MATH 5510 LECTURE NOTES EIGENFUNCTIONS: DEFINITIONS AND EIGENVALUE PROBLEMS

Topics covered

- Boundary value problems
  - Definitions
  - Boundary conditions and adjoint boundary conditions
  - Differential operators and adjoint operators
  - Formally self adjoint vs. self adjoint
- Eigenvalue problems
  - Mechanics: solving eigenvalue problems
  - LCC and Cauchy-Euler equations (cases where solution is exact)
  - The canonical examples (Fourier basis sets come from operators)

Main goals

- Introduce boundary value problems and important definitions
- Start thinking in terms of linear operators]
- Mechanics: using integration by parts with inner products; boundary terms
- Mechanics: solving eigenvalue problems explicitly (LCC)
- Connect back to Fourier series (where do the Fourier basis sets come from?)

1. Boundary value problems: introduction

A **boundary value problem** (BVP) is a differential equation (DE) with constraints specfified at more than one boundary. As a simple example, consider

$$\frac{d^2u}{dx^2} = 1, \quad u(0) = 1,\, u(1) = 2. \tag{1}$$

This is fundamentally different from an 'initial value problem' (IVP), e.g.

$$\frac{d^2u}{dx^2} = 1 \quad u(0) = 1,\, u'(0) = 1$$

where all the 'boundary data' is at one point ($x = 0$ here).

For the BVP (1) , the solution to the ODE (by integrating twice) is

$$u = \frac{1}{2}x^2 + ax + b$$

and applying the boundary conditions we get

$$u(x) = 1 + (x^2 + x)/2.$$

However, this 'direct' approach will not get us far (most BVPs cannot just be integrated like this). For myriad reasons, we will need better methods. To develop them, we will recast the problem in terms of linear algebra. The first step is to introduce the right language.

Definitions: Recall that a **linear operator** L on a vector space V is a function $V \to V$ that is **linear**, i.e.

$$L(c_1 u_1 + c_2 u_2) = c_1 L u_1 + c_2 L u_2 \text{ for all scalars } c_1, c_2 \text{ and } u_1, u_2 \in V.$$

Here we care about linear operators L acting on functions $u : [a, b] \to \mathbb{R}$. The **domain** $[a, b]$ of the *functions* is essential. However, the space V on which the operator is defined (e.g. L^2 functions on $[a, b]$) is often not needed (we will be precise where it is necessary).

A **linear differential operator** involves derivatives of the input function, such as

$$Lu = x^2 \frac{d^2 u}{dx^2} + x \frac{du}{dx} + 2u$$

A **boundary value problem** has three parts:
- A **domain** e.g. $[a, b]$ (possibly infinite)
- A differential equation (DE)

$$Lu = f, \qquad x \in (a, b)$$

  for a function u defined on the domain (the function to solve for) and some linear differential operator L (just 'linear operator' for short).
- **Boundary conditions** (BCs) that are some relations for u and its derivatives at the boundaries of the domain (see definitions below).

The DE is **homogeneous** if $f = 0$ (so $Lu = 0$) and **inhomogeneous** otherwise.

Defintions (types of BCs) For a BVP in $[a, b]$, the a boundary condition will have the form

$$Bu = c$$

where B is linear (see below) and involves u and its derivatives at $x = a$ or $x = b$.
There are several important types of boundary conditions to note:
- **Dirichlet:** $u(a) = C$ (or $u(a) = 0$)
- **Neumann:** $u_x(a) = C$ (or $u_x(a) = 0$)
- **Robin:**[1] $\alpha u(a) + \beta u_x(a) = C$
- ***Periodic:** $u(a) = u(b), u'(a) = u'(b), \cdots$

The BCs are called **separated** if each equation involves only a or b (not both) and **mixed otherwise**. E.g. Dirichlet at a and Neumann at b is separated; periodic is mixed.

Note that all the BCs here are 'linear' in that they have the form $Bu = c$ where the 'boundary function' B satisfies $B(c_1 u_1 + c_2 u_2) = c_1 Bu_1 + c_2 Bu_2$.

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[1]The name is French, so it is pronounced a bit like 'row-beh'. They are also know as BCs of 'the third kind', with Dirichlet/Neumann being the first and second.

The example (1), in the notation outlined above, has
$$L = d^2u/dx^2, \quad f(x) = 1, \quad B_1 u = u(0), \quad B_2 u = u(1).$$
For convenience, we denote by Bu the 'boundary operator' that combines all the BCs, e.g.
$$u(0) = 0, \quad u(1) = 1 \implies Bu = (u(0), u(1)).$$
The point of this is to write statements like 'u satisfies the boundary conditions' as $Bu = 0$.

Notation (BCs in physics): The main types of BCs have many names:
- Dirichlet: 'BCs of the first kind'
- Neumann: 'BCs of the second kind' or 'flux BCs' (when it represents a specified flux through the boundary)
- Robin: 'BCs of the third kind' or 'radiation BCs'

The names Dirichlet/Neumann are by far the most common (at least in mathematics), but 'flux' is often used for Neumann when it makes physical sense.

2. Adjoint operators

A linear operator for a BVP has associated boundary conditions. Because the BCs are so important, we often think of the two together as one entity. To be precise, define:
- **Formal operator**: The operator L itself, acting on any function defined on the interval $[a, b]$. This is usually what is called the 'operator'.
- **complete operator:** The operator L along with the boundary conditions, acting on functions in $[a, b]$ that satisfy the boundary conditions.

We'll use the latter to indicate that L has specified BCs.

The **complete adjoint operator** is an operator L^* along with **adjoint boundary conditions** B^* such that
$$\langle Lu, v \rangle = \langle u, L^* v \rangle \text{ for all } u \text{ s.t. } Bu = 0 \text{ and } v \text{ s.t. } B^* v = 0$$
The **formal adjoint** is just L^*, which is the operator such that
$$\langle Lu, v \rangle = \langle u, L^* v \rangle + \text{boundary terms}$$
which will make more sense with examples below. We say:
- L is **formally self-adjoint** if $L = L^*$ (roughyl, self-adjoint ignoring BCs)
- **self-adjoint** if the formal operators and BCs are equal ($L = L^*$ and $B = B^*$)

Important note: The BCs and adjoint BCs are always **homogeneous**, i.e. in the form $Bu = 0$ and not $Bu = c$; there is no hope of having an adjoint at all in the latter case.

As an example (notation: $u_x = du/dx$), we compute the (complete) adjoint operator for
$$Lu = u_{xx} + p(x)u_x$$
in the domain $[0, 1]$ with boundary conditions
$$2u(0) - u'(0) = 0$$
$$u'(1) = 0. \tag{2}$$

The trick to use integration by parts to move **all derivatives from** u **onto** v and get

$$\langle Lu, v\rangle = \text{`BC terms'} + \int_0^1 u(\cdots)\, dx.$$

The integral is $\langle u, L^*v\rangle$ for some operator L^* that can be identified. Setting the BC terms to zero for all u will give the adjoint boundary conditions for v.

The details for the example go as follows. First, integrate by parts repeatedly:

$$
\begin{aligned}
\langle Lu, v\rangle &= \int_0^1 u_{xx}v\, dx + \int_0^1 p(x)u_x v\, dx \\
&= u_x v\Big|_0^1 - \int_0^1 u_x v_x\, dx + puv\Big|_0^1 - \int_0^1 u(pv)_x\, dx \\
&= (u_x v - uv_x + puv)\Big|_0^1 + \int_0^1 u(v_{xx} - (pv)_x)\, dx \\
&= \text{`BC terms'} + \langle u, L^*v\rangle
\end{aligned}
$$

For the $\langle u, L^*v\rangle$ part, the integral is $\int_0^1 u L^*v\, dx$ for

$$L^*v = v_{xx} - (pv)_x$$

which is the formal adjoint. Now the 'BC terms' must vanish given adjoint BCs to be found. Since the BCs are separated we may look at $x = 0$ and $x = 1$ part of the 'BC terms' separately. Use the BCs for u to simplify (replace $u'(0)$ with $2u(0)$):

$$
\begin{aligned}
(\text{BC term, } x = 0) &= u_x(0)v - u(0)v_x(0) + p(0)u(0)v(0) \\
&= u(0)(2v(0) - v_x(0) + p(0)v(0)).
\end{aligned}
$$

This must vanish for all values of $u(0)$ (since this is not specified at $x = 0$) so

$$(2 + p(0))v(0) - v_x(0) = 0.$$

Now do the same for the other boundary term:

$$
\begin{aligned}
(\text{BC term, } x = 1) &= u_x(1)v(1) - u(1)v_x(1) + p(1)u(1)v(1) \\
&= u(1)(-v_x(1) + p(1)v(1))
\end{aligned}
$$

which must vanish for all values of $u(1)$ so

$$p(1)v(1) - v_x(1) = 0.$$

To summarize, the original and adjoint operators are

$$
\begin{aligned}
Lu &= u_{xx} + p(x)u_x \\
L^*v &= v_{xx} - (p(x)v)_x
\end{aligned}
$$

and the original and adjoint boundary conditions are

$$
\begin{aligned}
2u(0) - u'(0) &= 0 \\
u'(1) &= 0.
\end{aligned}
\tag{BC}
$$

$$
\begin{aligned}
(2 + p(0))v(0) - v_x(0) &= 0 \\
p(1)v(1) - v_x(1) &= 0.
\end{aligned}
\tag{BC^*}
$$

This ensures that $\langle Lu, v\rangle = \langle u, L^*v\rangle$ for all u satisfying (BC) and v satisfying (BC^*).

Example (self-adjoint operator): We show that the complete operator
$$Lu = -(pu_x)_x + qu, \quad u(0) = u(1) = 0$$
is self-adjoint (where $p(x)$ and $q(x)$ are functions). Integrate by parts twice:
$$\langle Lu, v \rangle = -\int_0^1 (pu')_x v \, dx + \int_0^1 quv \, dx$$
$$= -pu_x v \Big|_0^1 + \int_0^1 pu_x v_x \, dx + \int_0^1 quv \, dx$$
$$= -pu_x v \Big|_0^1 + puv_x \Big|_0^1 - \int_0^1 u(pv_x)_x \, dx + \int_0^1 quv \, dx.$$
The integral is $\langle u, L^*v \rangle$ for the operator
$$L^*v = -(pv_x)_x + qv$$
so $L = L^*$ (formally self-adjoint). For the BCs the second term cancels, leaving
$$p(0)u_x(0)v(0) = 0 \implies v(0) = 0$$
$$p(1)u_x(1)v(1) = 0 \implies v(1) = 0$$
since the boundary terms must vanish for all values of $u_x(0)$ and $u_x(1)$ (both arbitrary). Thus the adjoint BCs are $v(0) = v(1) = 0$, the same as the BCs for L, so L is self-adjoint.

3. Eigenvalue problems: three familiar examples

An operator L in $[a, b]$ with **homogeneous** boundary conditions $Bu = 0$ has an associated **eigenvalue problem** to find an eigenfunction ϕ in $[a, b]$ and an eigenvalue λ such that
$$L\phi = \lambda\phi, \qquad B\phi = 0. \tag{3}$$

Procedure for eigenvalue problems: The general procedure for solving the eigenvalue problem (3) is

a) In each range of λ where the DE has a certain form, find the general solution
$$\phi = c_1\phi_1 + \cdots + c_n\phi_n$$
where n is the order of the DE, using standard ODE solving techniques.

b) Use the boundary conditions $B\phi = 0$ to get $\approx n$ equations for the c's (plus any other constraints relevant to the problem from elsewhere)

c) Find all λ such that there are non-trivial solutions (c's not all zero) and identify the eigenfunction (basis for each set of solutions to $L\phi = \lambda\phi$)

There are three standard examples. Consider the operator
$$Lu = -\frac{d^2u}{dx^2}$$
in $[0, \pi]$ with three different boundary conditions. The eigenfunctions should look familiar.

3.1. **Dirichlet BCs.** The eigenvalue problem for $\phi(x)$ is

$$-\phi'' = \lambda\phi, \ x \in (0, \pi), \qquad \phi(0) = 0, \quad \phi(\pi) = 0.$$

To solve it, find the general solution to the DE, then use the boundary conditions and look for values of λ such that a non-trivial solution exists.

Case 1: $\lambda < 0$. We will show here that no solutions exist. The characteristic polynomial is

$$r^2 + \lambda = 0.$$

which has roots $\pm\mu$ where $\mu = \sqrt{-\lambda}$. Then the general solution is

$$\phi = c_1 e^{x\mu} + c_2 e^{-x\mu}.$$

Apply the boundary conditions to get a system for coefficients c_1, c_2:

$$0 = c_1 + c_2, \quad 0 = c_1 e^{\pi\mu} + c_2 e^{-\pi\mu}.$$

This system, in matrix form, is

$$\begin{pmatrix} 1 & 1 \\ e^{\pi\mu} & e^{-\pi\mu} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It has a non-trivial solution if and only if the determinant is zero. But

$$\det(\cdots) = e^{-\pi\mu} - e^{\pi\mu} = 2\sinh(\pi\mu) > 0,$$

so there is only the trivial solution (i.e. $c_1 = c_2 = 0$ is the only solution) for all μ. We conclude there are no negative eigenvalues.

Case 2: $\lambda = 0$. The general solution is $\phi = c_1 x + c_2$. Applying the boundary conditions, we need $b = c_2$ and $c_1\pi + c_2 = 0$ which forces $c_1 = c_2 = 0$. So again, no eigenvalues in this case.

Case 3: $\lambda > 0$. The general solution is

$$\phi = c_1 \sin(\sqrt{\lambda}x) + c_2 \cos(\sqrt{\lambda}x).$$

From the boundary conditions,

$$c_2 \cos 0 = 0$$

$$c_1 \sin(\sqrt{\lambda}\pi) + c_2 \cos(\sqrt{\lambda}\pi) = 0$$

so $c_2 = 0$. To have a non-trivial solution $(c_1 \neq 0)$ we need

$$\sin(\sqrt{\lambda}\pi) = 0.$$

This has a non-trivial solution when

$$\sqrt{\lambda}\pi = n\pi, \qquad n = 1, 2, \cdots$$

i.e. for eigenvalues $\lambda_n = n^2$ (note that λ had to be positive to get here, so the $-$ root is discarded!). Plugging back into the general solution (recall that $c_2 = 0$) we obtain the corresponding eigenfunctions

$$\phi_n = \sin nx.$$

Summary: Collecting the results of the three cases, we gfind that only the $\lambda > 0$ case yields eigenvalues. The total set of eigenvalues/functions is

$$\lambda_n = n^2, \ \phi_n = \sin nx, \qquad n = 1, 2, 3, \cdots,$$

which is exactly the basis from the Fourier sine series.

3.2. **Neumann BCs.** The eigenvalue problem is

$$-\phi'' = \lambda\phi, \ x \in (0, \pi), \qquad \phi'(0) = 0, \quad \phi'(\pi) = 0.$$

Case 1: $\lambda < 0$. Again let $\mu = \sqrt{-\lambda}$. The general solution is the same as before, and the boundary conditions require (check this!)

$$0 = \sqrt{\mu}(c_1 - c_2), \qquad 0 = \sqrt{\mu}\left(c_1 e^{\pi\mu} - c_2 e^{-\pi\mu}\right).$$

Again write

$$\begin{pmatrix} 1 & -1 \\ e^{\pi\mu} & -e^{-\pi\mu} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The determinant is $-2\mu\sinh(\pi\mu) < 0$. Note that $e^{\pi\mu} > 1$ and $e^{-\pi\mu} < 1$ since $\mu > 0$. Thus there are still no solutions.

Case 2: $\lambda = 0$. The general solution is

$$\phi = c_1 x + c_2.$$

The boundary conditions require only that $c_1 = 0$, so $\phi = 1$ is an eigenfunction for $\lambda = 0$.

Case 3: $\lambda > 0$. The general solution is the same as before; the boundary conditions require

$$c_1 = 0, \qquad -\sqrt{\lambda}c_2 \sin(\sqrt{\lambda}\pi) = 0.$$

Thus we need $\sqrt{\lambda}\pi = n\pi$, so the eigenvalues are

$$\lambda_n = n^2, \quad n = 1, 2, \cdots$$

with corresponding eigenfunctions

$$\phi_n = \cos nx$$

So in summary, the eigenfunctions/values are

$$\lambda_n = n^2, \qquad \phi_n = \cos nx, \qquad n = 0, 1, 2, \cdots.$$

which are the basis functions for the Fourier cosine series. Note that $n = 0$ is included here (from Case 2), which is absent from the Dirichlet version.

3.3. **Periodic BCs.** Take the domain to be $[0, 2\pi]$ for reasons to be clear. The problem is
$$-\phi'' = \lambda\phi \; x \in (0, 2\pi), \quad \phi(0) = \phi(2\pi), \quad \phi'(0) = \phi'(2\pi).$$
First, if $\lambda = 0$ then
$$\phi = c_1 x + c_2.$$
The boundary conditions are satisfied if
$$c_2 = c_1(2\pi) + c_2 \implies c_1 = 0, \qquad c_1 = c_1$$
so $\phi = \text{const}$ is an eigenfunction for $\lambda = 0$.

If $\lambda < 0$ the solution is, with $\mu = \sqrt{-\lambda}$,
$$\phi = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$
From the boundary conditions,
$$c_1 + c_2 = c_1 e^{2\pi\mu} + c_2 e^{-2\pi\mu}, \quad \mu c_1 + \mu c_2 = c_1\mu e^{2\pi\mu} - c_2\mu e^{-2\pi\mu}.$$
This becomes the system
$$\begin{bmatrix} 1 - e^{2\pi\mu} & 1 - e^{-2\pi\mu} \\ 1 - e^{2\pi\mu} & 1 + e^{-2\pi\mu} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$
There is a solution if and only if the determinant is zero, so
$$0 = (1 - e^{2\pi\mu})(1 + e^{-2\pi\mu}) - (1 - e^{2\pi\mu})(1 - e^{-2\pi\mu}) = 2(1 - e^{2\pi\mu})e^{-2\pi\mu}.$$
Thus $e^{2\pi\mu} = 1 \implies \mu = 0$ but μ has to be positive (since $\lambda < 0$) so there are no solutions.

If $\lambda > 0$ then, setting $\mu = \sqrt{\lambda}$, the solution is
$$\phi = c_1 \cos \mu x + c_2 \sin \mu x.$$
From the boundary conditions,
$$c_1 = c_1 \cos 2\pi\mu + c_2 \sin 2\pi\mu, \qquad c_2\mu = -\mu c_1 \sin 2\pi\mu + \mu c_2 \cos 2\pi\mu$$
As a system:
$$\begin{bmatrix} 1 - \cos(2\pi\mu) & -\sin(2\pi\mu) \\ \sin 2\pi\mu & 1 - \cos 2\pi\mu \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4}$$
Set the determinant to zero:
$$0 = (1 - \cos(2\pi\mu))^2 + \sin^2(2\pi\mu) = 2 - 2\cos(2\pi\mu).$$
Thus there is a solution for
$$\cos(2\pi\mu) = 1$$
which occurs for the values $\mu_n = n$ for $n = 1, 2, \cdots$.
   Notice that for such values, (4) becomes trivial and any c_1 and c_2 will work, so
$$\phi = c_1 \cos nx + c_2 \sin nx$$
is a solution for all $n \geq 1$ with eigenvalue $\lambda_n = n^2$. This means we have eigenfunctions
$$\phi_n = \cos nx, \quad \psi_n = \sin nx \text{ with } \lambda_n = n^2 \text{ for } n = 1, 2, \cdots$$
where ϕ_n and ψ_n are both eigenfunctions for λ_n. This is, of course, the basis for the full Fourier series (computed on $[0, 2\pi]$ rather than $[-\pi, \pi]$, but it is the same up to this translation).

3.4. **Cauchy-Euler equations.** Other than constant coefficient linear equations, there is one other useful type of ODE we can solve exactly: an **Euler equation** (or 'Cauchy-Euler' or 'equidimensional') ODE, which has the form

$$Lu := ax^2\frac{d^2u}{dx^2} + bx\frac{du}{dx} + cu = 0. \tag{5}$$

Assume that $x > 0$ for simpilicity. We see two basis solutions that span the solution set. To find them, first look for solutions of the form $u = x^r$. We get

$$L[x^r] = (ar(r-1) + br + c)x^r$$

so it follows that

$$x^r \text{ is a soln} \iff p(r) = ar(r-1) + br + c = 0.$$

If p has two real roots r_1, r_2 then the general solution is the span of the two solutions,

$$u(t) = c_1 x^{r_1} + c_2 x^{r_2}.$$

If p has complex roots $r = s \pm \omega i$ then write

$$x^r = e^{r\log x} = x^s e^{\omega i \log x} = x^s \left(\cos \omega \log x + i \sin \omega \log x \right).$$

Taking real/imaginary parts gives the two basis solutions so the general solution is

$$x = x^s \left(c_1 \cos \omega \log x + c_2 \sin \omega \log x \right).$$

Finally, if p has a repeated root r then the general solution is

$$x = x^r \left(c_1 + c_2 \log x \right).$$

That is, multiply the first solution by $\log x$ to get the second.

The 'eigenvalue problems' are solved the same way as the previous examples, now using this general solution. The procedure also works for higher-order 'equidimensional' equations, although the repeated roots case may need to be generalized if there is a triple root.