

MATH 5510 LECTURE NOTES

EIGENFUNCTIONS THE METHOD; SL THEORY

TOPICS COVERED

- Solving BVPs using eigenfunctions
 - Mechanics: the main solution procedure
 - Time dependent problems (a first look)
- Sturm-Liouville theory
 - Sturm-Liouville problems (definitions)
 - Main theorem for self-adjoint L ; weighted inner products
 - Regular and singular Sturm-Liouville problems
 - Tools: Rayleigh quotients, showing $\lambda > 0$
 - Much more to come on SL theory...

1. BOUNDARY VALUE PROBLEMS: THE METHOD

Here the procedure is detailed for solving

$$\begin{aligned}Lu &= f, & x \in (a, b) \\ B_a u &= c_1, & B_b u = c_2\end{aligned}$$

where B_a and B_b are separated BCs, with adjoint operator L^* and adjoint BCs B_a^*, B_b^* . If the operator is self-adjoint, note that you can simply skip calculating adjoints and replace the adjoint eigenfunctions ψ_n with ϕ_n . Most problems in practice will be self-adjoint.

Step 1: Get the eigenvalues and eigenfunctions (find the basis).

1a) Solve the eigenvalue problem for L with **homogeneous** BCs:

$$L\phi = \lambda\phi, \quad B_a\phi = 0, \quad B_b\phi = 0$$

for eigenvalues λ_j and eigenfunctions ϕ_j .

1b) Solve the eigenvalue problem for the adjoint L^* with **homogeneous** (adjoint) BCs:

$$L^*\psi = \lambda\psi, \quad B_a^*\psi = 0, \quad B_b^*\psi = 0$$

for adjoint eigenfunctions ψ_j (if eigenvalues are real, they are the same).

Step 2: Find the solution using eigenfunction expansion

2a) Take inner product of **inhomogeneous** DE with ψ_k and integrate by parts, **using the inhomogeneous BCs** on the boundary terms.

2b) Write u in terms of the eigenfunction basis

$$u = \sum_n c_n \phi_n,$$

plug into projected equation from (2a) and get equations for c_k (use bi-orthogonality).

2c) Solve (easy) equations for c_k .

Procedure for Step 2 (details): Once we have the eigenfunctions/values, project onto the n -th basis function:

$$\langle Lu, \psi_n \rangle = \langle f, \psi_n \rangle$$

The RHS is fine as is. For the LHS, note that u satisfies the **inhomogeneous** BCs. Integrate by parts (as in deriving the adjoint) to get

$$\begin{aligned} \langle Lu, \psi_n \rangle &= \beta_n + \langle u, L\psi_n \rangle \\ &= \beta_n + \lambda_n \langle u, \psi_n \rangle \end{aligned}$$

where β_n will be some expression left over from the boundary terms (that do not vanish since u has *inhomogeneous BCs* unlike the adjoint calculation!).

Now we write u in terms of the eigenfunction basis (assuming it is a basis)

$$u = \sum_n c_n \phi_n$$

so the equations become

$$\beta_n + c_n \lambda_n \langle \phi_n, \psi_n \rangle = \langle f, \psi_n \rangle \implies c_n = \frac{\langle f, \psi_n \rangle - \beta_n}{\lambda_n \langle \phi_n, \psi_n \rangle}.$$

The following example illustrates the process. Step (1a) and (2a) are the most involved. When L is self-adjoint, the process is obviously simpler.

1.1. Example with inhom BCs. Note that L is self-adjoint here but we leave the ‘adjoint’ eigenfunctions as ψ_j to illustrate the point that ‘self-orthogonality’ in the self-adjoint case and ‘bi-orthogonality’ in the general case work the same way.

Consider the DE

$$-\frac{d^2 u}{dx^2} = e^x, \quad u(0) = 1, \quad u(1) = 2;$$

Let $Lu = -d^2 u/dx^2$. Finding the eigenvalues is the same as before; the result is

$$\lambda_n = \pi^2 n^2, \quad \phi_n = \sin(n\pi x), \quad n \geq 1. \quad (1.1)$$

The adjoint operator is $L^* u = -d^2 u/dx^2$ with adjoint BCs $u(0) = u(1) = 0$ (the complete operator L is self-adjoint). Thus the eigenfunctions of L^* are

$$\psi_n = \sin(n\pi x), \quad n \geq 1.$$

Now let u be the solution and take the inner product with ψ_k :

$$\langle Lu, \psi_k \rangle = \langle e^x, \psi_k \rangle.$$

For the LHS, we need to integrate by parts:

$$\begin{aligned}\langle Lu, \psi_k \rangle &= - \int_0^1 u_{xx} \psi_k dx \\ &= (u\psi'_k - u_x\psi_k) \Big|_0^1 + \langle u, L^* \psi_k \rangle \\ &= (u\psi'_k - u_x\psi_k) \Big|_0^1 + \lambda_k \langle u, \psi_k \rangle\end{aligned}$$

again by calculations from before. Now we know $u(0) = 1, u(1) = 2$ and

$$\psi'_k(0) = \pi k, \quad \psi'_k(1) = \pi k(-1)^k, \quad \psi_k(0) = \psi_k(1) = 0$$

so evaluating the boundary terms, we get

$$\langle Lu, \psi_k \rangle = 2\pi k(-1)^k - \pi k + \lambda_k \langle u, \psi_k \rangle.$$

The eigenfunction series for the solution is

$$u = \sum_{n=1}^{\infty} c_n \phi_n$$

so plugging into the projected DE we get

$$\pi(2(-1)^k - 1) + \lambda_k c_k \langle \phi_k, \psi_k \rangle = \langle e^x, \psi_k \rangle$$

and solving for c_k gives

$$c_k = \frac{\langle e^x, \psi_k \rangle - \pi k(2(-1)^k - 1)}{\pi^2 k^2 \langle \phi_k, \psi_k \rangle} \quad (1.2)$$

where we compute

$$\langle e^x, \psi_k \rangle = \int_0^1 e^x \sin k\pi x dx = \frac{\pi k}{\pi^2 k^2 + 1} (1 - \cos \pi k), \quad \langle \phi_k, \psi_k \rangle = \int_0^1 \sin^2 k\pi x dx = 1/2.$$

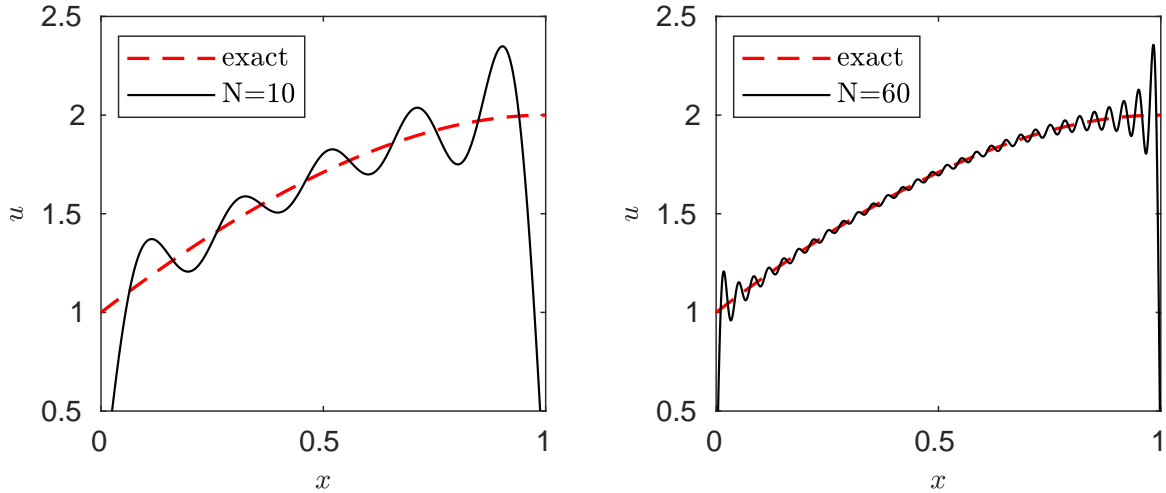
To summarize, the solution to the BVP is, in terms of the eigenfunction basis,

$$u = \sum_{n=1}^{\infty} c_n \sin n\pi x, \quad c_n \text{ given by (1.2)} \quad (1.3)$$

and ϕ_n, λ_n given by (1.1) and $\psi_n = \phi_n$. One could simplify the c_n 's a bit, but the above is a complete answer and substituting in all the expressions does not make the solution nicer. A plot of the exact solution (obtained just by integrating the equation)

$$u(x) = ex + 2 - e^x$$

against the series with $N = 10$ and $N = 60$ terms is shown below. Notice that this sine series (1.3) evaluates to zero at the endpoints since the eigenfunctions ϕ_n were constructed to satisfy the homogeneous BCs ($\phi(0) = \phi(1) = 0$). This is not a paradox; the series solution converges to $u(x)$ pointwise everywhere except $x = 0$ and $x = 1$; it exhibits Gibbs' phenomenon. We'll have more to say about this shortly.



1.2. **Remark on solvability.** What happens if $\lambda_k = 0$ for some k ? Suppose the DE is

$$-\frac{d^2u}{dx^2} = f, \quad u'(0) = 3, \quad u'(1) = 0$$

with Neumann BCs at both ends. There is an eigenvalue $\lambda_0 = 0$ with eigenfunction $\phi_0 = 1$ (and $\psi_0 = 1$); taking the inner product with $\psi_k = \cos k\pi x$, we get

$$\langle Lu, \psi_k \rangle = 3\psi_k(0) + \lambda_k \langle u, \psi_k \rangle$$

which leads to the formula

$$c_k = \frac{\langle f, \psi_k \rangle - 3}{\lambda_k \langle \psi_k, \phi_k \rangle} \text{ for } k \geq 1.$$

However, for $k = 0$ the projected DE becomes

$$3 = \langle f, \psi_0 \rangle = \int_0^1 f(x) dx.$$

The coefficient c_0 is **arbitrary**, and the RHS $f(x)$ must satisfy the condition above (a **solvability condition**) to have a solution. We'll generalize this observation later; for now, note that if u is a solution then $u + \text{const.}$ is a solution for any constant, so this result makes sense.

Remark (what's the benefit?): The eigenfunction series

$$u = \sum_j c_j \phi_j$$

will be a powerful tool for solving BVPs and partial differential equations. However, as we saw with Fourier series, differentiation can be a problem. Moreover, we often cannot differentiate u directly, or there are discontinuous functions in the DE (e.g. a force that is suddenly applied and then stopped).

The 'integration by parts' avoids ever differentiating the series, so we solve a version of the DE that *does not require the solution to be differentiable* - this is called the 'weak form', and is critical, for example, in understanding the finite element method in numerics.

2. INTERLUDE

2.1. **A motivating example.** We can use this method to solve **time dependent** partial differential equations (PDEs). The **heat equation** in one dimension is the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for a function $u(x, t)$. The heat equation describes **diffusion** processes, such as the heat distribution in a space, diffusion of particles in air and much more. The method of the previous section works here, except that c_n 's are now functions of t .

For example, consider the heat equation on $[0, \pi]$ with Dirichlet boundary conditions:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, & x \in (0, \pi), & t > 0 \\ u(0, t) &= g_1(t), & u(\pi, t) &= g_2(t) \\ u(x, 0) &= f(x) \end{aligned}$$

This models, for instance, the temperature in a metal bar where the values at the endpoints are controlled, and the initial temperature at $t = 0$ is $f(x)$. Defining $Lu = -u_{xx}$, we have

$$u_t = -Lu, \quad x \in (0, \pi), \quad t > 0.$$

The operator L is the same as before; the eigenfunctions/values are

$$\phi_n(x) = \sin nx, \quad \lambda_n = n^2, \quad n = 1, 2, \dots$$

Since $u(x, t)$ at each t is an L^2 function of x , it can be expressed in the eigenfunction basis:

$$u(x, t) = \sum_{n=1}^{\infty} c_n(t) \phi_n(x).$$

Projecting the PDE $u_t = -Lu$ onto ϕ_n , we get

$$\begin{aligned} \langle u_t, \phi_n \rangle &= -\langle Lu, \phi_n \rangle \\ &= \beta_n(t) - \langle u, L\phi_n \rangle \\ &= \beta_n(t) - \lambda_n \langle u, \phi_n \rangle \end{aligned}$$

where $\beta_n(t)$ comes from the boundary term of IBP; it depends on t since the BCs do as well. Now we can differentiate the series for u in t and project:

$$u_t = \sum_{n=1}^{\infty} c'_n(t) \phi_n(x) \implies \langle u_t, \phi_n \rangle = c'_n(t)$$

so the equations for the coefficients are now **ODEs in t** given by

$$c'_n(t) = \beta_n(t) - \lambda_n c_n(t).$$

What do eigenvalues mean? Now suppose $\beta_n = 0$. The solution for the c_n 's is

$$c_n(t) = c_n(0) e^{-\lambda_n t}$$

so the eigenvalues determines the **rate at which the system approaches equilibrium**. In particular, the **smallest eigenvalue** should give the rate, since we expect $u(x, t) \sim c_1(t) \phi_1$. This makes sense, because the diffusing heat should approach a steady state.

2.2. **Where are we going with this?** The sketch above and the contrast between linear algebra in \mathbb{R}^n and functions in L^2 raises some key questions that will motivate the topics to come. There are some equivalences, and many questions left to answer:

vectors in \mathbb{R}^n (or \mathbb{C}^n)	\iff	functions in $L^2[a, b]$
$n \times n$ matrices	\iff	linear operators L
$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i$	\iff	$\langle f, g \rangle = \int_a^b f(x)g(x) dx$
symmetric matrices	\iff	self-adjoint operators
linear systems $Ax = b$	\iff	linear DEs $Lu = f$
eigenvector basis	\iff	eigenfunction basis (series)
spectral thm: $L = L^* \rightarrow \{\phi_j\}$	\iff	*Sturm-Liouville theory

- **What is the right operator?** We want an orthogonal basis of eigenvectors for some linear operator L . This means identifying the right operator and understanding when it will do what we want.
- **When does an operator have nice eigenfunctions?** Self-adjoint matrices give a basis of eigenvectors. When is this true of operators in L^2 ? This is answered by Sturm-Liouville theory (coming up next).
- **Consequences of infinite dimensional basis?** The basis for the function space is infinite dimensional - so convergence is more subtle. Moreover, manipulating infinite series (e.g. differentiation) requires care. This has theoretical and practical consequences (like Gibbs' phenomenon). We will need the right analytical tools to derive solutions
- **What are the properties of the eigenvalues/functions?** To obtain and interpret solutions, we need to be able to solve the eigenvalue problem $L\phi = \lambda\phi$ (which is an ODE) and know the key properties of the λ 's and ϕ 's. Because they are solutions to an ODE, this can take some work.

3. STURM-LIOUVILLE THEORY

We now restrict our attention to **second-order** linear operators, which have a particularly nice structure. The classical theory of these operators is called ‘Sturm-Liouville theory’, and gives all the structure we need to solve important BVPs and PDEs (e.g. the heat equation).

Suggestion: When reviewing the theory, keep in mind an example like the Dirichlet problem

$$-\phi'' = \lambda\phi, \quad \phi(0) = \phi(\pi) = 0 \implies \phi_n = \sin nx, \quad \lambda_n = n^2.$$

In a sense, Sturm-Liouville theory says that solutions to the more general eigenvalue problems ‘behave like’ these standard examples, with some exceptions.

Definition: A **Sturm-Liouville (SL) operator** in $[a, b]$ has the form

$$Lu = -(p(x)u_x)_x + q(x)u \quad (3.1)$$

where $p(x) \geq 0$. A **Sturm-Liouville** eigenvalue problem (SLP) for a SL operator L and ‘weight function $\sigma(x)$ ’ has the form

$$Lu = \lambda\sigma(x)u, \quad x \in (a, b) \quad (3.2)$$

plus some **homogeneous** BCs. This a ‘generalized eigenvalue problem’ when $\sigma(x) \neq 1$.

The SLP is called **regular** if in addition:

i) The functions p, q, σ and p' are continuous on $[a, b]$ and

$$p(x) > 0 \text{ and } \sigma(x) > 0 \quad \text{for } x \in [a, b]$$

ii) There are two separated boundary conditions at $x = a$ and $x = b$, i.e. the BCs are

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0, \quad \beta_1 u(b) + \beta_2 u'(b) = 0. \quad (3.3)$$

Recall that for functions in $[a, b]$ we have the standard ‘ L^2 inner product’

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx.$$

For a weight function $\sigma(x)$, positive in (a, b) , we may define the **weighted inner product**

$$\langle f, g \rangle_\sigma = \int_a^b f(x)g(x)\sigma(x) dx$$

Sturm-Liouville operators are self-adjoint (see HW), which gives them useful structure.

Key result: In a domain $[a, b]$, a SL operator (3.1) satisfies **Green’s formula**

$$\langle Lu, v \rangle = p(uv_x - vu_x) \Big|_a^b + \langle u, Lv \rangle \quad (3.4)$$

for all smooth functions u, v in $[a, b]$. In particular, the operators with separated boundary conditions (3.3) is self-adjoint in the L^2 inner product, i.e.

$$\langle Lu, v \rangle = \langle u, Lv \rangle \text{ for all } u, v \text{ satisfying the BCs.}$$

3.1. Example (self-adjoint, not regular). A concentration c of tea particles in a cylindrical tea cup of radius R diffuses in the cup. If it is independent of θ and z then $c(r, t)$ obeys the heat equation

$$c_t = c_{rr} + \frac{1}{r}c_r, \quad c_r(R, t) = 0.$$

Look for a solution of the form $c(r, t) = u(r)e^{\lambda t}$ to find that

$$u_{rr} + \frac{1}{r}u_r + \lambda u = 0, \quad u_r(R) = 0.$$

for $u(r)$ in $[0, R]$. The operator

$$Lu = -u_{rr} - \frac{1}{r}u_r$$

is not self-adjoint, but can be made so using an integrating factor. Multiply by r to get

$$-(ru_r)_r = \lambda ru, \quad u_r(R) = 0$$

which is a SLP with the operator and weight function $\sigma(r)$ and

$$\tilde{L}u = -(ru_r)_r.$$

It is not regular as there is only one BC and the coefficient r in $(ru_r)_r$ can be zero.

However, we can show directly that it is self-adjoint. Let u, v be functions that satisfy the BC. Then (using the L^2 inner product $\langle f, g \rangle = \int_0^R fg \, dr$),

$$\begin{aligned} \langle Lu, v \rangle &= - \int_0^R (ru_r)_r v \, dr \\ &= r(uv_r - u_r v) \Big|_0^R - \int_0^R u(rv_r)_r \, dr \\ &= \langle u, Lv \rangle \end{aligned}$$

since the BC terms vanish at $r = 0$ (from the r factor) and at R (since $u(R) = v(R) = 0$).

3.2. The main theorem. The fundamental result of **Sturm-Liouville theory** is a description of the eigenvalues of eigenfunctions for a **regular** Sturm-Liouville operator (3.1) and their essential structure.

Theorem (Main theorem for regular operators): The Sturm-Liouville problem (3.2) for a **regular** SL operator L has infinitely many eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ and associated eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ with the following properties:

- i) The eigenvalues and eigenfunctions are real,
- ii) The eigenvalues form an infinite sequence with a smallest eigenvalue, tending to ∞ :

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots \rightarrow \infty.$$

- iii) There is exactly one eigenfunction ϕ_n for each eigenvalue
- iv) The eigenfunctions form a basis for $L^2_{\sigma}[a, b]$, and they are orthogonal in the weighted inner product. That is, the eigenfunctions satisfy

$$\langle \phi_m, \phi_n \rangle_{\sigma} = \int_a^b \phi_m(x)\phi_n(x)\sigma(x) \, dx = 0, \quad m \neq n.$$

and every $f \in L^2_\sigma[a, b]$ has a unique representation

$$f = \sum_{n=1}^{\infty} a_n \phi_n(x), \quad a_n = \frac{\langle f, \phi_n \rangle_\sigma}{\langle \phi_n, \phi_n \rangle_\sigma}. \quad (3.5)$$

iv) If f and f' are piecewise continuous, the series converges pointwise to the average of the left/right limits of $f(x)$ inside the interval. That is,

$$\lim_{N \rightarrow \infty} S_N(x) = \frac{1}{2}(f(x^-) + f(x^+)) \text{ for all } x \in (a, b).$$

Note that the result does not include the endpoints.

If the operator L is **self-adjoint** but **not regular** then the results may be different.

A few clarifying notes:

- Here L^2_σ is the space of functions with finite L^2 norm in the weighted inner product:

$$L^2_\sigma[a, b] = \{f(x) : [a, b] \rightarrow \mathbb{R} \text{ s.t. } \int_a^b |f(x)|^2 \sigma(x) dx < \infty\}.$$

In practice, this space typically contains all functions of interest so (iii) can be thought of as holding for all functions f on $[a, b]$.

- The phrase ‘exactly one eigenfunction’ means up to scaling by an arbitrary constant.
- To reiterate, the results may not hold when L is not regular - for instance, if the BCs are not separated or have a different form, or if $p(x)$ is not strictly positive.

3.3. Remarks on proof: The orthogonality is not too hard to show. The proof is the same as for self-adjoint operators in \mathbb{R}^n . Let ϕ_m and ϕ_n be eigenfunctions for $\lambda_m \neq \lambda_n$. Then both satisfy the BCs, so

$$\langle L\phi_m, \phi_n \rangle = \langle \phi_m, L\phi_n \rangle.$$

But $L\phi_m = \sigma\lambda\phi_m$ and the same for n so

$$\lambda_m \langle \sigma\phi_m, \phi_n \rangle = \lambda_n \langle \phi_m, \sigma\phi_n \rangle.$$

Both the inner products are equal to $\langle \phi_m, \phi_n \rangle_\sigma$, which gives

$$(\lambda_m - \lambda_n) \langle \phi_m, \phi_n \rangle_\sigma = 0 \implies \langle \phi_m, \phi_n \rangle_\sigma = 0.$$

The proof that eigenvalues/functions are real is also similar to the proof for self-adjoint matrices (see the book for details). The other assertions are not easy to prove, and require some functional analysis.

3.4. Positive eigenvalues via Rayleigh quotient. An integration by parts trick gives us a way to prove that eigenvalues are positive. In mathematics, we often refer to this technique as an ‘energy argument’.

For the SL operator (3.1) with eigenvalue/function λ and ϕ , take the L^2 inner product with ϕ (multiply by ϕ , integrate):

$$\begin{aligned} L\phi = \lambda\phi &\implies \langle L\phi, \phi \rangle = \lambda \langle \phi, \phi \rangle \\ &\implies \lambda = \frac{\langle L\phi, \phi \rangle}{\langle \phi, \phi \rangle}. \end{aligned} \quad (3.6)$$

Now use IBP on the numerator to get

$$\lambda \langle \phi, \phi \rangle = p\phi\phi_x \Big|_a^b + \int_a^b p\phi_x^2 + q\phi^2 dx.$$

We know $\langle \phi, \phi \rangle > 0$ and $p \geq 0$ by assumption. If $q \geq 0$ and the boundary term is positive, we can conclude that $\lambda > 0$.

Definition: For a self-adjoint L , the **Rayleigh quotient** is the ratio

$$\lambda = \frac{\langle L\phi, \phi \rangle}{\langle \phi, \phi \rangle}$$

where λ and ϕ are an eigenvalue/function. We also have the **minimization principle** that the smallest eigenvalue λ_1 is the minimum of the Rayleigh quotient over all functions v that satisfy the BCs (but are not necessarily eigenfunctions):

$$\lambda_1 = \min_{v \text{ satisfies BCs}} \frac{\langle Lv, v \rangle}{\langle v, v \rangle}.$$

This can be used to bound the smallest eigenvalue (see Sec. 5.6 of the book).

Example: We show that the eigenvalues are positive for

$$-\phi'' = \lambda\phi, \quad \phi'(0) = \phi(0), \quad \phi'(\pi) = -2\phi(\pi).$$

Multiplying by ϕ and integrating,

$$-\int_0^\pi \phi\phi'' dx = \lambda \int_0^\pi \phi^2 dx.$$

This gives the Rayleigh quotient formula for the eigenvalue,

$$\begin{aligned} \lambda \langle \phi, \phi \rangle &= -\int_0^\pi \phi\phi'' dx \\ &= -\phi\phi' \Big|_0^\pi + \int_0^\pi (\phi')^2 dx \\ &= 2\phi(\pi)^2 + \phi(0)^2 + \int_0^\pi (\phi')^2 dx. \end{aligned}$$

All terms are ≥ 0 and $\langle \phi, \phi \rangle > 0$ so $\lambda \geq 0$. Further, observe that

$$\lambda = 0 \iff \phi(0) = \phi(\pi) = 0 \text{ and } \phi'(x) = 0 \text{ for all } x$$

but the (trivial) ODE $\phi' = 0$ with $\phi(0) = 0$ only has $\phi = 0$ as a solution, so we conclude that λ cannot be zero. Thus all the eigenvalues are **strictly positive**.

Physical interpretation: In the context of the heat equation $u_t = u_{xx}$, the flux of heat is $-u_x$. The BCs say that heat leaks out at both boundaries ($-u_x(0) < 0$ and $-u_x(\pi) > 0$). Thus the heat of the system should decrease, so the eigenvalues should be positive.

3.5. Converting to self-adjoint form. Not every second order linear operator

$$Lu = -p(x)u_{xx} + q(x)u_x + r(x)u \quad (3.7)$$

is a SL operator. However, the theory can still be used because we can convert L into a SL operator. The claim here is that for any operator (3.7) there is a $\sigma(x)$ such that

$$Lu = \frac{1}{\sigma} \tilde{L}u$$

for some \tilde{L} that is self-adjoint in the L^2 inner product (**not in the weighted one!**), i.e.

$$\langle \tilde{L}u, v \rangle = \langle u, \tilde{v} \rangle \text{ for all } u, v \text{ satisfying the BCs.}$$

We then have equivalent eigenvalue problems

$$Lu = \lambda u \iff \tilde{L}u = \lambda \sigma u$$

which is the most common way a weight function arises (see examples in the next section).

To convert, notice that only the first-order term qu_x is a problem. We can use an integrating factor to ‘absorb’ it into the second order term:

$$\begin{aligned} Lu &= -p(x)u_{xx} + q(x)u_x + r(x)u \\ &= -p \left(u_{xx} - \frac{q}{p}u_x \right) + ru \\ &= -\frac{1}{\sigma} (p\sigma u_x)_x + ru \\ &= \frac{1}{\sigma} (-(p\sigma u_x)_x + r\sigma u) \\ &= \frac{1}{\sigma} \tilde{L}u \end{aligned}$$

where the integrating factor is $p\sigma = \exp(-\int q/p dx)$ and $\tilde{L} = -(p\sigma u_x)_x + r\sigma u$.

Example (self-adjoint-ifying an operator): For example, consider the operator

$$Lu = -u_{xx} + bu_x$$

and the associated eigenvalue problem

$$L\phi = \lambda\phi \quad (\dots + \text{BCs} \dots).$$

To use the results of Sturm-Liouville theory, we must convert to self-adjoint form (since L is not self-adjoint).

Using the integrating factor $e^{-\int b dx} = e^{-bx}$,

$$Lu = e^{bx}(e^{-bx}u_x)_x = e^{bx}\tilde{L}u$$

where $\tilde{L}u = (e^{-bx}u_x)_x$. Sturm-Liouville theory then applies for the eigenvalue problem

$$\tilde{L}\phi = \lambda e^{-bx}\phi, \quad (\dots + \text{BCs} \dots)$$

so the original eigenvalue problem has all the properties guaranteed by the theorem.