(A, k)$ for all models $\mathbf{K}$ of *Krit* and $k \in K$ (see [1] or [9]).

Bisimulations were introduced by Johan van Benthem in his dissertation [8]. In this thesis bisimulations were already employed to connect model theoretic properties with syntactic structure. A *bisimulation* between two transitive models $\mathbf{K}$ and $\mathbf{M}$ consists of a relation $R \subseteq K \times M$ satisfying:

- kRm implies $k \Vdash_{\mathbf{K}} p$ if and only if $m \Vdash_{\mathbf{M}} p$, for all atoms p;
- $k' \succ kRm$ implies $k'Rm'$ for some $m' \succ m$; and
- $kRm \prec m'$ implies $k'Rm'$ for some $k' \succ k$.

Let $A(x)$ be a formula of $\mathcal{L}_{Krit}$ with x as only possible free variable. We say that $A(x)$ is *preserved under bisimulations* over a theory $\Gamma \supseteq Krit$ if whenever R is a bisimulation between models $\mathbf{K}$ and $\mathbf{M}$ of Γ such that kRm and $\mathbf{K} \models A(k)$, then $\mathbf{M} \models A(m)$.

Proposition 3.1 *$I(B,x)$ is preserved under bisimulations over Krit, for all propositional formulas B.*

Proof. By induction on the complexity of B. The case for atoms, $\top$, and $\bot$ are immediate from the definitions, and the induction steps for $\wedge$ and $\vee$ are easy. Let $\mathbf{K} \models I(C \to D, k)$, and R be a bisimulation between $\mathbf{K}$ and $\mathbf{M}$ such that kRm. Suppose $m \prec m' \Vdash_{\mathbf{M}} C$. Then there exists k' satisfying $k \prec k'Rm'$. By induction $k' \Vdash_{\mathbf{K}} C$, and thus by definition of $I(C \to D, k)$, $k' \Vdash_{\mathbf{K}} D$. So by induction $m' \Vdash_{\mathbf{M}} D$. And thus $\mathbf{M} \models I(C \to D, m)$. ⊣

We call $A(x)$ *persistent* over a theory $\Gamma \supseteq Krit$ if $\Gamma \vdash A(x) \wedge x \prec y \to A(y)$. From [9] (and [1]) we have:

Proposition 3.2 *$I(B,x)$ is persistent over Krit, for all propositional formulas B.*

Proof. By induction on the complexity of B: Use the inductive definition of $I(B,x)$. ⊣

Persistence will be discussed in Section 4. We call $A(x)$ *strongly persistent* over a theory $\Gamma \supseteq Krit$ if $\Gamma \vdash A(x) \leftrightarrow \forall y(x \prec y \to A(y))$. Obviously a formula $I(B,x)$ is strongly persistent over Γ exactly when $\Gamma \vdash I(B,x) \leftrightarrow I(\top \to B, x)$. For each propositional formula B, define $\Xi(B)$ to be the formula

$$((\top \to B) \to B) \to (\top \to B).$$

Proposition 3.3 *Let $\mathbf{K}$ be a model of Krit, and B be a propositional formula. Then $\mathbf{K} \models I(B,x) \leftrightarrow I(\top \to B, x)$, if and only if $\mathbf{K} \models I(B,x) \leftrightarrow I(\Xi(B),x)$. Moreover, $\mathbf{K} \models I(\Xi(B),x) \leftrightarrow I(\top \to \Xi(B),x)$.*

Proof. Obviously, $Krit \vdash I(E,x) \to I(D \to E, x)$, for all propositional formulas D and E. So

$$\mathbf{K} \models I(B,x) \to I(\top \to B, x)$$

and

$$\mathbf{K} \models I(\top \to B, x) \to I(\Xi(B),x).$$

Suppose $\mathbf{K} \models I(\top \to B, x) \to I(B,x)$. Then

$$\begin{aligned} \mathbf{K} \models I(\Xi(B),x) &\leftrightarrow I((B \to B) \to B, x) \\ &\leftrightarrow I(\top \to B, x) \\ &\leftrightarrow I(B,x). \end{aligned}$$

The right-to-left half of the first claim is immediate from the second claim. So it suffices to prove the second claim. Suppose $\mathbf{K} \models I(\top \to \Xi(B), k)$ and $k \prec k' \Vdash_{\mathbf{K}} (\top \to B) \to B$. To show: $k' \Vdash_{\mathbf{K}} (\top \to B)$. We have $k' \Vdash_{\mathbf{K}} \Xi(B)$, so also $k' \Vdash_{\mathbf{K}} \top \to (\top \to B)$. Combining again with $k' \Vdash_{\mathbf{K}} (\top \to B) \to B$ gives us $k' \Vdash_{\mathbf{K}} \top \to B$. So $\mathbf{K} \models I(\Xi(B), k)$. ⊣

In [1] we have a short BPC proof of Proposition 3.3.

Our main result is:

Theorem 3.4 *Let $A(x)$ be a one-variable formula of $\mathcal{L}_{\rm Krit}$ which is strongly persistent and preserved under bisimulations over a theory $\Gamma \supseteq$ Krit. Then there is a propositional formula B such that*

$$\Gamma \vdash A(x) \leftrightarrow I(\Xi(B), x).$$

Proof. By Proposition 3.3 it suffices to find a propositional formula B such that

$$\Gamma \vdash A(x) \leftrightarrow I(B, x).$$

The remainder of the proof closely parallels the argument in [10, pp. 318–320]. Let $\Delta(x)$ be the set $\{I(B,x) \mid \Gamma \vdash A(x) \to I(B,x)\}$. If $\Gamma \cup \Delta(x) \vdash A(x)$, we are done by compactness. Suppose $\Gamma \cup \Delta(x) \nvdash A(x)$. There is a maximal set $\Delta'(x) \supseteq \Delta(x)$ of formulas of the form $I(B,x)$ such that $\Gamma \cup \Delta'(x) \nvdash A(x)$. By [3, Section 5.1] there exists an ω-saturated model $\mathbf{K}$ of Γ with an element $k \in K$ such that $\mathbf{K} \models \Delta'(k)$ and $\mathbf{K} \models \neg A(k)$. Let $\Theta(x)$ be the set

$$\Gamma \cup \{A(x)\} \cup \Delta'(x) \cup \{\neg I(C,x) \mid I(C,x) \notin \Delta'(x)\}.$$

Suppose $\Theta(x)$ were not consistent. Then there are finite sets X and Y of propositional formulas satisfying $k \Vdash_{\mathbf{K}} B$ for all $B \in X$ and $k \nVdash_{\mathbf{K}} C$ for all $C \in Y$, such that

$$\Gamma \vdash A(x) \to (\bigwedge_{B \in X} I(B,x) \to \bigvee_{C \in Y} I(C,x)).$$

So also

$$\Gamma \vdash \forall y[x \prec y \to A(y)] \to \forall y[x \prec y \to (I(\bigwedge X, y) \to I(\bigvee Y, y))].$$

By strong persistence of $A(x)$, and the definition of I,

$$\Gamma \vdash A(x) \to I(\bigwedge X \to \bigvee Y, x).$$

So $I(\bigwedge X \to \bigvee Y, x) \in \Delta(x) \subseteq \Delta'(x)$. Strong persistence of $A(x)$ implies that there exists $k' \succ k$ such that $\mathbf{K} \models \neg A(k')$. But $\mathbf{K} \models I(\bigvee Y, k')$, contradicting the maximality of $\Delta'(x)$. So $\Theta(x)$ is consistent, and therefore has an ω-saturated model $\mathbf{M}$ with $\mathbf{M} \models \Theta(m)$ for some m. Let $R \subseteq K \times M$ be defined by

$$k' R m' \text{ if and only if } k' \Vdash_{\mathbf{K}} B \text{ exactly when } m' \Vdash_{\mathbf{M}} B, \text{ for all propositional formulas } B.$$

We claim that R is a bisimulation. Assume $k' R m' \prec m''$. Let $\Lambda(x)$ be the set

$$\{k' \prec x\} \cup \{I(B,x) \mid m'' \Vdash_{\mathbf{M}} B\} \cup \{\neg I(C,x) \mid m'' \nVdash_{\mathbf{M}} C\}.$$

Suppose $\Lambda(x)$ were not satisfiable in $\mathbf{K}$. Then, by ω-saturatedness of $\mathbf{K}$, there are finite sets X and Y of propositional formulas satisfying $m'' \Vdash_{\mathbf{M}} B$ for all $B \in X$ and $m'' \nVdash_{\mathbf{M}} C$ for all $C \in Y$, and such that $k' \Vdash_{\mathbf{K}} \bigwedge X \to \bigvee Y$. By the definition of R, also $m' \Vdash_{\mathbf{M}} \bigwedge X \to \bigvee Y$, contradiction. So $\mathbf{K} \models \Lambda(k'')$ for some $k'' \succ k'$; so also $k'' R m''$. The case for $k'' \succ k' R m'$ is proven similarly. So R is a bisimulation between $\mathbf{K}$ and $\mathbf{M}$, with kRm. But $\mathbf{K} \models \neg A(k)$ and $\mathbf{M} \models A(m)$; contradiction. Thus $\Gamma \cup \Delta(x) \vdash A(x)$. $\dashv$

Theorem 3.4 immediately generalizes [10, pp. 318–320].

4 Persistence

Obviously, $A(x)$ is persistent over Γ exactly when $\Gamma \vdash A(x) \leftrightarrow \forall y(x \preceq y \to A(y))$. From the parallel result of Theorem 3.4 for Intuitionistic Propositional Calculus IPC and *Krip*, one should expect that equivalences of the form

$$Krit \vdash A(x) \leftrightarrow I(B, x)$$

for persistent $A(x)$ that are preserved under bisimulations require that formulas B allow for intuitionistic implications besides the BPC implication. This turns out to be the case. First let us establish the insufficiency of propositional formulas over BPC. There is the obvious embedding $A \mapsto A'$ of the language $\mathcal{L}$ of BPC into the language $\mathcal{ML}$ of K4, inductively defined by:

- $p' = p \wedge \Box p$;
- $\bot' = \bot$;
- $\top' = \top$;
- $(A \wedge B)' = A' \wedge B'$;
- $(A \vee B)' = A' \vee B'$; and
- $(A \to B)' = \Box(A' \to B')$.

Define the embedding I' of $\mathcal{ML}$ into $\mathcal{L}_{Krit}$ inductively by

- $I'(p, x)$ equals Px (we assume some bijection $p \mapsto P$);
- $I'(\bot, x)$ equals $\bot$;
- $I'(\top, x)$ equals $\top$;
- $I'(A \wedge B, x)$ equals $I'(A, x) \wedge I'(B, x)$;
- $I'(A \vee B, x)$ equals $I'(A, x) \vee I'(B, x)$;
- $I'(A \to B, x)$ equals $I'(A, x) \to I'(B, x)$; and
- $I'(\Box A, x)$ equals $\forall y(x \prec y \to I'(A, y))$ modulo a possible renaming of bound variables.

Let *Tran* be the theory over first-order logic with equality which is axiomatized by

- $(x \prec y) \wedge (y \prec z) \to (x \prec z)$ (transitivity).

So *Tran* is a proper subtheory of *Krit*. By the completeness theorem for K4 we have $\vdash_{\mathrm{K4}} A$ if and only if $\mathit{Tran} \vdash I'(A, x)$.

Proposition 4.1 *$\mathit{Krit} \vdash I(B, x) \leftrightarrow I'(B', x)$ for all propositional formulas B of $\mathcal{L}$.*

Proof. By an obvious induction on the complexity of propositional formulas. $\dashv$

The following counterexample essentially comes from [7]. Recall that $\neg A$ is short for $A \to \bot$.

Proposition 4.2 *Let A be the formula $(\neg p) \wedge \Box(\neg p) \in \mathcal{ML}$. Then $I'(A, x)$ is persistent and preserved under bisimulations, but there exists no $B \in \mathcal{L}$ such that*

$$\mathit{Krit} \vdash I'(A, x) \leftrightarrow I(B, x).$$

Proof. The preservation of A under bisimulations is well-known; see [8]. Moreover, the formula A essentially is intuitionistic negation when we replace $\prec$ by $\preceq$ in the transitive Kripke models. So A is persistent. Let $\mathbf{K}$ be a Kripke model with two irreflexive nodes k and k' as in the diagram below (open circles indicate irreflexive nodes, filled-in circles indicate reflexive nodes) with empty relation $\prec$, and with propositional variable p only forced in the right node.

$$k \circ \qquad k' \circ \; p$$

So $k \Vdash_{\mathbf{K}} A$ and $k' \not\Vdash_{\mathbf{K}} A$. As to the (non)existence of B, we may restrict ourselves to $B \in \mathcal{L}$ that are built from p using the connectives including $\bot$ and $\top$. But $\mathbf{K} \models I(C \to D, x)$ for all implications of $\mathcal{L}$, so $I(B, x)$ can only be valid on one of the sets $\emptyset$, $\{k'\}$, or $K = \{k, k'\}$. So $\mathbf{K} \not\models I'(A, x) \leftrightarrow I(B, x)$. $\dashv$

So Theorem 3.4 fails when we replace strong persistence by persistence and $\Xi(B)$ by arbitrary B. Our solution, extending the language (and theory) of BPC, essentially follows from [7]. Define the embedding I'' of $\mathcal{ML}$ into $\mathcal{L}_{Krit}$ inductively by

- $I''(p, x)$ equals Px (we assume some bijection $p \mapsto P$);
- $I''(\bot, x)$ equals $\bot$;
- $I''(\top, x)$ equals $\top$;
- $I''(A \wedge B, x)$ equals $I''(A, x) \wedge I''(B, x)$;
- $I''(A \vee B, x)$ equals $I''(A, x) \vee I''(B, x)$;
- $I''(A \to B, x)$ equals $\forall y(x \preceq y \wedge I''(A, y) \to I''(B, y))$; and
- $I''(\Box A, x)$ equals $\forall y(x \prec y \to I''(A, y))$ modulo a possible renaming of bound variables.

Notice the use of $\preceq$ in the inductive step for $\to$. An easy proof by induction on the complexity of propositional formulas yields:

Proposition 4.3 *Krit* $\vdash I(B, x) \leftrightarrow I''(B', x)$ *for all propositional formulas* B *of* $\mathcal{L}$.

So modulo the embedding $B \mapsto B'$, the language $\mathcal{ML}$ is a proper extension of the language $\mathcal{L}$. There is also a modal propositional theory that matches this extension: Following [7], let U be the theory with modus ponens as rule of inference, and with axioms

- an appropriate IPC axiomatization for $\bot$, $\top$, $\wedge$, $\vee$, and $\to$;
- $\Box(A \to B) \to (\Box A \to \Box B)$;
- $\Box A \to \Box\Box A$;
- $A \to \Box A$; and
- $\Box A \to (B \vee (B \to A))$.

An easy induction proof on the length of derivations shows that if $\vdash_{\mathrm{U}} A$, then *Krit* $\vdash I''(A, x)$; soundness. In [7] the authors prove the strong reverse direction; strong completeness.

Proposition 4.4 $I''(B, x)$ *is preserved under bisimulations, for all modal propositional formulas* B.

Proof. By induction on the complexity of B. A bisimulation for $\prec$ is also a bisimulation for $\preceq$, so the cases for atoms and the connectives minus $\Box$ are as for IPC. The case for $\Box A$ is as for $\top \to A$ in the proof of Proposition 3.1. $\dashv$

Proposition 4.5 $I''(B, x)$ *is persistent, for all modal propositional formulas* B.

Proof. By an easy induction on the complexity of B. $\dashv$

Theorem 4.6 *Let* $A(x)$ *be a one-variable formula of* $\mathcal{L}_{\mathrm{Krit}}$ *which is persistent and preserved under bisimulations over a theory* $\Gamma \supseteq$ *Krit. Then there is a modal propositional formula* B *such that*

$$\Gamma \vdash A(x) \leftrightarrow I''(B, x).$$

Proof. The proof closely parallels the arguments in the proofs of Theorem 3.4 and [10, pp. 318–320]. Let $\Delta(x)$ be the set $\{I''(B, x) \mid \Gamma \vdash A(x) \to I''(B, x)\}$. If $\Gamma \cup \Delta(x) \vdash A(x)$, we are done. Suppose $\Gamma \cup \Delta(x) \nvdash A(x)$. There exists an ω-saturated model $\mathbf{K}$ of Γ with an element $k \in K$ such that $\mathbf{K} \models \Delta(k)$ and $\mathbf{K} \models \neg A(k)$. Let $\Theta(x)$ be the set

$$\Gamma \cup \{A(x)\} \cup \{I''(B, x) \mid \mathbf{K} \models I''(B, k)\} \cup \{\neg I''(C, x) \mid \mathbf{K} \not\models I''(C, k)\}.$$

Suppose $\Theta(x)$ were not consistent. Then there are finite sets X and Y of modal propositional formulas satisfying $\mathbf{K} \models I''(B,k)$ for all $B \in X$ and $\mathbf{K} \not\models I''(C,k)$ for all $C \in Y$, such that

$$\Gamma \vdash A(x) \rightarrow (\bigwedge_{B\in X} I''(B,x) \rightarrow \bigvee_{C\in Y} I''(C,x)).$$

So also

$$\Gamma \vdash \forall y[x \preceq y \rightarrow A(y)] \rightarrow \forall y[x \preceq y \rightarrow (I''(\bigwedge X, y) \rightarrow I''(\bigvee Y, y))].$$

By persistence of $A(x)$, and the definition of I'',

$$\Gamma \vdash A(x) \rightarrow I''(\bigwedge X \rightarrow \bigvee Y, x).$$

So $I''(\bigwedge X \rightarrow \bigvee Y, x) \in \Delta(x)$, while $\mathbf{K} \not\models I''(\bigwedge X \rightarrow \bigvee Y, k)$, contradiction. So $\Theta(x)$ is consistent, and therefore has an ω-saturated model $\mathbf{M}$ with $\mathbf{M} \models \Theta(m)$ for some m. Let $R \subseteq K \times M$ be defined by

$k'Rm'$ if and only if
$\mathbf{K} \models I''(B,k')$ exactly when $\mathbf{M} \models I''(B,m')$, for all modal propositional formulas B.

We claim that R is a bisimulation. Assume $k'Rm' \prec m''$. Let $\Lambda(x)$ be the set

$$\{k' \prec x\} \cup \{I''(B,x) \mid \mathbf{M} \models I''(B,m'')\} \cup \{\neg I''(C,x) \mid \mathbf{M} \not\models I''(C,m'')\}.$$

Suppose $\Lambda(x)$ were not satisfiable in $\mathbf{K}$. Then, by ω-saturatedness of $\mathbf{K}$, there are finite sets X and Y of modal propositional formulas satisfying $\mathbf{M} \models I''(B,m'')$ for all $B \in X$ and $\mathbf{M} \not\models I''(C,m'')$ for all $C \in Y$, and such that $\mathbf{K} \models I''(\Box(\bigwedge X \rightarrow \bigvee Y), k')$. By the definition of R, also $\mathbf{M} \models I''(\Box(\bigwedge X \rightarrow \bigvee Y), m')$, contradiction. So $\mathbf{K} \models \Lambda(k'')$ for some $k'' \succ k'$; so also $k''Rm''$. The case for $k'' \succ k'Rm'$ is proven similarly. So R is a bisimulation between $\mathbf{K}$ and $\mathbf{M}$, with kRm. But $\mathbf{K} \models \neg A(k)$ and $\mathbf{M} \models A(m)$; contradiction. Thus $\Gamma \cup \Delta(x) \vdash A(x)$. $\dashv$

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