Using Markov chains to identify bifurcations in time series

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Domain and Features

- Scalar series being considered here – vector values also ok, but need much more data for that
- Discrete time
- Interested in cases where the usual approaches, ARIMA models and FFT, for instance, fail to describe the series in some sense – or one wishes to detect that the process generating the series has changed
- Would like an approach that would create a model that one could use to simulate other series with the same dynamic properties
- As few assumptions as possible – a nonparametric approach

Basic Assumptions

- Considering a time series, $\{x_n\}_{n=1}^{N}$, assume that the values are correlated – not independent as in a sequence of random numbers.
- The Markov assumption holds – only the current value is needed to determine the probability distribution for the next (order 1)
- There is sufficient data to approximate these probability distributions.

Approach

- Discretize the state space into *n* states. The choice of *n* is limited by the data available.
- Estimate 1 step transition probabilities by looking at relative frequencies.

$$p_{ij} = \left\{ \frac{\text{number of times that } x_{k+1} = j}{\text{total number of times that } x_k = i} \right\}$$

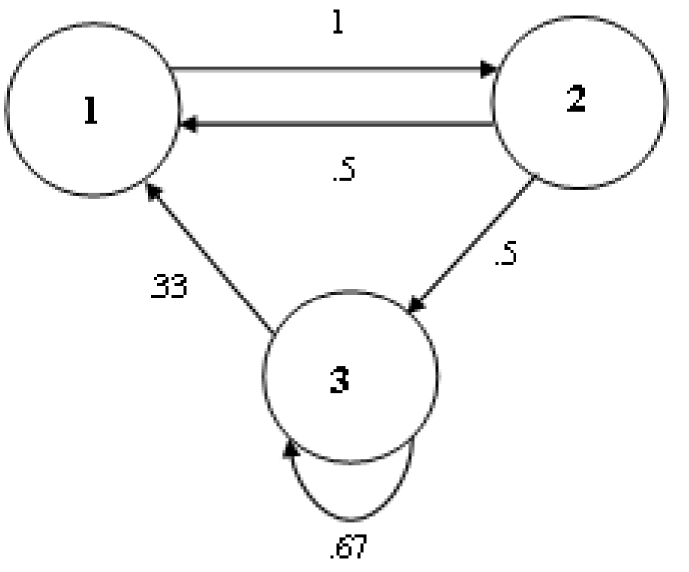
- Use the *nxn* matrix $P = \left(p_{ij} \right)$

for simulation and analysis

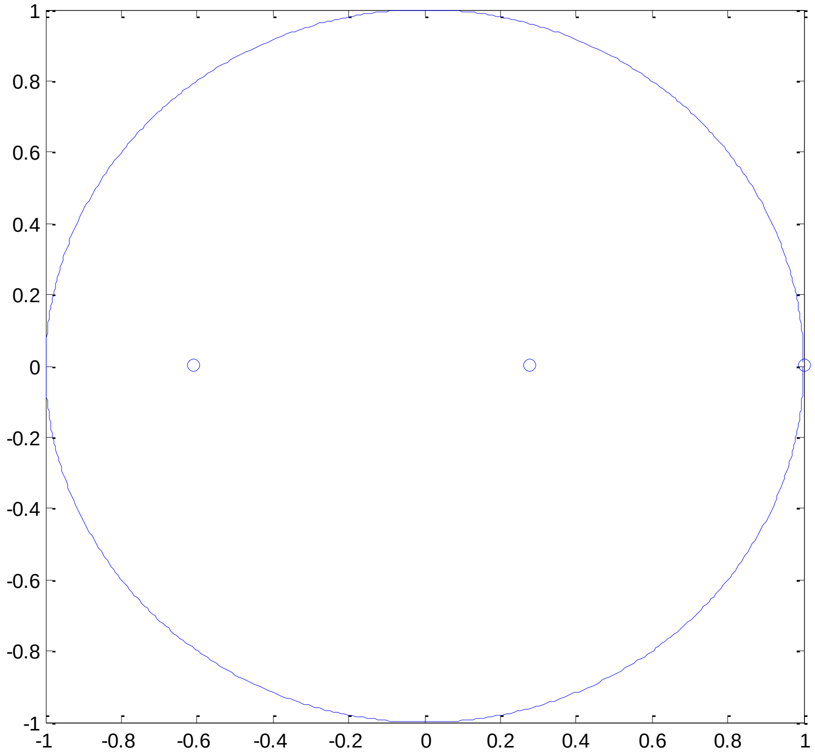
The Transition Matrix $\mathbf{P}$

- Current state -> row.
- Possible next state -> columns
- Each row represents a pdf given a current value in that state.

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ .5 & 0 & .5 \\ .33 & 0 & .67 \end{pmatrix}$$

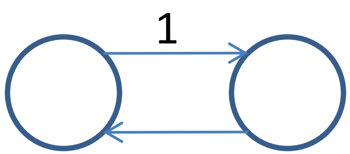


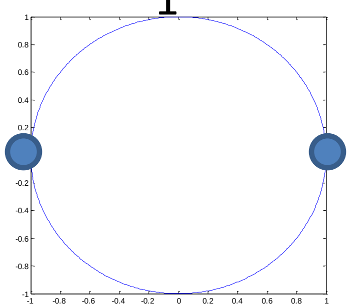
- Of interest will be the nature of the eigenvalues of $\mathbf{P}$. 1 is always an eigenvalue and all others are on or in the unit circle in the complex plane.

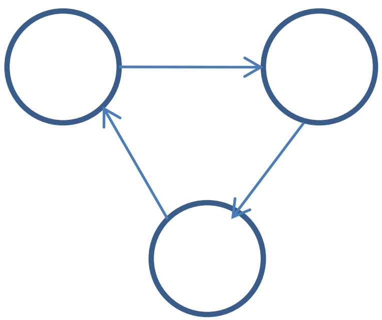


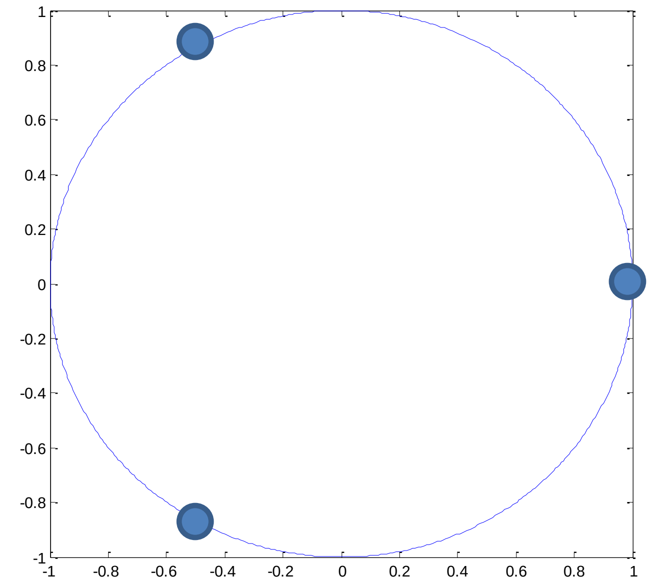
Meaning of the eigenvalues

- The eigenvalues describe the approach to a limit distribution (more on this later)
- The eigenvalues also describe the modes that are available – these depend on the nature of the digraph of transitions.









Simulation of Markov Chains

- Starting with an initial state, use the associated row of the transition matrix to determine the next row.
- Below are two sequences of length 10 generated from $\mathbf{P}$ starting at 1
- 1 2 3 1 2 3 1 2 3 3
- 1 2 1 2 3 3 3 1 2 3

Empirical Markov chains

- Now from data (a sample path of length 100 generated from $\mathbf{P}$), lets estimate $\mathbf{P}$ **(actual)** :

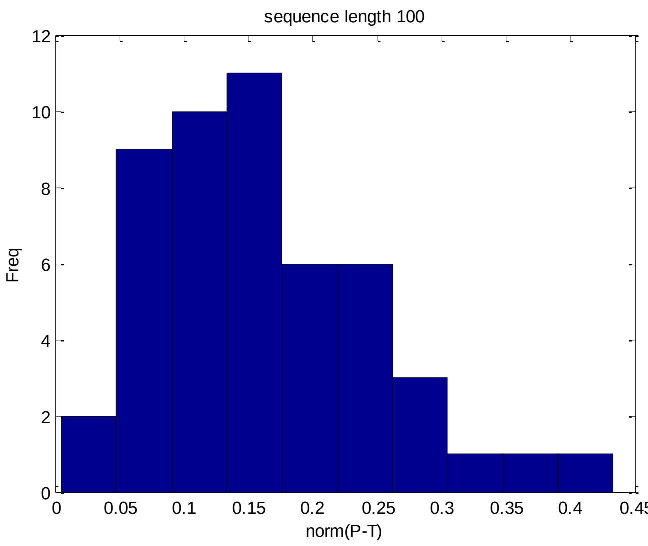
```
>> x=generate(1,P,100);
>> Dt=transi(x,3)
Dt =
        0        1.0000              0
0.5758 (.5)        0        0.4242 (.5)
0.3939 (.33)       0        0.6061 (.67)
```

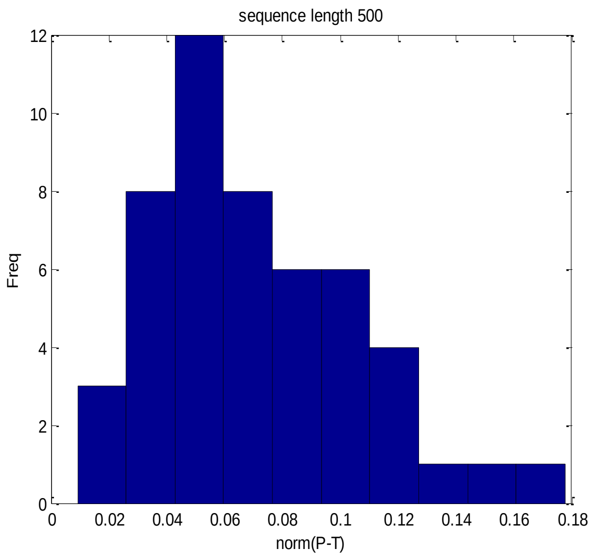
And the eigenvalues

```
>> eig(Dt)

ans =
   -0.6667
    1.0000
    0.2727

>> eig(P)

ans =
    -0.6091
     1.0000
     0.2791
```

What role does sequence length play?

- Look at the norm of the difference between the matrices as a function of sequence length

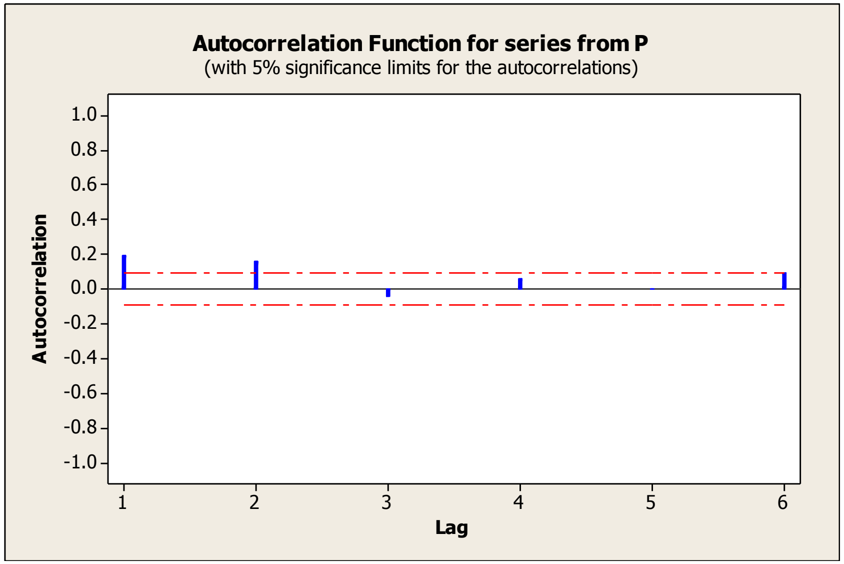


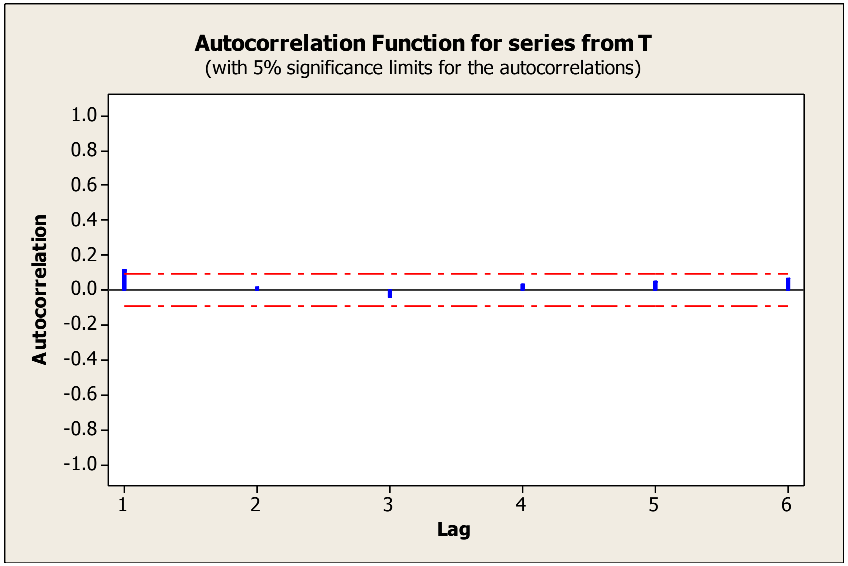


Distributions of norms of differences with different time series lengths to determine the approximate transition matrix.

Verifying the assumptions

- One way is to examine the autocorrelation function applied to the original sequence and a sequence generated from the empirical Markov chain. We have found using lags up to 5 or 6 is sufficient. As an example, we took a time series of length 500 from the original series and another of length 500 from the chain and computed the following results.


Autocorrelation Function for series from P
(with 5% significance limits for the autocorrelations)
Autocorrelation
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
1
2
3
4
5
6
Lag


Autocorrelation Function for series from T
(with 5% significance limits for the autocorrelations)
Autocorrelation
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
1
2
3
4
5
6
Lag

Detecting Bifurcations

- In this context, bifurcations are seen as a change in the nature of the eigenvalues of the transition matrix

→ the appearance or loss of complex eigenvalues

(change in the approach to the limit distribution)

→ the appearance of the roots of unity

(indicating new loops appearing)

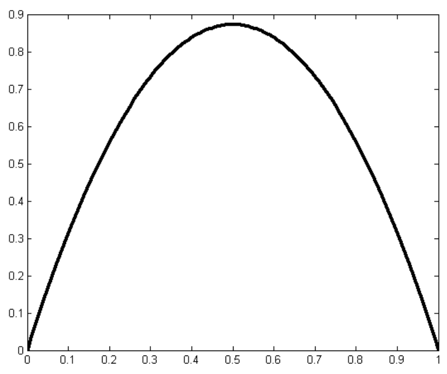
→ eigenvalues moving toward the unit circle

(a change is coming)

In general, bifurcation indicates a change in the topology of the digraph which determines $\mathbf{P}$

Bifurcation Example

- Bifurcation in the discrete logistic equation

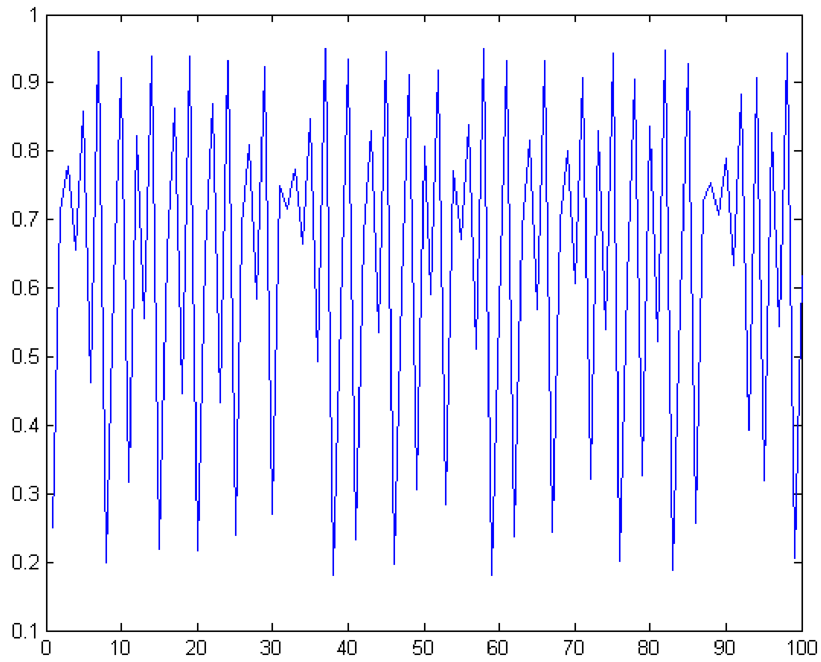


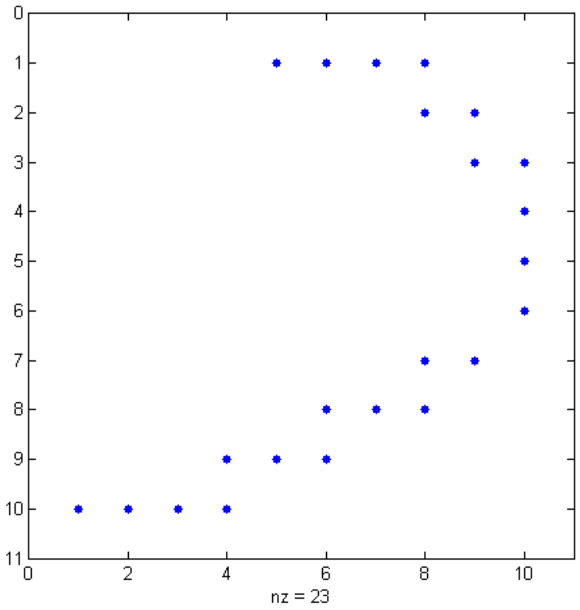
$$x_{n+1} = \mu\, x_n (1 - x_n) \quad 0 \le \mu \le 4$$

- Summary of behavior: $x = 0$ is always an equilibrium, stable if $\mu < 1$.

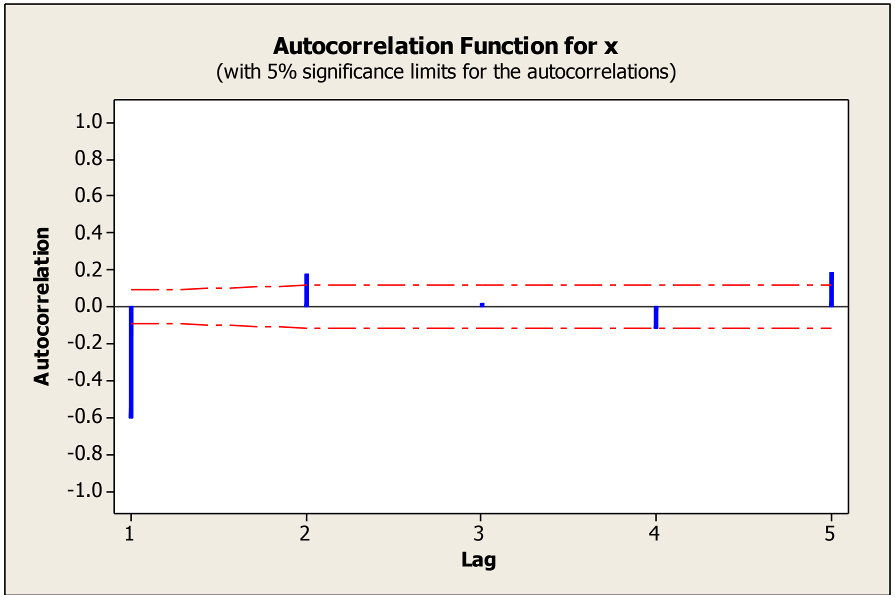
  $x = 1 - \dfrac{1}{\mu}$ is an equilibrium that is stable when $1 < \mu < 3$

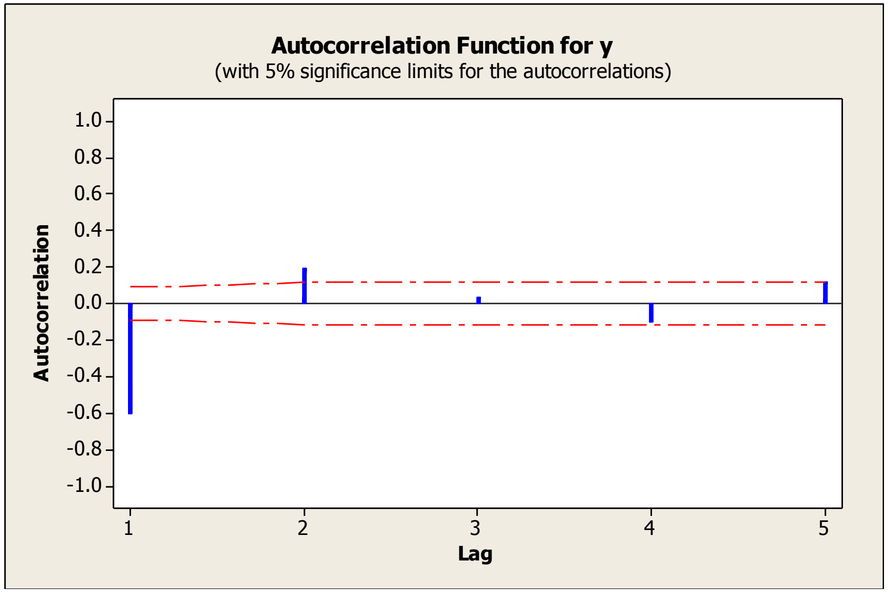
Transition Matrix $\mu = 3.8$





Nonzero entries in the transition matrix with 10 states (10x10 matrix)


Autocorrelation Function for x
(with 5% significance limits for the autocorrelations)
Autocorrelation
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
1
2
3
4
5
Lag


Autocorrelation Function for y
(with 5% significance limits for the autocorrelations)
Autocorrelation
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
1
2
3
4
5
Lag

Bifurcation to a period 2 point

- Transition matrix computed with 10 states at $\mu = 2.9, 3.0, \text{and } 3.05$ and eigenvalues computed
- At 2.9, {-.81, .81, 1, lots of 0’s}
- At 3.0, {-.92,.917, 1, lots of 0’s}
- At 3.05, {-1,1, lots of 0’s}
- From the above, it appears that a bifurcation took place just before 3.05. Moreover, because of the -1, it is a period 2 point.

The next bifurcation

- Transition matrix computed with 10 states at $\mu = 3.5,\ 3.503,\ \text{and}\ 3.6$ and eigenvalues computed
- At 3.5, {-1, 1, lots of 0’s}
- At 3.503, {-1,1, 0.0000 + 0.1010i, 0.0000 - 0.1010i, lots of 0’s}
- At 3.6, {-1,1, i, -i, lots of 0’s}
- From the above, it appears that a bifurcation took place just before 3.6. Moreover, because of the , it is a period 4 point.

Comments

- Similar to some applications Hidden Markov Models (HMM), although the dynamics get hidden in that process (it seems), and one can test to see if the description is faithful.
- There are some issues with the choices of state description (where the boundaries lie), the number of states, and the amount of data needed.

Typical Applications

- Changes in Heart Rate Variability (HRV)
- Cognitive changes with age
- Detection of epileptic seizure
- Bipolar disorder
- Dynamics of heart movement
- Testing effect of drugs (e.g. Parkinson’s)

References

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- Lancashire, I and Hirst, G. (2009) Vocabulary changes in Agatha Christie's mysteries as an indication of dementia: A case study, preprint.
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- Merrill, S.J. & Cochran, J.R. (1997) Markov chain methods in the analysis of heart rate variability, *Fields Inst Comm* **11**:241-252.
- Ratnakumar, S. (2008) Markov chain modeling of ECG gates live left atrial fluoroscopy variability to establish a well-defined basis for rigid registration to a 3D CT image*, Ph. D. Dissertation, Marquette University.*
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