

# MATH 5510 LECTURE NOTES EIGENFUNCTIONS: SOLVING PDES; MORE SL THEORY

## TOPICS COVERED

- Sturm-Liouville theory in practice
  - Procedure for eigenfunction expansion method ( $u_t = -Lu + f$ )
  - Example with weighted inner product (heat equation)
  - Comparing non self-adjoint (bi-orthogonal) approach
- Singular Sturm-Liouville problems
  - When the theory is the same or almost the same
  - When the theory does not apply

### 1. USING STURM-LIOUVILLE THEORY

We are now prepared to use the structure provided by SL theory to solve both BVPs and time-dependent problems. Consider the **initial boundary value problem** (IBVP)

$$\text{(PDE)} \quad u_t = -Lu + f, \quad x \in (a, b), \quad t > 0 \quad (\text{P})$$

$$\text{(boundary conditions)} \quad Bu = c \quad (\text{BC})$$

$$\text{(initial conditions)} \quad u(x, 0) = u_0(x) \quad (\text{IC})$$

where the boundary conditions  $Bu = c$  are **separated** and  $L$  is a second-order linear operator (not assumed self-adjoint).<sup>1</sup> The ‘IBVP’ has three parts: a PDE that in the domain  $(a, b)$  and in some range of times; BCs in  $x$  (space); and initial conditions (ICs) at an initial time.

**1.1. Self-adjoint approach (via Sturm-Liouville theory):** The process here is to convert the operator to self-adjoint form, then solve using the weighted inner product. We convert the **PDE in space and time** to a set of ODE **initial value problems** in time.

**Important reminder:** Projection onto the eigenfunction  $\phi_n$  will require taking the weighted inner product with  $\phi_n$ , i.e.  $\langle \cdots, \phi_n \rangle_\sigma = \int_a^b (\cdots) \phi_n \sigma \, dx$ . For instance, the series for  $f$  is

$$f = \sum_n f_n \phi_n.$$

Take the weighted inner product with  $\phi_n$  to get

$$\langle f, \phi_n \rangle_\sigma = f_n \langle \phi_n, \phi_n \rangle_\sigma \implies f_n = \frac{\langle f, \phi_n \rangle_\sigma}{\langle \phi_n, \phi_n \rangle_\sigma}.$$

Second, the converted operator  $\tilde{L}$  is self-adjoint in the  $L^2$  inner product (unweighted!):

$$\langle \tilde{L}u, v \rangle = \langle u, \tilde{L}v \rangle \text{ for all } u, v \text{ s.t. } Bu = 0, Bv = 0$$

<sup>1</sup>Note that the procedure here also works for BVPs  $Lu = f$  with no changes; just remove the  $u_t$  term.

**The main procedure (solving IBVPs/BVPs using Sturm-Liouville theory):**

1a) Convert (P) to SL form - find the factor  $\sigma(x)$  such that

$$L = \frac{1}{\sigma(x)} \tilde{L}.$$

The problem, in ‘self-adjoint’ form, is now

$$u_t = -\frac{1}{\sigma} \tilde{L}u + f, \quad Bu = c. \quad (\tilde{P})$$

1b) Get a basis for ‘all’ functions in  $[a, b]$  (technically  $L^2_\sigma[a, b]$ ) - solve the eigenvalue problem

$$\tilde{L}\phi = \lambda\sigma\phi, \quad B\phi = 0$$

for eigenvalues/functions  $\lambda_n$  and  $\phi_n$ . By the theory,  $\{\phi_n\}$  is a basis for functions in  $[a, b]$  and the  $\phi$ ’s are orthogonal in the weighted inner product:  $\langle \phi_m, \phi_n \rangle_\sigma = 0$  for  $m \neq n$ .

2a) From (1b), the solution to (P) (satisfying the **inhom.** BCs) can be written as

$$u(x, t) = \sum_n c_n(t) \phi_n. \quad (\text{S})$$

2b) Project the PDE onto  $\phi_n$  (take the weighted inner product with  $\phi_n$ ):

$$\langle u_t, \phi_n \rangle_\sigma = -\langle \frac{1}{\sigma} \tilde{L}u, \phi_n \rangle_\sigma + \langle f, \phi_n \rangle_\sigma. \quad (\tilde{P}_n)$$

For any time derivatives (e.g.  $u_t, u_{tt}$  etc.) differentiate the series term-wise, such as

$$\langle u_t, \phi_n \rangle_\sigma = \langle \sum_n c'_n(t) \phi_n, \phi_n \rangle_\sigma = c'_n(t) \langle \phi_n, \phi_n \rangle_\sigma.$$

For the term with  $\tilde{L}$ , integrate by parts carefully (note that  $u$  satisfies the **inhomogeneous BCs** and  $\phi_n$  satisfies the hom. BCs):

$$\begin{aligned} -\langle \frac{1}{\sigma} \tilde{L}u, \phi_n \rangle_\sigma &= -\langle \tilde{L}u, \phi_n \rangle \\ &= B_n - \langle u, \tilde{L}\phi_n \rangle && (\tilde{L} \text{ self-adj. in } L^2 \text{ inner product}) \\ &= B_n - \lambda_n \langle u, \sigma\phi_n \rangle && (\text{from eig. problem for } \tilde{L}) \\ &= B_n - \lambda_n c_n \langle \phi_n, \phi_n \rangle_\sigma. && (\text{def'n of weighted inner product}) \end{aligned}$$

where  $B_n$  contains the boundary terms from IBP. Once all the terms are projected,  $(\tilde{P}_n)$  becomes an ODE for each  $c_n$ :

$$c'_n(t) + \lambda_n c_n(t) = \frac{B_n + \langle f, \phi_n \rangle_\sigma}{\langle \phi_n, \phi_n \rangle_\sigma}. \quad (C_n)$$

2c) Project the initial condition (IC) onto  $\phi_n$  to get ICs for  $(C_n)$ :

$$\langle u(x, 0), \phi_n \rangle_\sigma = \langle u_0(x), \phi_n \rangle_\sigma \implies c_n(0) = \frac{\langle u_0, \phi_n \rangle_\sigma}{\langle \phi_n, \phi_n \rangle_\sigma}. \quad (C_n\text{-IC})$$

If the ODEs are higher order, one may also need to get  $c'_n(0)$  and so on.

2d) The solution is the series (S) with coefficients solving the ODEs  $(C_n)$  with initial conditions  $(C_n\text{-IC})$ . Suggestion: summarize the solution, collecting all the results together.

1.2. **Non self-adjoint approach (not required):** Alternately, we can use the adjoint operator  $L^*$  and bi-orthogonality. It is the same as the above but with  $\langle \cdots, \psi_n \rangle$  for projection instead of  $\langle \cdots, \phi_n \rangle_\sigma$ . To detail the procedure:

- First find the eigenfunctions for  $L$  and compute  $L^*$  and the adjoint eigenfunctions  $\{\psi_n\}$ .
- Project the PDE onto  $\phi_n$  using the projection  $\langle \cdots, \psi_n \rangle$ :

$$\langle u_t, \psi_n \rangle = -\langle Lu, \psi_n \rangle + \langle f, \psi_n \rangle. \quad (P_n)$$

- Write  $u$  as an eigenfunction expansion (and  $f$ ) and differentiate the series for  $u$  in  $t$ :

$$u = \sum_{n=1}^{\infty} c_n \phi_n \implies \langle u_t, \psi_n \rangle = c'_n(t) \langle \phi_n, \psi_n \rangle,$$

$$f = \sum_{n=1}^{\infty} f_n \phi_n \implies f_n \langle \phi_n, \psi_n \rangle = \langle f, \psi_n \rangle$$

- By the same computations as detailed for BVPs, we have from  $(P_n)$  that

$$\begin{aligned} (c'_n(t) - f_n) \langle \phi_n, \psi_n \rangle &= -\langle Lu, \psi_n \rangle \\ &= -B_n - \langle u, L^* \psi_n \rangle \\ &= -B_n - \lambda_n \langle u, \psi_n \rangle \\ &= -B_n - \lambda_n c_n(t). \end{aligned}$$

- This gives ODEs for the  $c_n$  just as in the self-adjoint approach, which are solved in the same way. For the initial condition,

$$\langle u(x, 0), \psi_n \rangle = \langle u_0(x), \psi_n \rangle \implies c_n(0) = \frac{\langle u_0, \psi_n \rangle}{\langle \phi_n, \psi_n \rangle}.$$

**Practical note (comparing approaches):** The two approaches are both viable:

- The self-adjoint approach requires us to find  $\tilde{L}$  (converting to self-adjoint form) and to use the weighted inner product. However, the operator is self-adjoint, so there is no separate adjoint to worry about.
- The non self-adjoint approach requires us to find the adjoint  $L^*$  and the adjoint eigenfunctions. However, we get to use the  $L^2$  inner product and there are no weights to worry about.

Typically, the self-adjoint approach is used since it only requires one set of eigenfunctions. The other approach is only necessary for problems outside of Sturm-Liouville theory where we are stuck with  $L$  and  $L^*$ .

1.3. **Simple example.** First, an example where most terms are zero. We solve the IBVP

$$\begin{aligned} u_t &= u_{xx}, \quad x \in (0, \pi), \quad t > 0 \\ u_x(0) &= A, \quad u_x(\pi) = 0 \\ u(x, 0) &= T_0 \end{aligned} \tag{1.1}$$

which describes heat in a metal rod insulated at one end and with a constant input flux at the other. The operator is  $Lu = -u_{xx}$ ,  $\sigma(x) = 1$ , and the eigenvalue problem is

$$-\phi'' = \lambda\phi, \quad \phi'(0) = \phi'(\pi) = 0 \implies \phi_n = \cos nx, \quad \lambda_n = n^2, \quad n = 0, 1, 2, \dots$$

Note  $\lambda_0 = 0$  is an eigenvalue, so keep in mind **to be careful with  $n = 0$  cases**.

**PDE solution:** Let  $u$  be the solution to the IBVP. Then

$$u(x, t) = \sum_{n=0}^{\infty} c_n(t) \phi_n(x).$$

Now take the inner product of the DE with  $\phi_n$  to get

$$\begin{aligned} c_n'(t) &= -\langle Lu, \phi_n \rangle \\ &= (\phi_n u_x - \phi_n' u) \Big|_0^{\pi} - \langle u, L\phi_n \rangle. \end{aligned}$$

By the boundary conditions,

$$\begin{aligned} \phi_n'(0) &= \phi_n'(\pi) = 0, \\ u_x(0) &= A, \quad u_x(\pi) = 0. \end{aligned}$$

Plugging this into the boundary terms, we get

$$c_n'(t) \langle \phi_n, \phi_n \rangle = -A\phi_n(0) - \lambda_n c_n \langle \phi_n, \phi_n \rangle.$$

Now from the IC,

$$f(x) = \sum_{n=0}^{\infty} c_n(0) \phi_n(x) \implies c_n(0) = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

But  $f = T_0 \phi_0$  so it follows that

$$c_0(0) = T_0, \quad c_n(0) = 0 \text{ for } n > 0. \tag{1.2}$$

This gives the IVP for the  $c_n$ 's:

$$c_n' + \lambda_n c_n = -A/\langle \phi_n, \phi_n \rangle, \quad c_n(0) \text{ given by (1.2)}.$$

**Solve the coeff. ODEs:** There are two cases. When  $\lambda_n \neq 0$ ,

$$c_n(t) = -\frac{A}{\lambda_n \langle \phi_n, \phi_n \rangle} (1 - e^{-\lambda_n t})$$

But for  $\lambda_0 = 0$ , we have  $c_n(0) = T_0$  and (note that  $\langle \phi_0, \phi_0 \rangle = \pi$ )

$$c_0' = -A/\langle \phi_0, \phi_0 \rangle \implies c_0(t) = T_0 - At.$$

**Summarize:** Thus, the solution is

$$u(x, t) = T_0 - At - A \sum_{n=1}^{\infty} \frac{(1 - e^{-\lambda_n t})}{\lambda_n \langle \phi_n, \phi_n \rangle} \phi_n$$

with  $\lambda_n = n^2$  and  $\phi_n = \cos nx$  and  $\langle \phi_n, \phi_n \rangle = \int_0^{\pi} \cos^2 nx \, dx$  (you could simplify more). Note that  $\langle \phi_0, \phi_0 \rangle = \pi$  and  $\langle \phi_n, \phi_n \rangle = \pi/2$ ; the integrals are different for  $n = 0$  and  $n \neq 0$ .

1.4. **Full example (cooling of a spherical shell: general).** Let  $u$  solve the IBVP

$$\begin{aligned} u_t &= u_{rr} + \frac{2}{r}u_r, \quad r \in (1, 2), \quad t > 0 \\ u(1) &= 0, \quad u(2) = e^{-t} \\ u(r, 0) &= r - 1 \end{aligned} \tag{1.3}$$

which can describe diffusion of heat (spherically symmetric) in a spherical shell between radius 1 and radius 2 (for instance, the mantle of the Earth). The BC has the inner layer at a fixed value and the outer layer at a value that decays over time.

We first follow the procedure to get a ‘complete solution’ but without solving the various equations explicitly (this is done afterwards).

**The eigenfunctions:** The operator, which is not self-adjoint, is

$$Lu = -u_{rr} - (2/r)u_r.$$

Using the integrating factor  $\exp(\int 2/r dr) = r^2$ , we get

$$Lu = -\frac{1}{r^2}(r^2u_r)_r = \frac{1}{\sigma(r)}\tilde{L}u, \implies \tilde{L}u := -(r^2u_r)_r, \quad \sigma(r) = r^2. \tag{1.4}$$

The eigenvalue problem is then

$$-(r^2\phi_r)_r = \lambda r^2\phi, \quad r \in (1, 2), \quad \phi(1) = \phi(2) = 0. \tag{1.5}$$

The operator  $\tilde{L}$  is self-adjoint with separated boundary conditions and  $\sigma(r) > 0$  and  $p(r) = r^2 > 0$  in  $[1, 2]$  so  $\tilde{L}$  is regular. By the main theorem, there is a set of eigenvalues  $\lambda_n$  and  $\phi_n$  for  $n \geq 1$  solving (1.5) with all the properties listed in the theorem.

This is enough to proceed with the method, even without an explicit solution.

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**Solving the PDE:** Let  $u$  be the solution to the IBVP. Since  $\{\phi_n\}$  is a basis for  $L^2_\sigma[a, b]$ ,

$$u(x, t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x), \quad c_n = \frac{\langle u, \phi_n \rangle_\sigma}{\langle \phi_n, \phi_n \rangle_\sigma}.$$

For convenience, define the constants

$$k_n = \langle \phi_n, \phi_n \rangle_\sigma = \int_1^2 r^2 \phi_n^2(r) dr. \tag{1.6}$$

Project the PDE onto  $\phi_n$  by taking the weighted inner product with  $\phi_n$ :

$$\langle u_t, \phi_n \rangle_\sigma = -\langle \tilde{L}u, \phi_n \rangle$$

noting that the RHS is  $-\langle \frac{1}{\sigma}Lu, \phi_n \rangle_\sigma$ . To emphasize, the boundary conditions for  $u$  and  $\phi_n$  are

$$u(1) = 0, \quad u(2) = e^{-t}, \quad \phi_n(1) = \phi_n(2) = 0.$$

By the general procedure,  $\langle u_t, \phi_n \rangle_\sigma = k_n c'_n(t)$  and so we get (using the BCs above to simplify)

$$\begin{aligned}
k_n c'_n(t) &= -\langle \tilde{L}u, \phi_n \rangle \\
&= \sigma(r)(u_r \phi_n - u \phi'_n) \Big|_1^2 - \langle u, L\phi_n \rangle \\
&= -\sigma(2)u(2)\phi'_n(2) - \lambda_n \langle u, \sigma \phi_n \rangle \\
&= -4\phi'_n(2)e^{-t} - \lambda_n k_n c_n(t).
\end{aligned}$$

This gives the ODE for the coefficients,

$$c'_n(t) + \lambda_n c_n(t) = b_n e^{-t} \text{ for } n \geq 1, \quad b_n := -4k_n \phi'_n(2). \quad (1.7)$$

For the initial condition, let  $f(r) = r - 1$ . Evaluating the series at  $t = 0$ :

$$u(x, 0) = f(x) \implies c_n(0) = k_n \langle f(x), \phi_n \rangle_\sigma. \quad (1.8)$$

**The answer:** We now have a well-defined solution. To summarize and make sure everything is defined and in the same place:

**Solution:** The solution (for the initial condition  $u(r, 0) = f(r)$ ) is

$$u(x, t) = \sum_{n=1}^{\infty} c_n(t) \phi_n(x) \quad (1.9)$$

where the eigenvalues/functions  $\lambda_n, \phi_n$  solve

$$-(r^2 \phi')' = r^2 \lambda \phi, \quad \phi(1) = \phi(2) = 0 \quad (1.10)$$

and the coefficients are given by the solution to the IVP

$$\begin{aligned}
c'_n(t) + \lambda_n c_n(t) &= b_n e^{-t} \\
c_n(0) &= k_n \int_1^2 r^2 f(r) \phi_n(r) dr.
\end{aligned} \quad (1.11)$$

The constants  $k_n, b_n$  are

$$\begin{aligned}
k_n &= \langle \phi_n, \phi_n \rangle_\sigma = \int_1^2 r^2 \phi_n^2(r) dr, \\
b_n &= -4k_n \phi'_n(2).
\end{aligned}$$

A plot of the solution is shown in [Figure 1](#).

**Note:** The IVP (1.10) has a unique solution (hence  $c_n$  is well-defined as written), but it can also be solved to get an explicit solution. With the integrating factor  $e^{\lambda_n t}$ , we get

$$c_n(t) = c_n(0)e^{-\lambda_n t} + b_n e^{-\lambda_n t} \int_0^t e^{(\lambda_n - 1)s} ds = \dots \quad (1.12)$$

Note that the integral is different in the cases  $\lambda_n \neq 1$  and  $\lambda_n = 1$ .

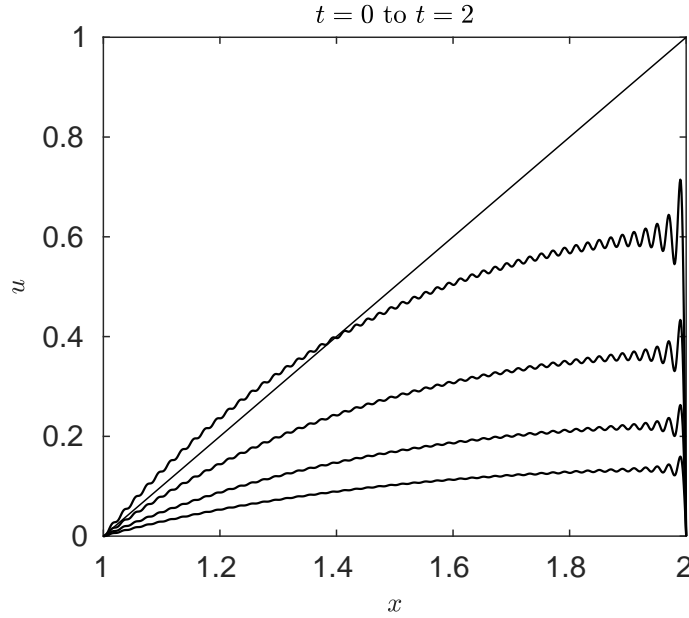


FIGURE 1. Solution (1.9) at  $t = 0, 0.5, 1, 1.5$  and  $2$ . Note that Gibbs' phenomenon occurs at  $x = 2$  due to the inhomogeneous BC  $u(2, t) = e^{-t}$ .

1.5. **Extracting some information.** This solution is good enough to do some analysis or to manipulate for theory. For example, suppose we want to show the system approaches a uniform state at temperature 0 and determine the 'decay rate'.

First, it is easy to show using the Rayleigh quotient that the  $\lambda$ 's are **strictly positive**. Take the  $L^2$  inner product with  $\phi$  of the eigenvalue problem for  $\tilde{L}$ :

$$\tilde{L}\phi = \lambda\sigma\phi \implies \langle \tilde{L}\phi, \phi \rangle = \lambda\langle \phi, \phi \rangle.$$

Now integrate by parts once to get

$$\lambda\langle \phi, \phi \rangle = \langle \tilde{L}\phi, \phi \rangle = \int_1^2 -(r^2\phi_r)_r\phi \, dr = \int_1^2 r^2\phi_r^2 \, dr \geq 0.$$

The boundary condition  $\phi(1) = 0$  also implies this is zero if and only if  $\phi = 0$ , so  $\lambda > 0$ .

For simplicity, assume  $\lambda_n \neq 1$  (this is true; and even if not, the calculations are similar). Then from the solution (1.12),

$$c_n(t) = c_n(0)e^{-\lambda_n t} + \frac{b_n}{\lambda_n - 1} (e^{-t} - e^{-\lambda_n t}).$$

**What can we conclude?** The coefficients decay like  $\max\{e^{-t}, e^{-\lambda_n t}\}$  as  $t \rightarrow \infty$ . In particular, all of them go to zero: the system approaches an equilibrium state  $u = 0$  and the decay rate is  $\min\{\lambda_1, 1\}$ . Finding the value of  $\lambda_1$  requires more work or numerical approximation.

**1.6. Solving the eigenvalue problem.** In general, eigenvalue problems are difficult or impossible to solve exactly, and we must resort to a mixed bag of analytical techniques or approximation. For this example, there is a trick. The goal is to solve

$$-\phi'' - \frac{2}{r}\phi' = \lambda\phi, \quad \phi(1) = \phi(2) = 0.$$

We use a ‘convenient’ substitution to convert to a LCC equation. Let  $y = r\phi$ . Then

$$-y'' = -r\phi'' - 2\phi' = -r\left(\phi'' + \frac{2}{r}\phi'\right) = \lambda r\phi = \lambda y$$

so the general solution for  $\phi$  is (solve for  $y$ , set  $\phi = y/r$ )

$$\phi = c_1 \frac{\sin \mu r}{r} + c_2 \frac{\cos \mu r}{r}$$

where  $\mu = \sqrt{\lambda}$ . Using the boundary conditions, we get the system for  $c_1, c_2$ :

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin \mu & \cos \mu \\ (\sin 2\mu)/2 & (\cos 2\mu)/2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \implies \cos 2\mu \sin \mu - \sin 2\mu \cos \mu = 0 \implies \sin \mu = 0.$$

The eigenvalues/functions (note: the Rayleigh quotient argument above ensures  $\lambda > 0$ ) are

$$\lambda_n = \pi^2 n^2, \quad \phi_n = \frac{1}{r} \sin \pi r, \quad n = 1, 2, \dots \quad (1.13)$$

Note that these are the same as the eigenvalues/functions for  $\tilde{\phi} = \lambda\sigma\phi$  (this and  $L\phi = \lambda\phi$  are equivalent).

From here, we can calculate the various constants in the solution (1.9). For instance,

$$k_n = \int_1^2 r^2 \phi_n^2 dr = \int_1^2 \sin^2 n\pi r dr = 2,$$

$$b_n = -4k_n \phi_n'(2) = -2nk_n$$

and so on to get an explicit, computed solution (this is how [Figure 1](#) was produced), numerical values for the decay rate (etc). However, other than avoiding the  $\lambda_n = 1$  case in the integral (1.12), the computations do not give us any new structure.



## 2. SINGULAR SLPs: WHEN THE THEORY STILL WORKS...

When the operator is not regular (called **singular**), either some parts of the theorem do not hold (we lose some structure) or the theorem fails entirely (other theory is required).

**Theorem (self-adjoint, not regular):** Suppose  $L$  is (almost) a regular SL operator, except that it has BCs other than the separated type (but is still self-adjoint!). Then the results of the Main Theorem still hold, except that:

- There may be more than one eigenfunction per eigenvalue
- The smallest eigenvalue still has one eigenfunction

**Eigenspace has dimension  $> 1$ :** Consider the operator with periodic BCs

$$Lu = -u_{xx}, \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi).$$

The operator  $L$  is self-adjoint. However, the BCs are not separated, so it is not regular. We know that each eigenvalue has two eigenfunctions:

$$\lambda_n = n^2, \quad \phi_{n,1} = \cos nx, \quad \phi_{n,2} = \sin nx$$

leading to the Fourier series. All the other conclusions of the theorem hold; the only difference is that the solution to

$$L\phi = \lambda_n\phi, \quad \phi \text{ satisfies the BCs}$$

for a given eigenvalue  $\lambda_n$  is spanned by **two** eigenfunctions rather than one (except for the smallest eigenvalue  $\lambda_0 = 0$ , which still has only one). The properties that do hold are enough to still use this basis for solving PDEs with periodic boundary conditions.

**Highlight:** For this course, periodic BCs will be one of the only cases we run into with multiple eigenfunctions per eigenvalue. However, it is important to be aware that non-separated or more exotic BCs can lead to these more complicated eigenfunction structures.

**Missing a BC:** Consider the heat equation in a full sphere of radius  $\pi$  (for a solution  $u(r, t)$ ) with a prescribed value on the boundary:

$$u_t = u_{rr} + \frac{2}{r}u_r, \quad r \in [0, \pi] \quad u(\pi, t) = a.$$

There is now no inner boundary! Recall that the eigenvalue problem is

$$-\phi'' - \frac{2}{r}\phi = \lambda\phi, \quad \phi(\pi) = 0 \tag{2.1}$$

and it has the general solution

$$\phi = c_1 \frac{\sin \sqrt{\lambda}r}{r} + c_2 \frac{\cos \sqrt{\lambda}r}{r}. \tag{2.2}$$

With only one boundary condition, (2.1) cannot be solved. However, to have a physically reasonable solution the function  $\phi(x)$  should be finite. The true eigenvalue problem is then

$$-\phi'' - \frac{2}{r}\phi = \lambda\phi, \quad \phi \text{ bounded in } [0, \pi], \quad \phi(\pi) = 0. \tag{2.3}$$

This forces  $c_2 = 0$  in (2.2), and then we can proceed to get  $\lambda_n = n^2$  and  $\phi_n = \sin nr/r$  for  $n \geq 1$ . There is one eigenfunction per eigenvalue; the results of the theorem all hold.

**Highlight (boundedness):** Often, a ‘bounded’ constraint will eliminate an eigenfunction that is not bounded in the interval (like  $1/x$  in  $[0, x]$ ). Such constraints are common in singular problems where  $p(x)$  can be zero at an endpoint and the usual boundary condition is missing at this endpoint.

When well behaved, the structure tends to be the same as the regular case, even though the theorem as stated does not quite apply.

### 3. ...AND WHEN IT DOESNT.

Only equations of a certain form can be solved with this approach. There has to be a certain amount of ‘separation’ that lets us have an operator  $L$  depending only on  $x$ -derivatives and reasonably nice BCs. Here are some examples of equations where this fails (exercise: try to make the method work):

**Infinite domain:** Suppose we have an eigenvalue problem for  $Lu = -u_{xx}$  in an infinite domain such as the problem on the half-line

$$-\phi'' = \lambda\phi \text{ in } (0, \infty), \quad \phi(0) = 0, \quad \phi \text{ bounded as } x \rightarrow \infty.$$

The function  $\phi(x) = \sin(kx)$  is a solution for any **real** number  $k$ . A different theory is required as the eigenvalue set is not discrete (we’ll see this as the **Fourier transform** later).

**Time cannot be separated:** The approach here depends on being able to write  $u$  in terms of a basis of eigenfunctions that is ‘separated’ from  $t$ :

$$u = \sum c_n(t)\phi_n(x)$$

and not  $\phi_n(x, t)$ ; the basis must stay the same for all  $t$ . A ‘time-dependent’ Robin BC like

$$u(a, t) + f(t)u_x(a, t) = 0$$

makes the eigenvalue problem for  $\phi(x)$  impossible to form. The boundary conditions for  $\phi$  would have to be  $\phi(a) = -f(t)\phi'(a)$  which does not work.

Similarly, for the PDE

$$u_t = u_{xx} + c(t)u_x$$

(which is diffusion with a time dependent velocity term) we cannot take the operator

$$Lu = -u_{xx} - c(t)u_x$$

because then the eigenvalue problem would depend on  $t$ . Unless some change of variables can disentangle  $x$  from  $t$ , some other method is needed. Fortunately, it is often true in practice that the PDE does not have much explicit time dependence.