Empirical Markov Chains as Models of Dynamic Processes

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Dynamic Modeling

- Constructing a mathematical model for a dynamic process can follow several paths
  - **First Principle Models** - where the "physics" of the system dictates the form of the equations.
    - Example : equation of motion of a spring based on Hooke's Law (restoring force proportional to amount of deformation) and F=ma or a birth-and-death process.
  - **Empirical Models** – where data or observations dictates the model within a class of possible models.
    - Examples : classical Box-Jenkins time series (ARIMA) models or models based on a catastrophe surface.

Deterministic / Stochastic

- Another decision is to whether the process is essentially deterministic (and stochastic effects can be ignored) or one in which stochastic effects are embedded in the dynamics.
  - The difference is usually in the assumption that $x_{n+1} = f(x_n)$ where the x_n's are measured with error (but that error does not affect the dynamics) or $x_{n+1} = f(x_n + \varepsilon_n)$ where it does.

Another Possibility

- Stochastic models can also be described where there is no such function *f*. In these cases, the probability of the next value is given by a probability density function depending on x_n.
  - Examples: The extreme case is where the sequence is a run of **random numbers** from the same distribution. Here, the distribution does not depend on the previous value.

    Markov chains, where the distributions are specified – in the empirical case by the history of the data values.

This Talk

- Brief description of Markov chains
- Estimating transition probabilities from a time series
- Applications of this technique
  - Heart rate variability
  - Cardiac imaging registration
  - Study of chaotic systems
- Summary and References

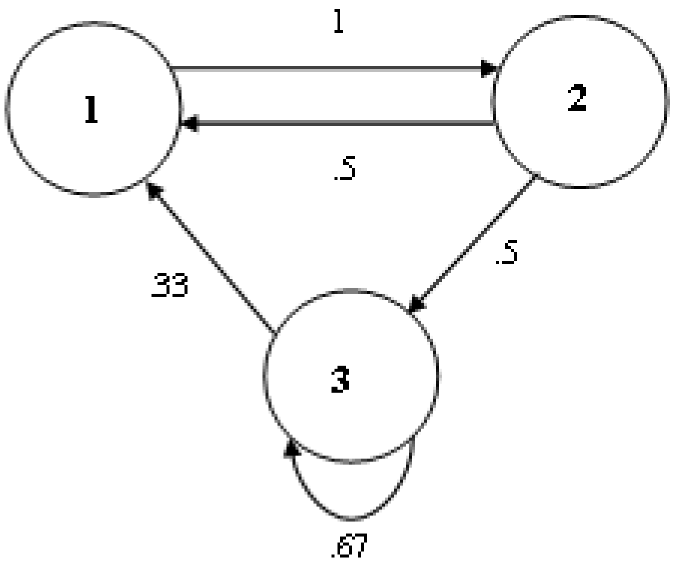
Markov Chain Model Assumptions

- **Discrete time – discrete state**
- **Order 1** (only the past state value determines the distribution) – this is the Markov assumption (previous history not relevant).
- If system is continuous state, **need to define what the states are** (think histogram).
- (for time homogeneous chains) The model is specified **by determining the transition probabilities** (by first principles or through data).

The Transition Matrix $\mathbf{P}$

- Current state -> row.
- Possible next state -> columns
- Each row represents a pdf given a current value in that state.

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ .5 & 0 & .5 \\ .33 & 0 & .67 \end{pmatrix}$$



Generating a time series from a transition matrix

- Using a Matlab m-file, starting with 1, 10 random numbers based on the matrix P can be generated.
- >> x=generate(1,P,10)

  x = 1 2 3 3 3 1 2 1 2 3
- Similarly, given a sequence of state values, P can be approximated.
- In the next slide, an AR(1) process generates a time series.

$$x_{n+1} = .5x_n + \varepsilon_n$$

Example of Empirical Transition Matrix

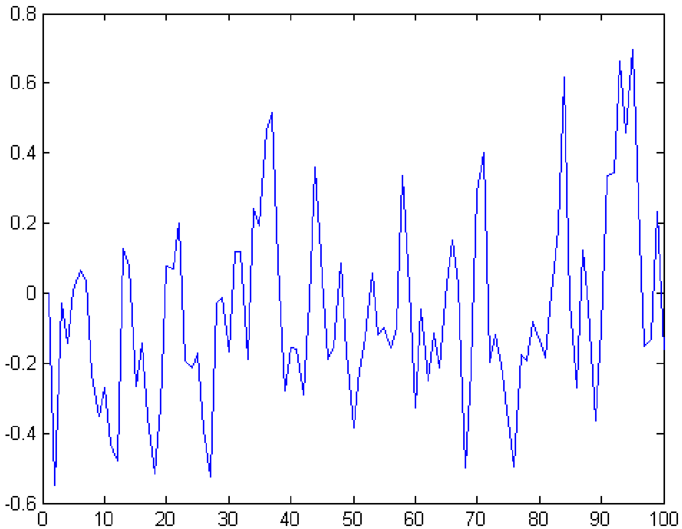
>> Dt=transi(d,5)

Dt =

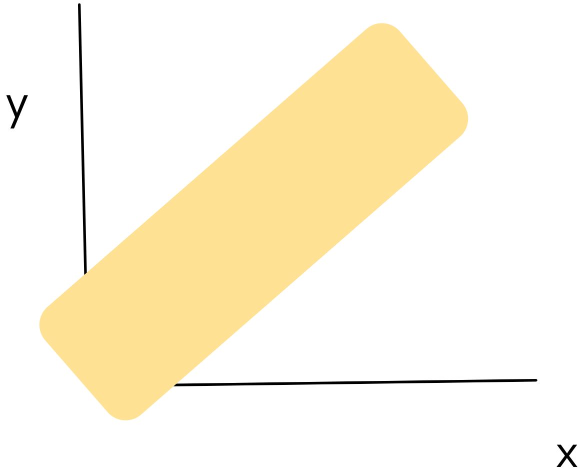
y →

x ↓

0.3333	0.3333	0.3333	0	0
0.1538	0.5128	0.2051	0.1282	0
0.1379	0.3793	0.3448	0.0690	0.0690
0	0.4000	0.3000	0.2000	0.1000
0	0	0.3333	0.1667	0.5000



Note that the pattern of nonzero values in the transition matrix suggests that the form of the relationship f

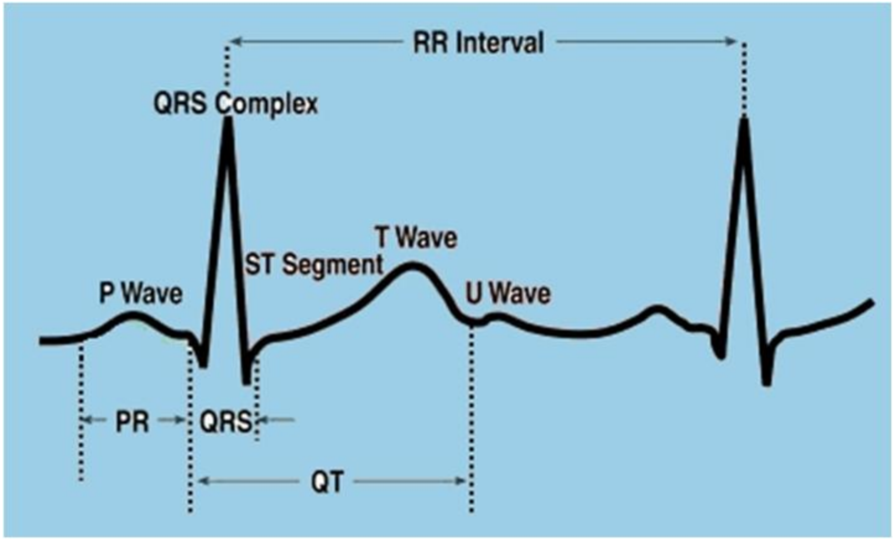


Heart Rate Variability

- Heart Rate Variability (HRV) describes the beat-to-beat variation in the time interval between beats as seen on ECG. It is described by many different indices.
- The variability is due to several different control mechanisms in the systems.
- Operation of the controls are affected by drugs (specifically here, anesthesia)

Motivation and Background

- Pediatric patients undergoing surgery
- The goal was to design a real-time monitor to anticipate sudden cardiac arrest . Data had been collected on several patients and standard indices did not behave well as measures of HRV in several patients



The HRV data

2.5 year old girl

7.5 year old boy -- early phase with halothane
late phase with the addition of atropine

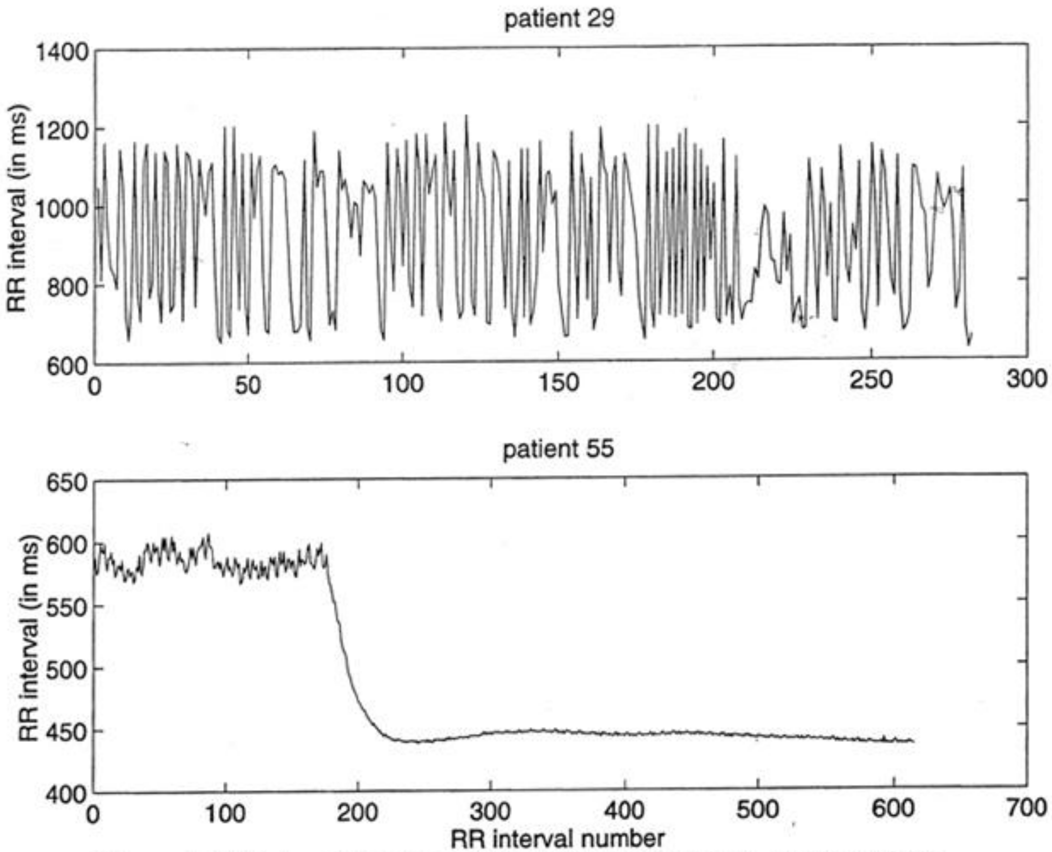


Figure 1 RR interval data from Patient 29 and Patient 55. In the analysis, the second set will be divided into an early and late phase.

Lag 1 Maps – HRV data

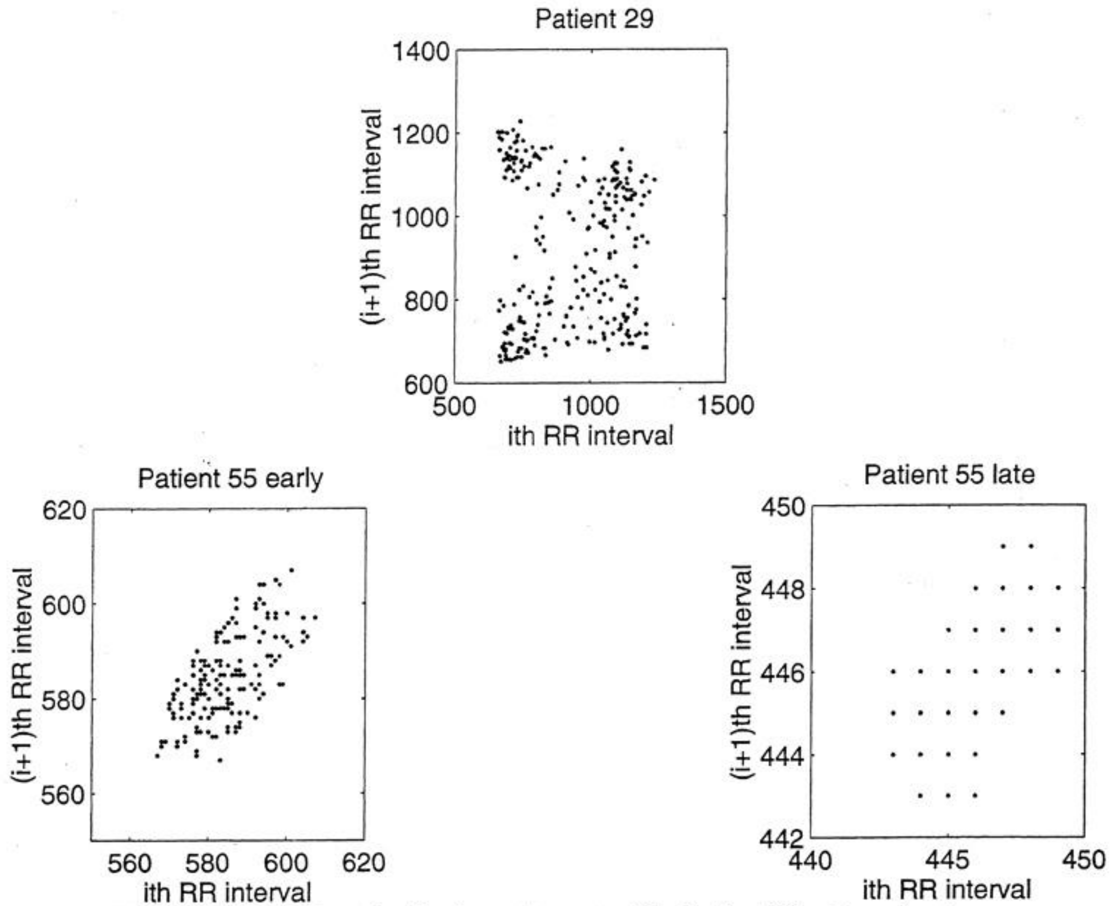


Figure 3 Lag 1 plots for the three data sets. The Patient 29 pattern has been described previously as a "complex pattern" (Woo et al. [1992]). Patient 55 early and late data plots would be classified as "torpedo patterns" by Woo et al. [1992].

Advantages/disadvantages of the Markov chain model

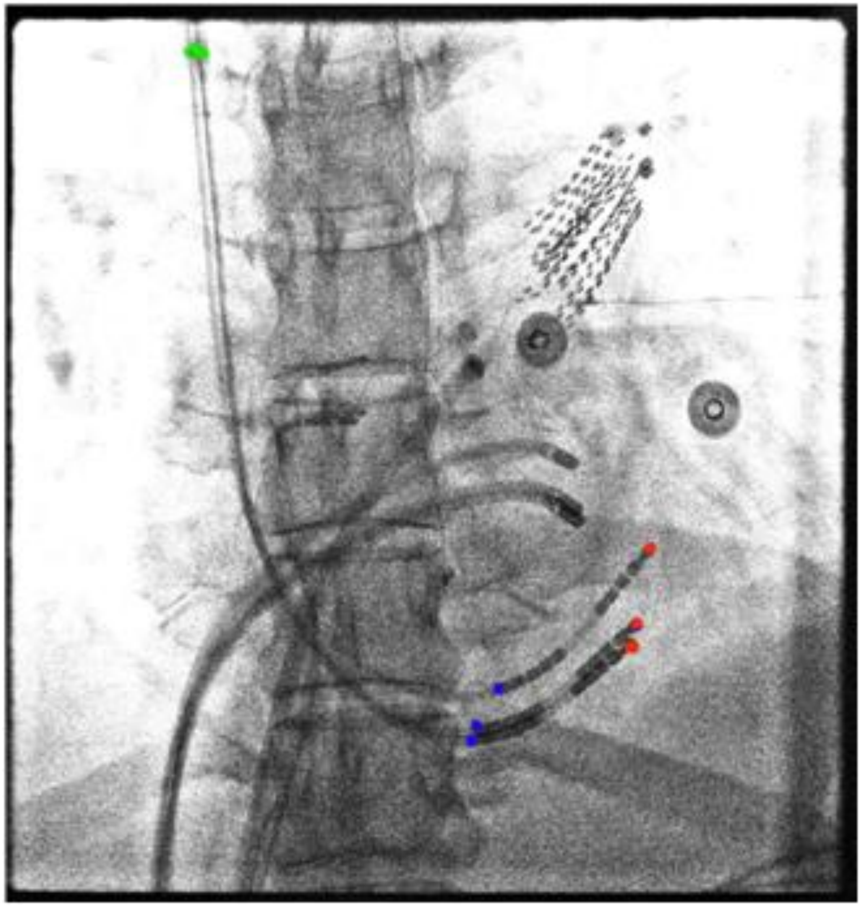
- Qualitative properties of the chain (e.g. eigenvalues or limit distribution) can be used to characterize the data set or identify changes in the data set.
- Simulation of the chain can result in an unlimited number of sample paths with the same dynamic behavior of the original set.
- Transition matrix depends on the definition of the "bins" and the number of them.
- N states requires estimating N^2 probabilities

Cardiac Image Registration

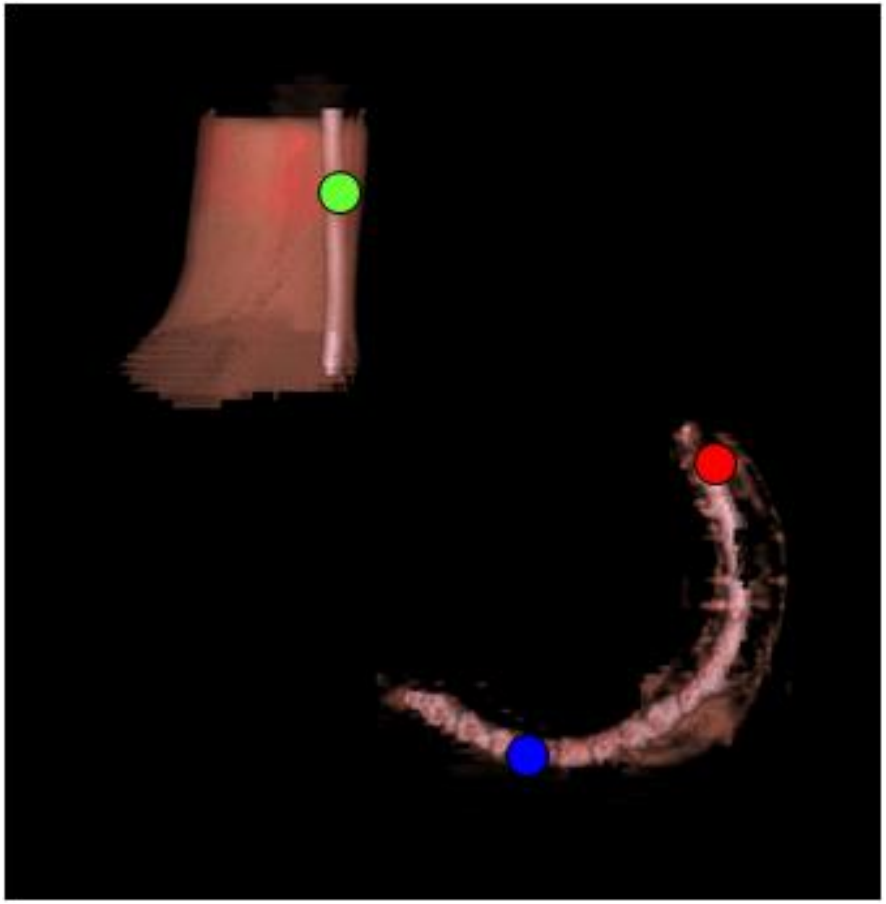
Shivani Ratnakumar

- Register 2-d (real-time fluoroscope) to 3-d (static 3-d CT) image to guide ablation catheter in treatment of atrial fribrillation.
- The selection of corresponding fiducial points is difficult in the moving image (even with gating for cardiac cycle and breathing)
- The chaotic movement of points in the heart make a model problematic. The thought was to create an empirical Markov chain to describe this movement.

The Registration Problem



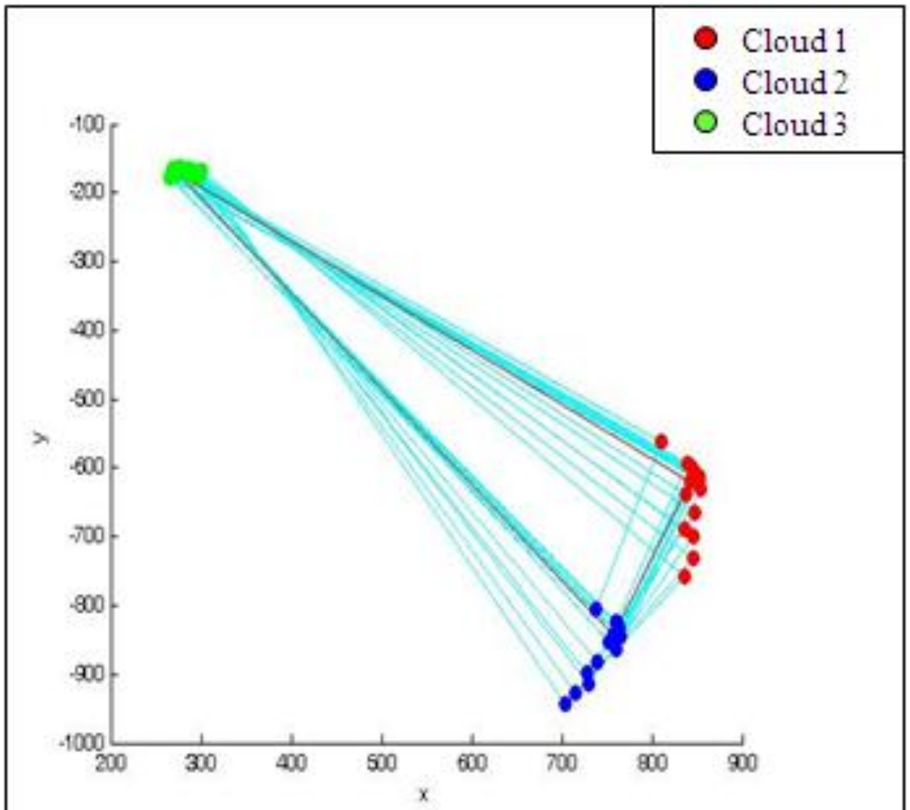
Three layered ECG gated fluoro frames



Segmented CT image that will be registered

Concentrating on the dynamics

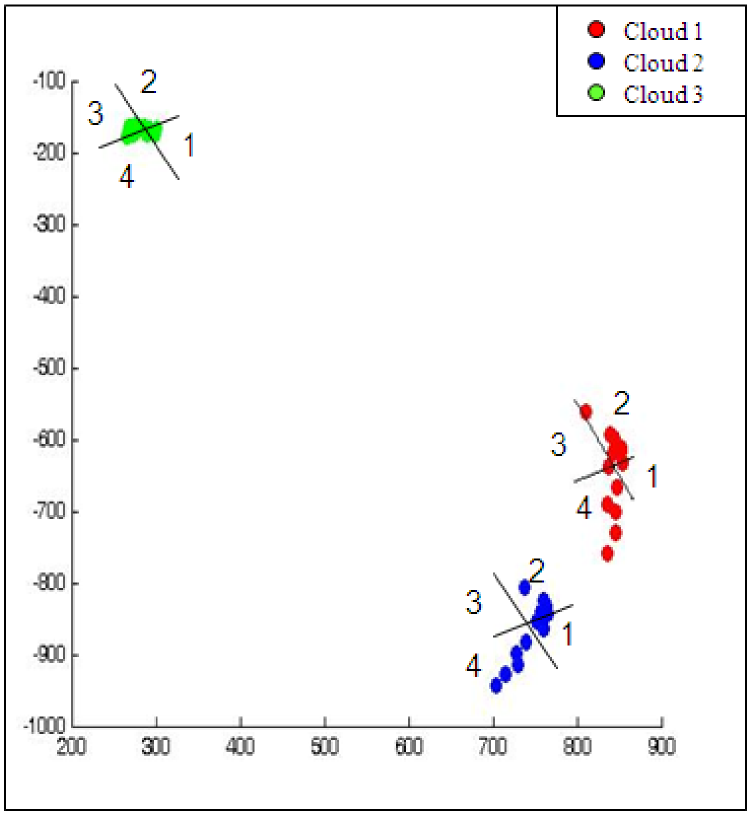
ECG gated fluoro data -- triples of points, each forming a "cloud"
2d coordinate locations of three points were recorded across fluoro sequences



$$
\begin{array}{ccc}
1 & 2 & 3
\end{array}
$$

$$
\begin{bmatrix}
(852,-618) & (766,-844) & (266,-177) \\
(810,-562) & (738,-806) & (268,-164) \\
\vdots & \vdots & \vdots \\
(848,-608) & (760,-836) & (273,-174)
\end{bmatrix}
$$

Discretize the state space



$$\begin{matrix} 1 & 2 & 3 \end{matrix}$$

$$\begin{bmatrix} 2 & 2 & 4 \\ 2 & 2 & 3 \\ \vdots & \vdots & \vdots \\ 2 & 2 & 4 \end{bmatrix}$$

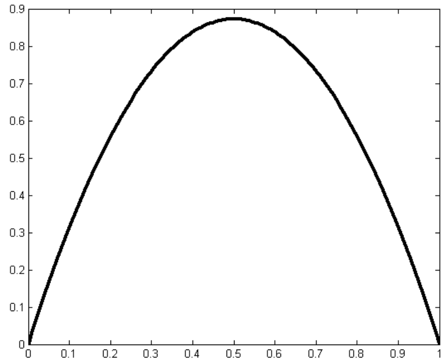
Sequence of coordinate points triplets can be written as a sequence of state triplets

Building a Transition Matrix

- **For each row**, there is a probability that a "1" in cloud 1 is associated with a "1" in cloud 2. Similarly for cloud 2 to cloud 3 and cloud3 to cloud 1 in the next row (next time). Compute these probability from the data (~30 rows).
- From these matrices, **compute cloud 1 to cloud 1 one step transition probabilities** (details omitted).
- **This 1 -> 1 matrix is a description of the motion of that area of the heart.**
- This was used (through the limit distribution of that matrix) to find a well-defined fiducial point.

Using Markov Chains to study chaotic systems

- Example: bifurcation in the discrete logistic equation:



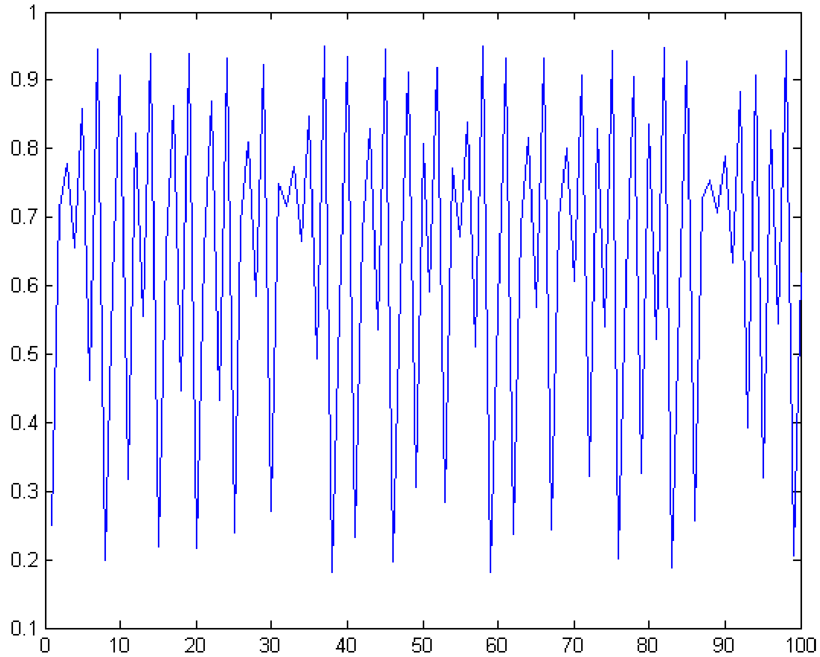
$$x_{n+1} = \mu\, x_n (1 - x_n) \quad 0 \le \mu \le 4$$

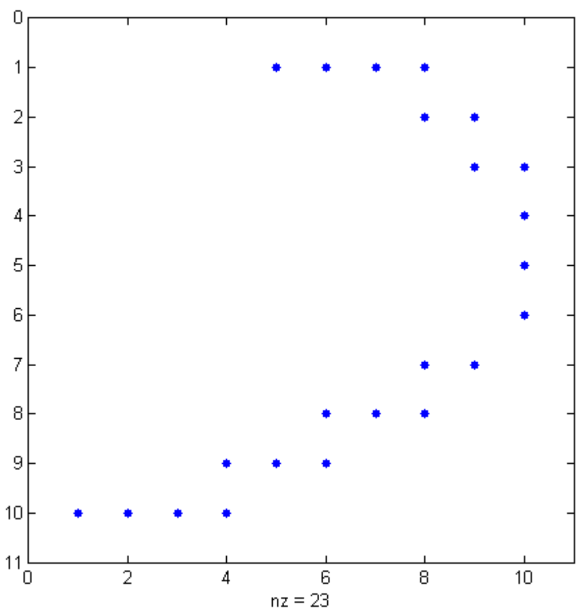
- Summary of behavior: $x = 0$ is always an equilibrium, stable if $\mu < 1$.

$x = 1 - \frac{1}{\mu}$ is an equilibrium that is stable when $1 < \mu < 3$

-- then the fun starts.

Transition matrix





Nonzero entries in the transition matrix with 10 states $\mu = 3.8$

Bifurcation in Markov Chains

- One can use the empirical chain to detect bifurcations. In this case, the bifurcation at $\mu = 3$ to a period 2 point. This is done through the eigenvalues of the transition matrix.
- Facts: 1 is always an eigenvalue and others are on or inside the unit circle. If 1 is the only eigenvalue of magnitude 1, there is a unique limit distribution.
- Loss of the unique limit distribution and establishment of a new one is the indication of a bifurcation in the system.
- As a parameter changes, look for **eigenvalue(s) approaching the unit circle**.

Bifurcation to a period 2 point

- Transition matrix computed with 10 states at $\mu = 2.9, 3.0, \text{and } 3.05$
- At 2.9, {-.81, .81, 1, lots of 0's}
- At 3.0, {-.92,.917, 1, lots of 0's}
- At 3.05, {-1,1, lots of 0's}
- From the above, it appears that a bifurcation took place just before 3.05. Moreover, because of the -1, it is a period 2 point.

Summary

- Empirical Markov chains can be employed without specifically knowing the (functional) nature of the dynamics.
- Having a Markov chain model enables one to simulated the process – generating new time series with the same properties as the original.
- The transition matrices especially the limit distribution and eigenvalues contain useful information on the process.

References

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