PRODUCTS OF IDEMPOTENT MATRICES OVER HERMITE DOMAINS

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0. Introduction

In [Fountain 1991] John Fountain shows, among other things, that if R is a discrete valuation ring or the ring of integers, then all $n \times n$ matrices of rank less than n are products of idempotent matrices. We generalize this result by characterizing those rings among the Hermite domains that satisfy this same result: All $n \times n$ matrices of rank less than n over a Hermite domain are products of idempotents if and only if all invertible matrices are products of elementary matrices. Since we work over a larger class of rings, many results of [Fountain 1991] have to be redone. Fountain's methods are an important guide to us on how to proceed.

1. Preliminaries

In the Preliminaries we state some basic facts of ring theory on behalf of experts in semigroup theory, and some basic facts of semigroup theory on behalf of experts in ring theory.

Recall that an *integral domain* is a ring R with unit such that $r \neq 0$ and $s \neq 0$ implies $rs \neq 0$, for all $r, s \in R$. For each left submodule N of a module R^n, define the *pure closure* $\overline{N} = \{\mathbf{m} \mid r\mathbf{m} \in N \text{ for some nonzero } r \in R\}$. Note that direct summands are pure, but in general pure closures need not even be submodules. A *Hermite domain* is an integral domain R such that for all matrices α there are invertible square matrices σ and ρ such that both $\sigma\alpha$ and $\alpha\rho$ are upper triangular matrices. This is equivalent to: For all integers $n > 0$, each finitely generated left submodule $N \subseteq R^n$ has a unique finite rank $\operatorname{rk} N$, and its pure closure $\overline{N}$ is a direct summand of R^n and has equal finite rank.

Clearly, for each $\mathbf{m} \in R^n$ there exist bases $\mathbf{f}_1, \ldots, \mathbf{f}_n$ and $d \in R$ such that $\mathbf{m} = d\mathbf{f}_1$. It is well-known that if $M \subseteq N \subseteq R^n$ are finitely generated left modules, then $\operatorname{rk} M \leq \operatorname{rk} N$, and $\overline{M}$ is a direct summand of $\overline{N}$; left invertible and right invertible square matrices are invertible; and for each $m \times n$ matrix α there are square matrices σ and ρ such that σ's rows are linearly independent and ρ is invertible, and a diagonal matrix δ, such that $\sigma\alpha\rho = \delta$. If σ can be chosen invertible, then we call α diagonalizable. The number of nonzero elements of δ is unique, and is called the *rank* of α, written $\operatorname{rk} \alpha$. This number is equal to the rank of the left module generated by the row space of α.

Examples of Hermite domains include division rings and principal ideal domains. More generally, if R is an integral domain such that all matrices are diagonalizable, then R is Hermite.

Recall that an *elementary matrix* is an invertible matrix that is identical to the identity except for one entry. Permutation matrices are products of elementary matrices. For Euclidean domains all invertible matrices are products of elementary matrices.

1.1 Proposition. *All invertible matrices over a Hermite domain are products of elementary matrices if and only if its invertible* 2×2 *matrices are products of elementary matrices.*

Proof: Immediate from [Kaplansky 1949, Theorem 7.1]. $\dashv$

For a semigroup S, let S^1 be obtained by adding an identity (if S already has an identity, we may set $S^1 = S$). Each $a \in S$ defines a transformation $\rho_a \in T(S^1)$ by right multiplication, and a transformation $\lambda_a \in T^*(S^1)$ by left multiplication. Define equivalence relations $\mathcal{R}^*$ and $\mathcal{L}^*$ on S by $a\mathcal{R}^*b$ if and only if ρ_a and ρ_b introduce the same equivalence relation on their domains ($xa = ya$ if and only if $xb = yb$), and $a\mathcal{L}^*b$ if and only if λ_a and λ_b introduce the same equivalence relation on their domains. Let $\mathcal{H}^* = \mathcal{R}^* \cap \mathcal{L}^*$ and $\mathcal{D}^* = \mathcal{R}^* \vee \mathcal{L}^*$, the meet and join equivalence relations of $\mathcal{R}^*$ and $\mathcal{L}^*$.

1.2 Lemma. *For* $S = \mathrm{End}(R^n)$, R *a Hermite domain, we have*

$$\alpha\mathcal{R}^*\beta \iff \operatorname{Ker}\alpha = \operatorname{Ker}\beta \quad \textit{and}$$
$$\alpha\mathcal{L}^*\beta \iff \overline{\operatorname{Im}\alpha} = \overline{\operatorname{Im}\beta}.$$

Proof: The case for $\mathcal{R}^*$ versus Ker is trivial.
The case for $\mathcal{L}^*$ versus $\overline{\mathrm{Im}}$: $\alpha\gamma = 0$ if and only if $\operatorname{Im}\alpha \subseteq \operatorname{Ker}\gamma$ if and only if $\overline{\operatorname{Im}\alpha} \subseteq \operatorname{Ker}\gamma$. Since $\overline{\operatorname{Im}\alpha}$ is a direct summand, there is an (idempotent) γ with $\operatorname{Ker}\gamma = \overline{\operatorname{Im}\alpha}$. $\dashv$

Note that $\operatorname{Ker}\alpha$ is a (finitely generated) direct summand, for if $x_1\alpha, \ldots, x_m\alpha$ is a basis of $\operatorname{Im}\alpha$, then $\operatorname{Ker}\alpha \oplus Rx_1 \oplus \cdots \oplus Rx_m = R^n$.
For each pair of finitely generated pure submodules $N, P \subseteq R^n$ over a Hermite domain R such that $\operatorname{rk} N + \operatorname{rk} P = n$, there exist $\alpha \in \mathrm{End}(R^n)$ with $\operatorname{Ker}\alpha = N$ and $\operatorname{Im}\alpha = P$. All $\mathcal{H}^*$ classes are uniquely determined by such pairs. For each such pair N, P, we denote by $(N : P)$ the $\mathcal{H}^*$-class of a corresponding α.
An element a of a semigroup is (von Neumann) *regular* if there exist b such that $a = aba$. If R is a Hermite domain, then $\alpha \in \mathrm{End}(R^n)$ is regular exactly when $\operatorname{Im}\alpha$ is pure, for if $\alpha\beta\alpha = \alpha$ and $x\alpha = ry$, then $r(y\beta\alpha) = ry$. Moreover, each equivalence class $\mathcal{H}^*$ of $\mathrm{End}(R^n)$ contains a regular element.

1.3 Lemma. *Let R be a Hermite domain, and let $\varepsilon_1, \ldots, \varepsilon_k \in End\ (R^n)$ be of rank r. For all $k \geq j \geq i \geq 1$, define $\alpha_{j,i} = \varepsilon_j\varepsilon_{j-1}\ldots\varepsilon_i$, and $\alpha_i = \alpha_{k,i}$. If α_1 is regular and of rank r, then so is $\alpha_{j,i}$ for all $j \geq i$.*

Proof: If $j' \geq j \geq i \geq i'$, then $\operatorname{rk}\alpha_{j',i'} \leq \operatorname{rk}\alpha_{j,i} \leq r$. So $\operatorname{rk}\alpha_{j,i} = r$ for all $j \geq i$. Now $\alpha_{j,i}$ is regular if and only if $\operatorname{Im}\alpha_{j,i}$ is pure. So it suffices to show that all α_i are regular. We complete the proof by induction on i. Suppose that α_i is regular. Then $\operatorname{Im}\alpha_i = \operatorname{Im}\varepsilon_i$, and the module $\operatorname{Im}\alpha_{i+1}$ has a free basis $\mathbf{x}_1, \ldots, \mathbf{x}_r$ such that $\mathbf{x}_1\varepsilon_i, \ldots, \mathbf{x}_r\varepsilon_i$ is a basis for $\operatorname{Im}\varepsilon_i$. Let $\mathbf{x}_{r+1}, \ldots, \mathbf{x}_n$ be a basis for $\operatorname{Ker}\varepsilon_i$. Then $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is a free basis for R^n. Thus $\operatorname{Im}\alpha_{i+1}$ is a direct summand, hence pure. $\dashv$

Green's equivalence relations $\mathcal{R}$ and $\mathcal{L}$ on a semigroup S are defined by

$$a\mathcal{R}b \iff aS^1 = bS^1 \quad \text{and}$$
$$a\mathcal{L}b \iff S^1a = S^1b.$$

So $a\mathcal{R}b$ if and only if there are $x, y \in S^1$ such that $ax = b$ and $by = a$ (similarly for $\mathcal{L}$). Let $\mathcal{D} = \mathcal{R} \vee \mathcal{L}$. Then $\mathcal{D}$ is an equivalence relation such that $a\mathcal{D}b$ if and only if there are $x_1, y_1, x_2, y_2 \in S^1$ such that $x_1ay_1 = b$ and $x_2x_1a = ay_1y_2 = a$. So $\mathcal{D} = \mathcal{L}\mathcal{R} = \mathcal{R}\mathcal{L}$. Obviously, if $a\mathcal{R}b$, then $a\mathcal{R}^*b$ (similarly, $\mathcal{L} \subseteq \mathcal{L}^*$). If a and b are regular, then the reverse holds too: If $axa = a$ and $byb = b$, and $a\mathcal{R}^*b$, then $axb = b$ and $bya = a$ (similarly, $\mathcal{L}^* = \mathcal{L}$ with restriction to regular elements).

A semigroup S is *abundant* if each $\mathcal{R}^*$-class and each $\mathcal{L}^*$-class contains an idempotent.

1.4 Lemma. *If S is a semigroup in which each $\mathcal{H}^*$-class contains a regular element, then $\mathcal{D}^* = \mathcal{R}^*\mathcal{L}^* = \mathcal{L}^*\mathcal{R}^*$, and S is abundant.*

Proof: [Fountain 1991]. $\dashv$

Lemma 1.4 applies to $\mathrm{End}(R^n)$, R a Hermite domain: So it is abundant, and $\mathcal{D}^* = \mathcal{R}^*\mathcal{L}^* = \mathcal{L}^*\mathcal{R}^*$. For $\alpha, \beta \in \mathrm{End}(R^n)$ we have

$$\alpha\mathcal{D}^*\beta \iff \operatorname{rk}\alpha = \operatorname{rk}\beta.$$

For if $\alpha\mathcal{D}^*\beta$, then $\alpha\mathcal{R}^*\gamma\mathcal{L}^*\beta$ for some γ with $\operatorname{Ker}\alpha = \operatorname{Ker}\gamma$ and $\overline{\operatorname{Im}\gamma} = \overline{\operatorname{Im}\beta}$. So $\operatorname{rk}\alpha = \operatorname{rk}\gamma = \operatorname{rk}\beta$. Conversely, if $\operatorname{rk}\alpha = \operatorname{rk}\beta$, then $\operatorname{rk}(\operatorname{Ker}\alpha) + \operatorname{rk}(\overline{\operatorname{Im}\beta}) = n$. So there exist γ in the $\mathcal{H}^*$-class $(\operatorname{Ker}\alpha : \overline{\operatorname{Im}\beta})$ such that $\alpha\mathcal{R}^*\gamma\mathcal{L}^*\beta$.

For each $a \in S$, let $\mathcal{H}^*_a$ be the $\mathcal{H}^*$-class containing a. Similarly define $\mathcal{L}^*_a$, $\mathcal{R}^*_a$, $\mathcal{L}_a$, and $\mathcal{R}_a$. If f is idempotent and $x\mathcal{L}^*f$, then $xf = x$ because $ff = f1$. Similarly, if f is idempotent and $x\mathcal{R}^*f$, then $fx = x$. If f is idempotent and $x\mathcal{L}f$, then $f = x_1x$ for some x, and $xx_1x = x$. So x is regular. Similarly, if f is idempotent and $x\mathcal{R}f$, then $f = xx_2$, and $xx_2x = x$. The following essentially is a special case of [Fountain 1982, Proposition 1.13].

1.5 Proposition. *Let S be a semigroup in which every $\mathcal{H}^*$-class contains a regular element. Let $e, c, d, a \in S$ be such that e is idempotent, c, d are regular, $e\mathcal{L}^*c\mathcal{R}^*a$, and $e\mathcal{R}^*d\mathcal{L}^*a$. Then the function $\theta\colon \mathcal{H}^*_e \to \mathcal{H}^*_a$ given by $b\theta = cbd$ is a bijection.*

Proof: We may assume that a is regular. So $e\mathcal{L}c\mathcal{R}a$, and $e\mathcal{R}d\mathcal{L}a$.

Suppose $b, b' \in S$ are such that $cbd = cb'd$. Then $ebe = eb'e$. If $b, b' \in \mathcal{H}^*_e$, then $b = b'$. Thus $\theta\colon \mathcal{H}^*_e \to S$ is one-to-one.

Note that $\mathcal{H}^*_a = \mathcal{L}^*_a \cap \mathcal{R}^*_a = \mathcal{L}^*_d \cap \mathcal{R}^*_c$.

Let $x \in \mathcal{H}^*_e$. We wish to show that $cxd\mathcal{L}^*d$. Obviously, if $dp = dq$, then $cxdp = cxdq$. Conversely, suppose $cxdp = cxdq$. From $c\mathcal{L}^*e$ it follows that $exdp = exdq$, so, because $ex = x$, $xdp = xdq$. Now $x\mathcal{L}^*e$, so $edp = edq$, so, because $ed = d$, $dp = dq$. Similarly, $cxd\mathcal{R}^*c$. Thus $cxd \in \mathcal{L}^*_d \cap \mathcal{R}^*_c$, and thus $\operatorname{Im}\theta \subseteq \mathcal{H}^*_a$.

Let $y \in \mathcal{L}^*_d$. By Lemma 1.4 there is an idempotent f such that $y\mathcal{L}^*d\mathcal{L}f$. So $y = yf$, $d = df$, and $f = d'd$ for some d'. So $y = yd'd$. Similarly, if $y \in \mathcal{R}^*_c$, then $y = cc'y$ for some c' such that $g = cc'$ is idempotent, and $y\mathcal{R}^*c\mathcal{R}g$. So if $y \in \mathcal{H}^*_a$, then $y = cc'yd'd = cec'yd'ed$. Set $x = ec'yd'e$. We wish to show that $x\mathcal{L}^*e$. Obviously, if $ep = eq$, then $xp = xq$. Conversely, suppose $xp = xq$. Since $e\mathcal{L}^*c$ we have $cc'yd'ep = cc'yd'eq$. Since $cc'y = gy = y$ this implies $yd'ep = yd'eq$. Now $y\mathcal{L}^*d$, so $dd'ep = dd'eq$. Now $e = dd_1$ for some d_1, so $dd'dd_1p = dd'dd_1q$. So $dfd_1p = dfd_1q$, so $dd_1p = dd_1q$. Thus $ep = eq$. Similarly, $x\mathcal{R}^*e$. So $x \in \mathcal{H}^*_e$, and thus θ maps onto $\mathcal{H}^*_a$. $\dashv$

1.6 Lemma. *If one element of an $\mathcal{D}$-class is regular, then so are all elements of that $\mathcal{D}$-class.*

Proof: [Clifford, Preston 1961, Theorem 2.11]. $\dashv$

1.7 Lemma. *Let a, b be elements of a semigroup S. Then the following are equivalent:*

(1) $ab \in \mathcal{R}_a \cap \mathcal{L}_b$.
(2) $\mathcal{L}_a \cap \mathcal{R}_b$ *contains an idempotent.*

If (1) *or* (2) *holds, then $a\mathcal{D}b$, and a and b are regular.*

Proof: [Clifford, Preston 1961, Theorem 2.17]. $\dashv$

2. Products of Idempotents

Define E^n_{n-r} to be the set of idempotents of $\mathrm{End}(R^n)$ of rank r, and let $E^n = \bigcup_{s>0} E^n_s$, the set of all non-identity idempotents. We write E_m instead of E^n_m if confusion isn't likely. For subsets X of a semigroup S, let $\langle X \rangle$ denote the subsemigroup generated by X.

Let $\mathcal{D}^*_r \subseteq \mathrm{End}(R^n)$ denote the $\mathcal{D}^*$-class of endomorphisms of rank r, and let $K(n, r)$ be the semigroup ideal of $\mathrm{End}(R^n)$ of matrices of rank at most r. So $K(n, r) = \bigcup_{s \le r} \mathcal{D}^*_s$.

A pair of finitely generated pure submodules $A, B \subseteq M$ is called *complementary* if $A \oplus B = M$. If R is a Hermite domain and $A, B \subseteq R^n$ are complementary, then the projection on B with kernel A is an idempotent element of the $\mathcal{H}^*$-class $(A : B)$. In fact, this class contains an idempotent if and only if A and B form a complementary pair. A pair of finitely generated pure submodules $A, B \subseteq M$ is called *weakly complementary* if there is a finite sequence of finitely generated pure submodules $C_1, \ldots, C_k, D_1, \ldots, D_k \subseteq M$ with $A = C_1$, $B = D_k$, and such that (C_i, D_i) and (C_{j+1}, D_j) are complementary pairs for all $1 \le i \le k$ and $1 \le j \le k - 1$. Obviously weak complementarity is the smallest symmetric relation containing complementarity such that if (A, B), (C, B), and (C, D) are related, then so is (A, D). A module R^n is *weakly complementary* if all pairs of pure submodules $A, B \subseteq R^n$ such that $\mathrm{rk}\, A = n - 1$, $\mathrm{rk}\, B = 1$, and $A \cap B = 0$, are weakly complementary.

2.1 Lemma. *Let R be a Hermite domain, and let A, B be pure submodules of R^n of ranks r and $n - r$ respectively. Then (A, B) is weakly complementary if and only if the $\mathcal{H}^*$-class $(B : A)$ of* $\mathrm{End}(R^n)$ *contains a regular element that is a product of idempotents or rank r.*

Proof: Suppose (A, B) is a weakly complementary pair. There are submodules

$$C_1, \ldots, C_k, D_1, \ldots, D_k \subseteq R^n$$

with $A = C_1$, $B = D_k$, and all pairs (C_i, D_i) and (C_{j+1}, D_j) are complementary. So $\mathrm{rk}\, C_i = r$ and $\mathrm{rk}\, D_i = n - r$, for all i. So each $\mathcal{H}^*$-class $(D_i : C_i)$ contains an idempotent $\varepsilon_i \in \mathcal{D}^*_r$, and each $\mathcal{H}^*$-class $(D_j : C_{j+1})$ contains an idempotent $\eta_j \in \mathcal{D}^*_r$. For all $j \le k - 1$ we have $\varepsilon_j \mathcal{R} \eta_j \mathcal{L} \varepsilon_{j+1}$, so by Lemma 1.7,

$$\varepsilon_{j+1}\varepsilon_j \in \mathcal{R}_{\varepsilon_{j+1}} \cap \mathcal{L}_{\varepsilon_j}.$$

Let $\alpha_i = \varepsilon_k \cdot \ldots \cdot \varepsilon_i$. Then $\alpha_j \mathcal{R} \alpha_{j+1}$, and if $x \mathcal{L} \varepsilon_{j+1}$, then $x\varepsilon_j \mathcal{L} \varepsilon_j$. So by induction we have

$$\alpha_j \in \mathcal{R}_{\varepsilon_k} \cap \mathcal{L}_{\varepsilon_j}$$

and, by Lemma 1.6, all α_j are regular. Hence $\mathrm{Ker}\, \alpha_i = \mathrm{Ker}\, \varepsilon_k = D_k$ and $\mathrm{Im}\, \alpha_i = \mathrm{Im}\, \varepsilon_i = C_i$. So α_1 is a regular element of $(B : A)$ that is a product of idempotents of rank r.

Conversely, let α be an element in the $\mathcal{H}^*$-class $(B : A)$ that is regular and a product $\varepsilon_k \cdot \ldots \cdot \varepsilon_1$ of idempotents of rank r. Set $C_i = \operatorname{Im}\varepsilon_i$ and $D_i = \operatorname{Ker}\varepsilon_i$. Then $A = C_1$, $B = D_k$, and the pairs (C_i, D_i) are complementary. By Lemma 1.3 $\varepsilon_{i+1}\varepsilon_i$ is regular and of rank r, so $\operatorname{Im}\varepsilon_{i+1}\varepsilon_i = \operatorname{Im}\varepsilon_i$. The submodules $\operatorname{Ker}\varepsilon_{i+1} \subseteq \operatorname{Ker}\varepsilon_{i+1}\varepsilon_i$ are also pure and of the same rank, so $\operatorname{Ker}\varepsilon_{i+1} = \operatorname{Ker}\varepsilon_{i+1}\varepsilon_i$. So $\varepsilon_{i+1}\mathcal{R}\varepsilon_{i+1}\varepsilon_i\mathcal{L}\varepsilon_i$, and thus, by Lemma 1.7, there is an idempotent

$$\eta_i \in \mathcal{L}_{\varepsilon_{i+1}} \cap \mathcal{R}_{\varepsilon_i}.$$

So the pairs (C_{i+1}, D_i) are complementary. $\dashv$

2.2 Corollary. *Let R be a Hermite domain. If each $\mathcal{H}^*$-class contained in $\mathcal{D}^*_{n-1} \subseteq \operatorname{End}(R^n)$ contains a regular element that is a product of idempotents, then R^n is weakly complementary.*

Proof: If $a \in \mathcal{D}^*_{n-1}$ is a product of idempotents, then it is a product of idempotents of rank $n-1$. $\dashv$

2.3 Lemma. *Let R be a Hermite domain. If $\alpha = \varepsilon_k \ldots \varepsilon_1$ is a product of idempotents of $\operatorname{End}(R^n)$, and D is a direct summand such that both D and $D\alpha$ have rank r, then there is a product $\beta = \eta_k \ldots \eta_1$ of idempotents of rank r such that $\operatorname{Ker}\beta \supseteq \operatorname{Ker}\alpha$ and $\mathbf{d}\beta = \mathbf{d}\alpha$ for all $\mathbf{d} \in D$.*

Proof: We complete the proof by induction on k. The case for $k = 1$ is easy, so assume that $k > 0$, and that the result is true for $k-1$. Let $\alpha' = \varepsilon_{k-1} \ldots \varepsilon_1$, and let $\mathbf{d}_1, \ldots, \mathbf{d}_r$ be a basis for D. Then $\mathbf{d}_1\varepsilon_k, \ldots, \mathbf{d}_r\varepsilon_k$ generates a submodule of rank r with pure closure D', and $D'\alpha'$ has rank r. By induction there is a product $\beta' = \eta_{k-1} \ldots \eta_1$ of idempotents of rank r such that $\operatorname{Ker}\beta' \supseteq \operatorname{Ker}\alpha'$ and $\mathbf{d}'\beta' = \mathbf{d}'\alpha'$ for all $\mathbf{d}' \in D'$. There is a direct summand $K \supseteq (\operatorname{Ker}\beta')\varepsilon_k^{-1} = \operatorname{Ker}(\varepsilon_k\beta')$ such that $D' \oplus K = R^n$. Choose for η_k the projection on D' with kernel K, and set $\beta = \eta_k\beta'$. So $\operatorname{Ker}\beta \supseteq \operatorname{Ker}\alpha$. For all $\mathbf{x}$, $\mathbf{x}\eta_k = (\mathbf{x} - \mathbf{x}\varepsilon_k + \mathbf{x}\varepsilon_k)\eta_k = \mathbf{x}\varepsilon_k\eta_k$. So $\eta_k = \varepsilon_k\eta_k$, and $\mathbf{d}\beta = \mathbf{d}\alpha$ for all $\mathbf{d} \in D$. $\dashv$

2.4 Lemma. *Let R be a Hermite domain, and let $\alpha \in \operatorname{End}(R^n)$ be of rank $r \le n-2$. Then $\alpha = \beta\gamma$ for some $\beta, \gamma \in \operatorname{End}(R^n)$ of rank $r+1$ such that γ is idempotent. If α is an idempotent, then we can choose β to be idempotent too.*

Proof: There is a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of R^n such that $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ is a basis of $\overline{\operatorname{Im}\alpha}$. The sequence $\langle \mathbf{x}_1\alpha, \ldots, \mathbf{x}_n\alpha\rangle$ generates $\operatorname{Im}\alpha$, so contains a subsequence Y of length r that is linearly independent. Let j be such that $\mathbf{x}_j\alpha$ is not selected for the subsequence Y. If α is idempotent, then we may assume Y to be the first r elements, and $j = r+1$. Define β by $\mathbf{x}_j\beta = \mathbf{x}_{r+1}$, and $\mathbf{x}_i\beta = \mathbf{x}_i\alpha$ for $i \ne j$. Define γ by $\mathbf{x}_i\gamma = 0$ if $i \ge r+3$, $\mathbf{x}_{r+1}\gamma = \mathbf{x}_j\alpha$, and $\mathbf{x}_i\gamma = \mathbf{x}_i$ for the remaining i. Clearly, β and γ have rank $r+1$, $\alpha = \beta\gamma$, and γ is an idempotent such that $\operatorname{Im}\gamma$ is the pure submodule generated by $\{\mathbf{x}_1, \ldots, \mathbf{x}_r, \mathbf{x}_{r+2}\}$. If α is idempotent, then so is β. $\dashv$

Lemma 2.4 implies that $\mathcal{D}^*_r$ generates $K(n, r)$, for all $r < n$. Combining Lemmas 2.3 and 2.4, we get:

2.5 Corollary. *Let R be a Hermite domain. Then $\langle E\rangle \cap K(n, r) = \langle E_{n-r}\rangle$ for $1 \le r \le n-1$. In particular, $\langle E\rangle = \langle E_1\rangle$.* $\dashv$

2.6 Lemma. *Let R be a Hermite domain, and let $\mathcal{H}^*_\varepsilon$ be the $\mathcal{H}^*$-class of an idempotent ε of rank r in $\mathrm{End}(R^n)$. If every $\mathcal{H}^*$-class of $\mathcal{D}^*_r$ contains a regular element that is a product of idempotents of rank r, then $K(n,r) = \langle \mathcal{H}^*_\varepsilon \cup E_{n-r}\rangle$.*

Proof: Let $\alpha \in \mathcal{D}^*_r$. Then the $\mathcal{H}^*$-classes $\mathcal{L}^*_\varepsilon \cap \mathcal{R}^*_\alpha$ and $\mathcal{R}^*_\varepsilon \cap \mathcal{L}^*_\alpha$ are contained in $\mathcal{D}^*_r$, and contain regular elements that are products of idempotents, say γ and δ. By Proposition 1.5 the map $\theta\colon \mathcal{H}^*_\varepsilon \to \mathcal{H}^*_\alpha$, defined by $\beta\theta = \gamma\beta\delta$, is a bijection. Since γ and δ are products of idempotents of rank r, this implies that $\alpha \in \langle \mathcal{H}^*_\varepsilon \cup E_{n-r}\rangle$. So $\mathcal{H}^*_\varepsilon \cup E_{n-r}$ generates $\mathcal{D}^*_r$, and thus $K(n,r)$. ⊣

2.7 Lemma. *Let R be a Hermite domain. Then for all $n \geq 2$ and $\alpha \in \mathrm{End}(R^n)$ of rank 1, if α is $\mathcal{H}^*$-equivalent to an idempotent, then there are idempotents $\varepsilon, \eta \in \mathrm{End}(R^n)$ such that $\alpha = \varepsilon\eta$.*

Proof: Let $n \geq 2$, and let $\alpha \in \mathrm{End}(R^n)$ have rank 1. There is a basis $\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}, \mathbf{x}$ of R^n such that $\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}$ is a basis of $\mathrm{Ker}\,\alpha$, and $\mathbf{x}\alpha = a\mathbf{x}$ for some $a \in R$. If $a = 1$, then α is idempotent. In general, define $\varepsilon, \eta \in \mathrm{End}(R^n)$ by:

$\mathbf{x}\varepsilon = \mathbf{x} + (a-1)\mathbf{y}_{n-1}$, and $\mathbf{y}_i\varepsilon = 0$ for all $\mathbf{y}_i$.

$\mathbf{x}\eta = \mathbf{y}_{n-1}\eta = \mathbf{x}$, and $\mathbf{y}_i\eta = 0$ for all $i < n-1$.

Then ε and η are idempotents, and $\varepsilon\eta = \alpha$. ⊣

2.8 Proposition. *Let R be a Hermite domain. If for all $m \leq n$ every $\mathcal{H}^*$-class in $\mathcal{D}^*_{m-1} \subseteq \mathrm{End}(R^m)$ contains a regular element that is a product of idempotents of $\mathrm{End}(R^m)$, then $K(n, n-1) = \langle E^n_1\rangle$.*

Proof: By induction on n. The case for $n = 2$ follows from Lemmas 2.6 and 2.7. Suppose, then, that $n \geq 3$, and that the result is true for $n-1$. Let $\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}, \mathbf{y}$ be a basis for R^n, and let A and B be the subspaces generated by $\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}$ and $\mathbf{y} - \mathbf{x}_1$ respectively. Then $A \oplus B = R^n$, so the $\mathcal{H}^*$-class $(B : A)$ contains an idempotent. By Lemma 2.6 it suffices to show that all elements in $(B : A)$ are products of idempotents (of rank $n-1$). Let $\alpha \in (B : A)$. Then $\mathbf{x}_1\alpha = \mathbf{y}\alpha$, and $\mathrm{Im}\,\alpha$ is generated by the sequence of linearly independent elements $\mathbf{x}_1\alpha, \ldots, \mathbf{x}_{n-1}\alpha$. Let $\beta \in \mathrm{End}(A) \cong \mathrm{End}(R^{n-1})$ be defined by $\mathbf{x}_1\beta = \mathbf{x}_2\alpha$, and $\mathbf{x}_i\beta = \mathbf{x}_i\alpha$ for all $i \geq 2$. Then $\mathrm{rk}\,\beta = n-2$, so, by induction, $\beta = \varepsilon_1 \cdot \ldots \cdot \varepsilon_k$ for some idempotents $\varepsilon_i \in \mathrm{End}(A)$ of rank $n-2$. Extend each ε_i to an idempotent $\varepsilon'_i \in \mathrm{End}(R^n)$ of rank $n-1$ by putting $\mathbf{y}\varepsilon'_i = \mathbf{y}$. Now define idempotents φ, η of rank $n-1$ by putting $\mathbf{y}\varphi = \mathbf{x}_1\varphi = \mathbf{y}$, and $\mathbf{x}_i\varphi = \mathbf{x}_i$ for all $i \geq 2$, and by putting $\mathbf{y}\eta = \mathbf{x}_1\alpha$, and $\mathbf{x}_i\eta = \mathbf{x}_i$ for all i. One easily verifies that $\alpha = \varphi\varepsilon'_1 \cdot \ldots \cdot \varepsilon'_k\eta$. So every element of $(B : A)$, hence every element of $K(n, n-1)$, is a product of idempotents of rank $n-1$. ⊣

2.9 Theorem. *The following conditions are equivalent for a Hermite domain R:*

1. *$K(m, m-1) = \langle E^m\rangle$ for all $m \leq n$;*
2. *$K(m, m-1) = \langle E^m_1\rangle$ for all $m \leq n$;*
3. *$K(m, r) = \langle E^m_{m-r}\rangle$ for all $m \leq n$ and $1 \leq r \leq m-1$;*
4. *For all $m \leq n$, every $\mathcal{H}^*$-class contained in $\mathcal{D}^*_{m-1} \subseteq \mathrm{End}(R^m)$ contains a regular element that is a product of idempotents.*
5. *For all $m \leq n$ the module R^m is weakly complementary, that is, all pairs of pure submodules $A, B \subseteq R^m$ such that rk $A = m-1$, rk $B = 1$, and $A \cap B = 0$, are weakly complementary.*
6. *Every invertible 2×2 matrix over R is a product of elementary matrices.*
7. *Every invertible matrix over R is a product of elementary matrices.*

PROOF: The equivalence of (1), (2), and (3) follows from Corollary 2.5. Clearly, (2) implies (4), and (5) follows from (4) by Corollary 2.2. Assume (5), and let

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

be an invertible matrix. If one of the coefficients is invertible or equal to 0, then A is a product of elementary matrices. So we may assume that p is nonzero and not invertible. Then $A = R(p,q)$ and $B = R(0,1)$ are pure submodules of R^2 of rank 1 such that $A \cap B = 0$. So there are vectors $(c_{1,1}, c_{1,2}), \ldots, (c_{k,1}, c_{k,2})$ and $(d_{1,1}, d_{1,2}), \ldots, (d_{k,1}, d_{k,2})$ such that $(p,q) = (c_{1,1}, c_{1,2})$ and $(0,1) = (d_{k,1}, d_{k,2})$, and

$$E_i = \begin{pmatrix} c_{i,1} & c_{i,2} \\ d_{i,1} & d_{i,2} \end{pmatrix} \text{ and } F_j = \begin{pmatrix} c_{j+1,1} & c_{j+1,2} \\ d_{j,1} & d_{j,2} \end{pmatrix}$$

are invertible matrices, for all $1 \leq i \leq k$ and $1 \leq j \leq k-1$. Set $F_0 = A$. There are matrices $P_1, \ldots, P_{k-1}$ and $Q_1, \ldots, Q_{k-1}$ such that $E_i = P_i F_i$ and $F_j = Q_j E_{j+1}$ for all $1 \leq i \leq k-1$ and $0 \leq j \leq k-1$. All P_i and Q_j, as well as E_k, have at least one 0 entry, hence are products of elementary matrices. So A is a product of elementary matrices.

The equivalence of (6) and (7) follows from Proposition 1.1. Assume (7). We wish to establish (2). Let $m > 0$, and let $(B : A)$ be an $\mathcal{H}^*$-class of $\mathcal{D}^*_{m-1} \subseteq \mathrm{End}(R^m)$. By Lemma 2.1 and Proposition 2.8 it suffices to show that A and B are weakly complementary. A and B are pure submodules of R^n with bases $\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}$ and $\mathbf{b}$ respectively. We may assume that $\mathbf{b} = (0, \ldots, 0, 1)$. There is an invertible matrix M such that $(\mathbf{a}_1, \ldots, \mathbf{a}_{n-1})^T$ forms the upper $(m-1) \times m$ submatrix of M. Then M is a product $E_k \ldots E_1$ of elementary matrices. Let $M_i = E_i \ldots E_1$. Write $M_i = (\mathbf{c}_{i,1}, \ldots, \mathbf{c}_{i,n-1}, \mathbf{d}_i)^T$. Let C_i and D_i be the submodules generated by $\mathbf{c}_{i,1}, \ldots, \mathbf{c}_{i,n-1}$ and $\mathbf{d}_i$ respectively. Then (C_i, D_i) and (C_{j+1}, D_j) are complementary pairs for all $0 < i < k$ and $0 \leq j \leq k-1$, and $B = D_0$ and $A = C_k$. So A and B are weakly complementary. ⊣

Examples of Hermite domains satisfying (6) include division rings and Euclidean domains, so Theorem 2.9 generalizes [Fountain 1991]. Since (6) doesn't depend on n, we may conclude from Theorem 2.9 that if R is a Hermite domain such that $K(2,1) = \langle E^2 \rangle$, then $K(n, n-1) = \langle E^n \rangle$ for all $n \geq 2$.

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