MATH 5510 LECTURE NOTES
PDES: THE MAIN TYPES AND PROPERTIES

Topics covered

- The heat equation
  - Domain; irreversibility
  - Coefficient decay: smoothing properties
  - Convergence to a steady state
- Separation of variables
  - As applied to the heat equation
  - Remarks on its limitations
- Wave equation
  - Physical interpretation (Fourier modes)
  - Coefficients: non-smoothing
  - Where are the waves?
- Laplace's equation
  - Pure boundary value problems: separable case
  - Superposition, solvability

1. The heat equation

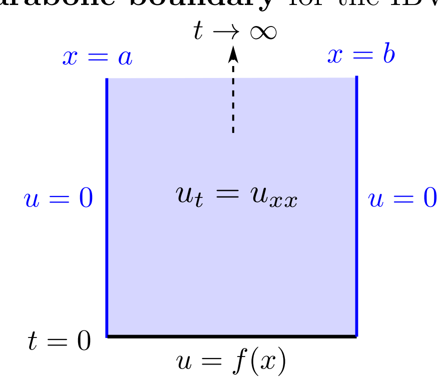
With the structure of eigenfunctions and the solution procedure, we may now study some important PDEs and their fundamental properties. Recall that an **initial boundary value problem** (IBVP) for the heat equation consists of the PDE itself plus BCs and an IC e.g.

$$u_t = u_{xx} \qquad t > 0 \text{ and } x \in (a,b), \tag{1.1a}$$

$$u(a,t) = 0 \text{ and } u(b,t) = 0 \qquad \text{for } t > 0 \tag{1.1b}$$

$$u(x,0) = f(x). \tag{1.1c}$$

The initial condition can be thought of as a boundary at $t = 0$; the term 'initial' is used because it is treated differently than the x-boundaries: the time domain $(0,\infty)$ is **infinite** and the time derivative $\frac{\partial}{\partial t}$ is not self-adjoint. The heat equation is the prototypical **parabolic** PDE, which describes diffusion processes. The three-sided boundary in the (x,t) plane, shown below, is called the **parabolic boundary** for the IBVP.



Well-posedness: This initial boundary value problem (1.1a)-(1.1c) has a unique solution in the domain for all $t > 0$ and small changes in the initial condition lead to small changes in the solution. We say the problem is **well-posed** in this case.

Note that the condition $t > 0$ is necessary; the equation does not have a solution (ill-posed) for $t < 0$. The heat equation **cannot be solved backward in time** (see HW). Physically, this reflects the irreversibility of diffusion (it acts only forward in time).

1.1. **In physical problems.** Suppose $u(x,t)$ is the temperature in a solid. If the material properties vary in space and there is an external 'source' of heat $g(x,t)$, then the heat equation for u is

$$c(x)u_t = (k(x)u_x)_x + g(x,t)$$

where $c(x)$ is the specific heat and $k(x)$ is the thermal diffusivity. The **flux** of heat $q(x,t)$ through the point x (directed towards the + direction) is

$$q = -ku_x.$$

The equation then has the **conservation law** form

$$e_t + q_x = g(x,t)$$

where $q = -ku_x$ is the **flux of heat** and $e = cu$ is the heat energy; this is a statement of conservation of heat in the system. A Robin boundary condition like $-u_x(a) = u(a)$ then states that there is a flux into the system proportional to the temperature; Neumann BCs say the flux is zero.

The term $(ku_x)_x$ is used to describe diffusion in general (not just heat). For example, a concentration $u(x,t)$ of a chemical in a pipe, carried by a fluid with velocity v_0, may satisfy the **advection-diffusion** equation

$$u_t + v_0 u_x = (ku_x)_x.$$

where k is the diffusivity of the chemical. If the pipe is closed at one end, the flux $q = 0$, so the BC is $v_0 u = ku_x$ (another place where Robin boundary conditions arise).

1.2. **Conservation of mass.** Suppose u satisfies the heat equation in an interval,

$$u_t = (k(x)u_x)_x, \quad x \in (a,b).$$

We can integrate from a to b to find that

$$\frac{\partial}{\partial t}\left(\int_a^b u\,dx\right) = k(b)u_x(b,t) - k(a)u_x(a,t). \tag{1.2}$$

Letting $q(x,t) = -ku_x$ be the flux and $M(t) = \int_a^b u(x,t)\,dx$,

$$\frac{\partial M}{\partial t} = q(a,t) + (-q(b,t)) = \text{flux in at } a + \text{flux in at } b.$$

This says that the 'mass' of u in the domain is conserved: its rate of change is given by the flux of u into the system. With some information about the boundary terms, (1.2) can be useful - for instance, if $u_x(a,t) = u_x(b,t) = 0$ (Neumann BCs) then $M(t)$ is constant.

1.3. **Superposition.** A linear, homogeneous DE $Lu = 0$ obeys the superposition principle: linear combinations of solutions are also solutions. This is a restatement of linearity in the differential operators.

Superposition principle: For a linear homogeneous DE (i.e. $Lu = 0$),

$$u_1, u_2 \text{ are solutions} \implies c_1u_1 + c_2u_2 \text{ is a solution for all } c_1, c_2 \in \mathbb{R}. \tag{1.3}$$

More generally, for an inhomogeneous DE

$$Lu = f,$$

if u_1 and u_2 solve the DE with inhomogeneous terms f_1 and f_2 then $c_1u_1 + c_2u_2$ solves the DE with $f = c_1f_1 + c_2f_2$.

Important remark on notation: In the context of the heat equation

$$u_t = u_{xx} + f$$

we have written $Lu = -u_{xx}$ and $u_t = -Lu + f$ (only the x-derivatives in L). This is also in the form $\hat{L}u = f$ for $\hat{L}u = u_t - u_{xx}$, which is the L in the superposition definition above. Essentially, we have split the full operator $\hat{L}$ into a self-adjoint part $(-u_{xx})$ and left u_t alone.

Using superposition: The BCs and ICs, if non-zero, add in the same way as f. For instance, if u, v both solve

$$\begin{aligned} u_t &= u_{xx}, \qquad t > 0 \text{ and } x \in (a, b), \\ u(a, t) &= u(b, t) = 0 \quad \text{ for } t > 0 \end{aligned}$$

where u, v have initial conditions

$$u(x, 0) = f_1(x), \qquad v(x, 0) = f_2(x)$$

then $w = c_1u + c_2v$ solves the IBVP

$$\begin{aligned} w_t &= w_{xx}, \qquad t > 0 \text{ and } x \in (a, b), \\ w(a, t) &= w(b, t) = 0 \quad \text{ for } t > 0 \\ w(x, 0) &= c_1f_1(x) + c_2f_2(x). \end{aligned}$$

Superposition allows us to split problems into simpler parts. Suppose we wish to solve

$$\begin{aligned} u_t &= u_{xx} + g(x, t), \\ u(0, t) &= u(1, t) = 0, \\ u(x, 0) &= f(x). \end{aligned}$$

We can split this into $u = v + w$ where v solves an *inhomogeneous* problem with *zero* ICs and $w(x, t)$ solves a *homogeneous* problem with non-zero ICs:

$$\begin{aligned} v_t &= v_{xx} + g(x, t), \\ v(0, t) &= v(1, t) = 0, \\ v(x, 0) &= 0 \end{aligned} \qquad\qquad \begin{aligned} w_t &= w_{xx}, \\ w(0, t) &= w(1, t) = 0, \\ w(x, 0) &= f(x), \end{aligned}$$

We'll use this idea in other ways to simplify solutions. Here, they have physical significance: one is the response to the source term g; the other is the response if there was no forcing.

2. Steady states; smoothness of solutions

Recall that positive eigenvalues implies that the solution to the heat equation will converge to some equilibrium (a **steady state**) as $t \to \infty$. To further discuss, recall the IBVP

$$\begin{aligned} u_t &= u_{xx}, \qquad x \in (0,1),\, t > 0 \\ u_x(0,t) = 0, &\quad u_x(1,t) = 0, \quad t > 0 \\ u(x,0) &= f(x) \end{aligned} \tag{2.1}$$

with eigenvalues/functions $\lambda_n = n^2\pi^2$ and $\phi_n = \cos n\pi x$ for $n = 0, 1, 2, \cdots$ and

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n t} \cos n\pi x, \qquad a_n = \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle} = \begin{cases} 2\int_0^1 \cos n\pi x\, dx, & n \geq 1 \\ \int_0^1 f(x)\, dx & n = 0 \end{cases}.$$

From the solution (plotted in Equation 2), we see that (assuming $a_1 \neq 0$):

$$u(x,t) = a_0 + a_1 e^{-\lambda_1 t} \cos \pi x + \text{ smaller terms.}$$

It follows (omitting a rigorous proof) that:

- The solution approaches the constant 'steady-state' solution $\overline{u}(x) = a_0$ as $t \to \infty$.
- This 'steady state' $\overline{u}(x)$ is a **time-independent** solution to the PDE and BCs.
- $a_0 = \int_0^1 f(x)dx$ is the average value of $f(x)$. That is, $u(x,t)$ converges to the average of the initial data (which makes physical sense, e.g. for diffusion in a closed container).

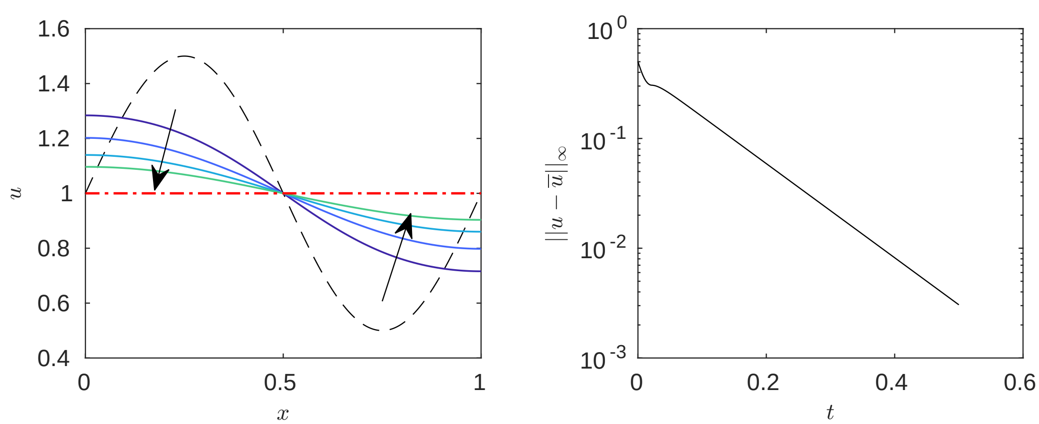
Moreover, so long as $a_1 \neq 0$ (what if $a_1 = 0$?), the convergence has exponential rate λ_1:

$$\max_{x\in[0,1]} |u(x,t) - a_0| \sim a_1 e^{-\lambda_1 t} \text{ as } t \to \infty.$$

Steady states: A 'steady state' or 'equilibrium solution' $\overline{u}(x)$ is a time-independent solution to the PDE and the BCs. For the heat equation with the typical BCs, if a steady state exists then it is unique; if the eigenvalues are all non-negative then

$$\lim_{t\to\infty} u(x,t) = \overline{u}.$$

A word of caution: To show that this 'steady state' really is the limit, we must verify that the eigenvalues are positive and check that inhomogeneous BCs or other complications do not change the limit.



Using steady states: If a time-independent solution can be found in advance, we can simplify the solution procedure. Consider, for instance,

$$\begin{aligned} u_t &= u_{xx} + 2A(x+1), \quad x \in [0,1], \;\; t>0 \\ u_x(0,t) = 0, &\quad u(1,t) = B, \quad t>0 \\ u(x,0) &= f(x) \end{aligned} \tag{2.2}$$

A time-independent solution $\overline{u}(x)$ to the PDE+BCs (ignoring the IC) solves the BVP

$$\overline{u}_{xx} + 2A(x+1) = 0, \quad \overline{u}(0) = 0, \, \overline{u}(1) = 1.$$

Solving this by integrating twice,

$$\overline{u}_x = c_1 - A(x+1)^2 \implies \overline{u} = B + \frac{5A}{3} + Ax - \frac{A}{3}(x+1)^3.$$

Now observe that $v = u - \overline{u}$ (by superposition) solves the homogeneous IBVP

$$\begin{aligned} v_t &= v_{xx} \quad x \in [0,1], \;\; t>0 \\ v_x(0,t) = 0, &\quad v(1,t) = 0, \quad t>0 \\ v(x,0) &= f - \overline{u}. \end{aligned} \tag{2.3}$$

The solution for v is (with $\phi_n = \cos((n-1/2)\pi x)$ and $\lambda_n = \pi^2(n-1/2)^2$)

$$v(x,t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\phi_n, \qquad a_n = \frac{\langle f - \overline{u}, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}.$$

The solution to the original IBVP (2.2) is then

$$u(x,t) = \overline{u} + v(x,t). \tag{2.4}$$

Direct approach: Suppose we instead solve inhomogeneous problem directly. Let

$$2(x+1) = \sum_{n=1}^{\infty} g_n\phi_n, \quad f = \sum_{n=1}^{\infty} f_n\phi_n$$

so $f_n = \langle f, \phi_n\rangle / \langle \phi_n, \phi_n\rangle$ and $g_n = \langle x+1, \phi_n\rangle / \langle \phi_n, \phi_n\rangle$. Then

$$c_n'(t) = Ag_n + Bh_n - \lambda_n c_n, \quad h_n = \frac{1}{B\langle \phi_n, \phi_n\rangle}\left(\phi_n u_x - \phi_n' u\right)\Big|_0^1 = \frac{(-1)^{n+1}\pi(n-1/2)}{\langle \phi_n, \phi_n\rangle}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x), \quad c_n(t) = f_n e^{-\lambda_n t} + \frac{Ag_n + Bh_n}{\lambda_n}(1 - e^{-\lambda_n t}) \tag{2.5}$$

Both solutions are, in fact the same; the inhomogeneous terms are accounted for inside the series for (2.5) and outside the series for (2.4). Rewriting (2.5) in parts:

$$u(x,t) = A\sum_{n=1}^{\infty} \frac{g_n}{\lambda_n}\phi_n + B\sum_{n=1}^{\infty}\frac{h_n}{\lambda_n}\phi_n + \sum_{n=1}^{\infty}\left(f_n + \cdots\right)e^{-\lambda_n t}\phi_n.$$

The first two terms are the eigenfunction series for $\overline{u}$. Note that

$$g_n \sim C/n^2, \quad h_n \sim C/n$$

so the 'B' part has poor convergence (and is the source of the Gibbs' phenomenon at $x = 1$). Separating out the steady state removes this term, yielding a smooth solution (Figure 1).

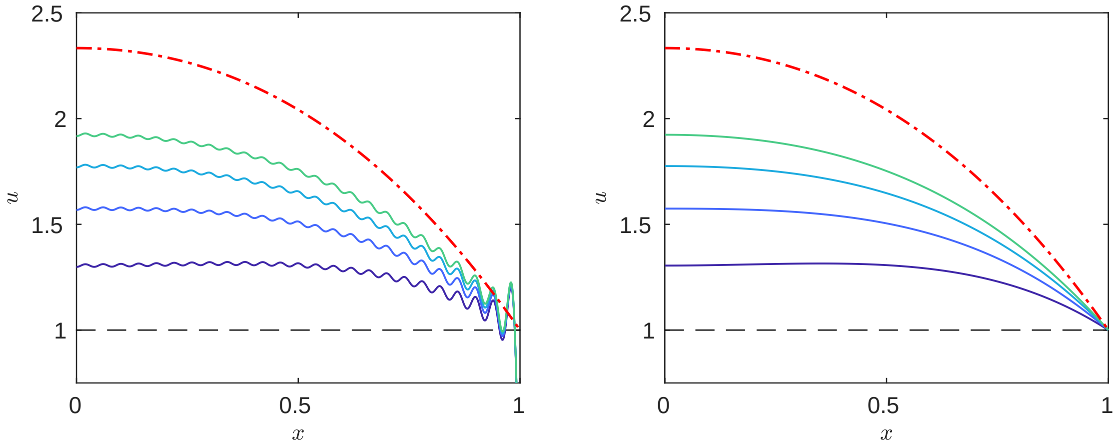


FIGURE 1. Improving smoothness for (2.2) via steady state. Left: solution (2.4) solved directly (50 terms). Right: solution (2.5) using a steady state.

2.1. **Smoothness.** Consider a (possibly non-uniform) heat equation

$$\begin{aligned} u_t &= (k(x)u_x)_x \quad x \in [0,1], \;\; t > 0 \\ u(x,0) &= f(x) \end{aligned} \tag{2.6}$$

with **homogeneous** BCs and a solution

$$u = \sum c_n(t)\phi_n(x),$$

The coefficients for the 'homogeneous' part will satisfy an ODE of the form

$$c_n'(t) + \lambda_n c_n(t) = 0 \implies c_n(t) = f_n e^{-\lambda_n t}.$$

where $f_n = \langle f, \phi_n\rangle / \langle \phi_n, \phi_n\rangle$. It follows that

at each $t > 0$, the coefficients $c_n(t)$ decay 'exponentially' with n

where 'exponentially' depends on the growth of λ_n (typically $\sim e^{-an^2}$). Note that the series at $t = 0$,

$$u(x,0) = \sum_n f_n \phi_n$$

may not have fast decay. The smoothness of $u(x,t)$ improves once some diffusion has occured.

Smoothing: In particular, this is much faster than the $1/n^k$ we get for a function that has a discontinuous k-th derivative. This is a 'smoothing' property of diffusion: the solution to the homogeneous problem is smooth for all $t > 0$ (the series can be differentiated an infinite number of times), **even if the initial condition is not**.

It follows that the series converges quickly to $u(x,t)$ for positive times. The Gibbs' phenomenon issues encountered earlier are the result of inhomogeneous BCs, which introduce discontinuities that persist for all t.

2.2. **Improving smoothness.** If a steady state is not available, the non-smoothness can still be mitigated by use of a similar superposition trick. Consider, for instance,

$$\begin{aligned} u_t &= u_{xx} \quad x \in [0,1], \;\; t > 0 \\ u_x(0,t) = \sin t, \quad u(1,t) &= e^{-t}, \quad t > 0 \\ u(x,0) &= f(x) \end{aligned} \tag{2.7}$$

No steady state exists due to the time-dependence in the BCs. The solution can be found as

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x) \tag{2.8}$$

using the usual eigenfunction expansion of u. However, due to the inhomogeneous BCs,

$$c_n(t) \sim C/n \implies \text{Gibbs' phenomenon at boundaries.}$$

To remedy this, we look for a function $w(x,t)$ such that

$$w(x,t) \text{ satisfies the inhom. BCs.} \tag{2.9}$$

Then we can write

$$u = v + w$$

so that $v = u - w$ has **homogeneous BCs**. However, it is now the case that

$$v_t = v_{xx} + g(x,t).$$

To calculate the new source $g(x,t)$,

$$v_t - v_{xx} = u_t - u_{xx} - (w_t - w_{xx}) \implies g = w_t - w_{xx}$$

which is **non-zero** in general since w does not solve the PDE. There are many choices for w since (2.9) is not a strong condition. A linear function is simplest (quadratic if necessary):

$$w(x,t) = A(t) + xB(t) \implies w(x,t) = x\sin t + (e^{-t} - \sin t)$$

which then gives $g = x\cos t - e^{-t} - \cos t$ since $w_t = 0$, so v solves

$$\begin{aligned} v_t &= v_{xx} + g(x,t) \quad x \in [0,1], \;\; t > 0 \\ v_x(0,t) = 0, \quad v(1,t) &= 0, \quad t > 0 \\ v(x,0) &= f - 1 \end{aligned} \tag{2.10}$$

since $w(x,0) = 1$. Solving (2.10) for v, we get the solution to (2.6):

$$\tilde{c}_n'(t) + \lambda_n \tilde{c}_n = g_n(t), \quad g(x,t) = \sum g_n(t)\phi_n(x).$$

$$u = w + \sum_{n=1}^{\infty} \tilde{c}_n(t)\phi_n(x) \tag{2.11}$$

Now note that we know since the series for $g(x,t)$ converges and $g(1,t) \neq 0$ that

$$g_n(t) \sim C/n \implies \tilde{c}_n \sim \frac{C}{n\lambda_n} \sim \frac{C}{n^3}.$$

This is better than the original solution (even though they are equal), which has $c_n \sim C/n$.

Note that unlike the steady state case where v was smooth (so c_n had exponential decay), there is still a 'higher order' Gibbs' phenomenon (Figure 2) in the second derivative:

$$v_{xx} = \sum_{n=1}^{\infty} \lambda_n c_n(t)\phi_n(x) \implies \lambda_n c_n \sim \frac{C}{n}.$$

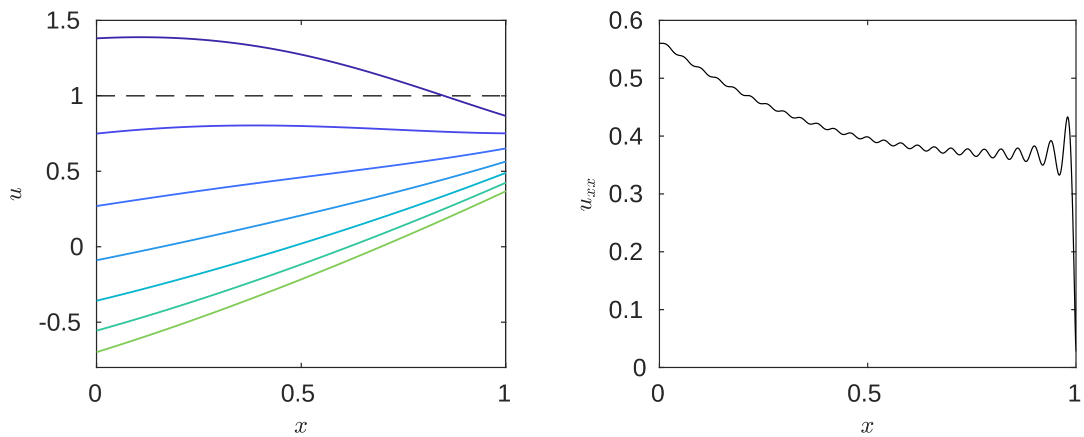


FIGURE 2. Left: Solution to (2.10) by using superposition to improve smoothness. Right: second x-derivative u_{xx} at $t = 1$ with 50 terms.

3. Separation of variables

For **homogeneous** problems, we can exploit this independence to obtain solutions quickly. The following is a **useful shortcut** for eigenfunction expansions. To illustrate, consider

$$\begin{gathered} u_t = u_{xx} + u_x + tu, \quad x \in [0, \pi],\, t > 0 \\ u(0,t) = 0, \quad u(1,t) + u_x(1,t) = 0\, t > 0 \\ u(x,0) = f(x). \end{gathered}$$

Note that this PDE is homogeneous. We look for a **separated solution**

$$u = F(t)G(x). \tag{3.1}$$

Substitute this ansatz into the PDE to get

$$\frac{F'(t)}{F(t)} = \frac{G'' + G'}{G} + (t+1).$$

Now separate t and x variables:

$$\frac{F'(t)}{F(t)} - (t+1) = \frac{G''(x) + G'(x)}{G(x)}.$$

This has the form (function of t) = (function of x) so both must equal a constant:

$$\frac{F'(t)}{F(t)} - (t+1) = \frac{G''(x) + G'(x)}{G(x)} = -\lambda. \tag{3.2}$$

Now plug (3.1) into the PDE to get

$$F(t)G(0) = 0, \quad F(t)(G(1) + G'(1)) = 0$$

This must hold for all t, which gives BCs for G. This and (3.2) gives two ODEs (one for each function) plus boundary conditions:

$$G'' + G' = -\lambda G, \quad G(0) = G(1) + G'(1) = 0$$
$$F' + (\lambda + t + 1)F = 0.$$

The first line is an eigenvalue problem with solutions G_n and eigenvalues λ_n. Then solve

$$F_n' + (\lambda_n + t)F_n = 0$$

to get $F_n(t) = a_n(\cdots)$ for arbitrary a_n. We conclude that

$$u_n(x,t) = F_n(t)G_n(x)$$

is a solution for each n. By superposition, so is any linear combination, so we get

$$u(x,t) = \sum_n F_n(t)G_n(x).$$

The constants a_n are then found in the usual way. Note that we know this process will yield the complete solution due to Sturm-Liouville theory.

3.1. **The method: summary.** The procedure is straightforward. Below is an outline; it is important to view as a strategy and not a rigid procedure. To be concrete, suppose the PDE is for $u(x,t)$ (this could be different, of course) with homogeneous BCs. To solve:

- Guess a separated solution (a product of functions of one variable):

$$u(x,t) = F(t)G(x).$$

* Plug into the PDE and separate independent variables to get an expression like

$$\text{function of } t = \text{function of } x.$$

Conclude that they are equal to a constant (this will be $\pm$ the eigenvalue).

* Use any hom. BCs to get boundary conditions for the separated functions

- Solve the eigenvalue problem to get eigenfunctions/values, then solve the other ODE(s) to get the general form of separated solutions $u_n(x,t)$.
- Assume the general solution is an eigenfunction series (add up separated solutions):

$$u = \sum_n u_n(x,t).$$

- Solve for unknown coefficients using remaining ICs, BCs etc.

The starred steps can easily fail, in which case another method must be used.

Practical note: When it works (for **homogeneous** problems), separation of variables is the easiest way to solve PDEs. For some non-homogeneous problem, superposition tricks sand other techniques can reduce the problem to a homogeneous one. The disadvantage is that it hides the underlying theory and structure (of eigenfunctions) and it fails when inhomogeneous terms are introduced.

Often, SoV is a good place to start when looking for solutions to more complicated problems (if the eigenfunction structure is not known).

3.2. **SoV, superposition and modes.** Consider again the example problem

$$\begin{aligned} u_t &= u_{xx} + g(x,t), \quad x \in [0,\pi],\ t > 0 \\ u(0,t) &= u(\pi,t) = 0,\ t > 0 \\ u(x,0) &= f(x). \end{aligned} \tag{P}$$

Let $\phi_n = \sin nx$ and $\lambda_n = n^2$ and

$$f = \sum_{n=1}^{\infty} f_n \phi_n, \quad g(x,t) = \sum_{n=1}^{\infty} g_n(t)\phi_n \quad f_n = \langle f, \phi_n\rangle / \langle \phi_n, \phi_n\rangle,\ g_n = \langle g, \phi_n\rangle / \langle \phi_n, \phi_n\rangle.$$

We can 'project' this equation onto ϕ_n to get an n-th problem

$$\begin{aligned} u_t &= u_{xx} + g_n(t)\phi_n, \quad x \in [0,\pi],\ t > 0 \\ u(0,t) &= u(\pi,t) = 0,\ t > 0 \\ u(x,0) &= f_n\phi_n. \end{aligned} \tag{P_n}$$

Let $u_n(x,t)$ be the solution to (P_n). This is sometimes referred to as the n-th **mode** of the solution (or **Fourier mode** if it is part of a Fourier series). Then the full solution u is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t).$$

That is, u is the superposition of the solutions to all the projected equations (P_n). Moreover, each mode evolves independently (according to (P_n)). To solve (P_n), we need only look for a solution with a ϕ_n term (not a full series!)

$$u_n(x,t) = c_n(t)\phi_n$$

and plug into the PDE and ICs to get

$$c_n'(t) = -\lambda_n c_n(t) + g_n(t), \quad c_n(0) = f_n.$$

This is exactly the process we used before, but using the actual projection $\langle \cdot, \phi_n\rangle\phi_n$ (returning a function) rather than $\langle \cdot, \phi_n\rangle$ (returning the coefficient).

Practical note: If the initial condition and source are a finite sum of modes, then the solution and process can be simplified since we only need to solve a finite number of problems (P_n) for one term solutions. This requires homogeneous BCs.

For example, $u_t = u_{xx} + \sin x$ with $u(0,t) = u(\pi,t)$ and $u(x,0) = \sin 2x$ has a solution with two modes, $u = c_1(t)\sin x + c_2(t)\sin 2x$. For a real example, see subsection 5.7.

Connection to SoV: The method of separation of variables is simply solving for (P_n) first, then adding the solutions together. The process finds the appropriate eigenvalue problem along the way. Note that it only works with no source ($g = 0$); otherwise the eigenfunctions must be found first.

Comparison to inhom. BCs: Note also that if there are inhomogeneous BCs then (P_n) is not the right projected equation, because it does not take the BCs into account (and we cannot project the BCs in the same way). In this case, the individual terms $c_n(t)\phi_n(x)$ are not quite solutions to the projected PDE (P_n); only the full series is a solution.

4. Wave equation

The wave equation, in one dimension, has the form

$$u_{tt} = c^2 u_{xx}$$

for $u(x,t)$. Here c is the 'wave speed'. This is the fundamental equation for describing propagation of (physical) waves e.g. elecromagnetic, seismic, sonic and so on. As with the heat equation, the wave speed may vary in space. For a vibrating string with variable density $\rho(x)$ and tension T (constant), we have, for instance,

$$\rho(x) u_{tt} = T u_{xx}.$$

4.1. **Vibrating string.**

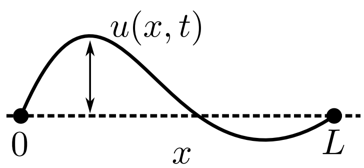
Consider, for example a string that is fixed at ends $x = 0$ and $x = L$ with constant tension T and density ρ. Let $c = \sqrt{T/\rho}$. Then the displacement $u(x,t)$ of the string can be described by the wave equation:

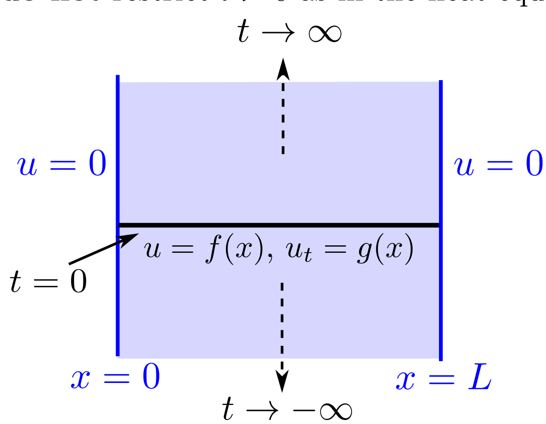
$$u_{tt} = c^2 u_{xx}, \qquad x \in (0, L)$$

The derivation is standard (see e.g. the book). Suppose that the string has, at $t = 0$, an initial displacement $f(x)$ and speed $g(x)$. The IBVP for $u(x,t)$ is

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad x \in (0, L),\ t \in \mathbb{R} \\ u(0,t) &= 0, \quad u(L,t) = 0, \\ u(x,0) &= f(x), \quad u_t(x,0) = g(x). \end{aligned} \tag{4.1}$$

Note that there are two ICs needed because of the two t-derivatives. A sketch and the domain (in the (x,t) plane) is shown below. We **do not** restrict $t > 0$ as in the heat equation.





4.2. **Solution (separation of variables).**

Look for a separated solution

$$u = h(t)\phi(x).$$

Substitute into the PDE and rearrange terms to get

$$\frac{1}{c^2}\frac{h''(t)}{h(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda.$$

The eigenvalue problem and solution are:

$$\phi'' + \lambda\phi = 0, \quad \phi(0) = \phi(L) = 0,$$

$$\implies \phi_n = \sin\frac{n\pi x}{L}, \quad \lambda_n = n^2\pi^2/L^2.$$

Define the **fundamental frequency**[1] and its multiples

$$\omega_0 = \pi c/L, \quad \omega_n = n\pi c/L.$$

Fr each λ_n, we solve for the solution $h_n(t)$ (ICs to be applied later):

$$h_n'' + c^2\lambda_n h_n = 0$$
$$\implies h_n = a_n \cos\omega_n t + b_n \sin\omega_n t.$$

The full solution to the PDE is then the series

$$u(x,t) = \sum_{n=1}^{\infty} h_n(t)\phi_n(x) = \sum_{n=1}^{\infty}(a_n \cos\omega_n t + b_n \sin\omega_n t)\sin\frac{n\pi x}{L}. \tag{4.2}$$

To find the coefficients, project onto ϕ_n to get

$$a_n = h_n(0) = \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}, \quad \omega_n b_n = h_n'(0) = \frac{\langle g, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}.$$

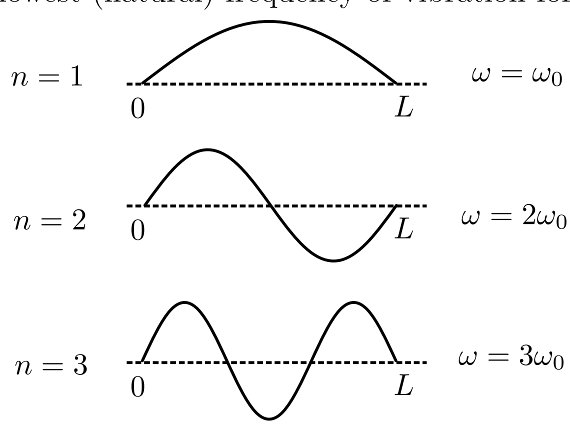
Explicitly, the formulas are (note that this is just a Fourier sine series)

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \qquad b_n = \frac{2}{nc\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}.\,dx. \tag{4.3}$$

4.3. **Standing waves.** The separated solutions (the '**modes**') in (4.2) have the form

$$u_n(t) = (a_n \cos\omega_n t + b_n \sin\omega_n t)\sin(n\pi x/L).$$

These solutions are **standing waves**, because they have points that stay fixed ('nodes'). The frequency ω_0 is the lowest (natural) frequency of vibration for the string.



Where is the wave? So far, it is not clear why the full solution describes a propagating wave. With some effort we can show that the solution to the wave equation is really a superposition of **two** superimposed waves traveling in opposite directions. Using

$$\cos nct \sin nx = \frac{1}{2}(\sin n(x+ct) + \sin n(x-ct)) = \frac{1}{2}h_n(x+ct) + \frac{1}{2}h_n(x-ct)$$

we can rewrite the solution in the form $F(x+ct) + G(x-ct)$ (**D'Alembert's formula**). This hints at key structure for the wave equation (propagation along **characteristics**) that is outside of the scope of the eigenfunction method; we will not pursue it here.

[1] Definitions vary by a factor of 2π; typically $\omega_0 = c/2L$ instead.

Example: plucking a string. Suppose a string from a guitar or harp is plucked. The initial displacement will be something like a triangular shape, such as

$$f = A \cdot \begin{cases} 2x/L & 0 \le x < L/2 \\ 2(L-x)/L & L/2 < x < L \end{cases}$$

where A is the initial displacement at $x = L/2$. The initial speed is $g = 0$. In that case, it is straightforward to show that $b_n = 0$ and

$$a_n = \frac{8A}{\pi^2 n^2} \sin \frac{n\pi}{2}.$$

In terms of the frequencies ω_n, the response of the string (Figure 3) is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(2\pi\omega_n t) \sin \frac{n\pi x}{L}. \tag{4.4}$$

Since $\sin n\pi/2 = 0$ for n even, the string, when plucked exactly at the center, vibrates with only the odd harmonics, and the amplitude of the harmonics decay quadratically with n. Note that because the initial displacement is not an eigenfunction, there are an infinite number of harmonics present. For a musical instrument, this is ideal, since the sound is much better when it is a mix of frequencies (a pure tone of one frequency is not pleasant).

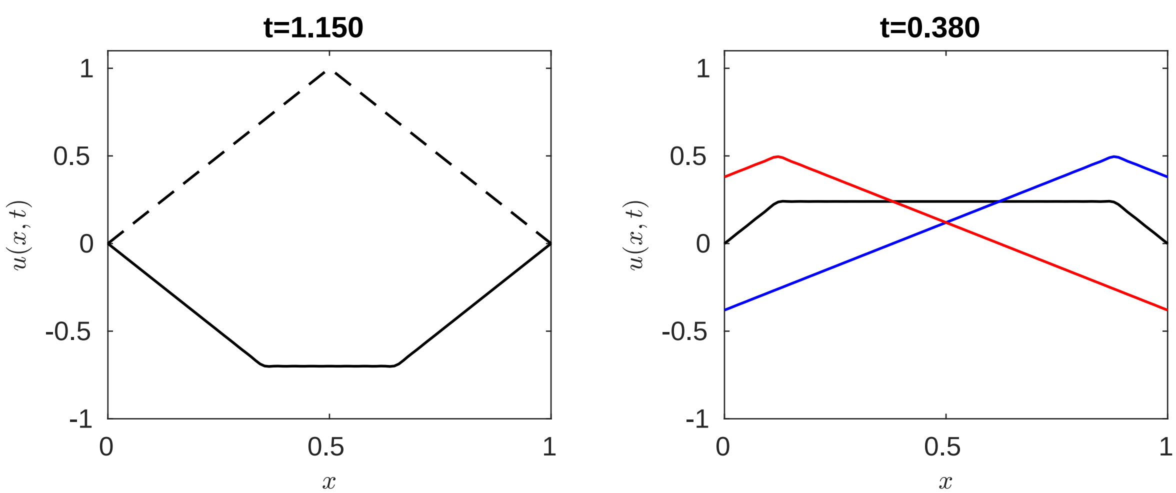


FIGURE 3. Left: solution (4.4) and initial condition (dashed). Right: solution and its two waves $h_n(x \pm ct)$ (red and blue).

Superposition of initial conditions: We can split the IBVP (4.1) into one part with zero initial speed ($u_t = 0$) and one with zero initial displacement ($u = 0$). That is, let v solve

$$\begin{aligned} v_{tt} &= c^2 v_{xx}, \quad x \in (0,L), \\ v(0,t) = 0&, \quad v(L,t) = 0, \\ v(x,0) = f(x)&, \quad v_t(x,0) = 0 \end{aligned} \tag{4.5}$$

and let w solve

$$\begin{aligned} w_{tt} &= c^2 w_{xx}, \quad x \in (0,L), \\ w(0,t) = 0, &\quad w(L,t) = 0, \\ w(x,0) = 0, &\quad w_t(x,0) = g(x). \end{aligned} \tag{4.6}$$

Then the solution $u(x,t)$ to (4.1) is

$$u = v + w.$$

Notice that the two pieces correspond to the sine/cosine terms in the full solution:

$$u(x,t) = v + w = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi c t}{L} \sin \frac{n\pi x}{L} \right) + \sum_{n=1}^{\infty} \left(b_n \sin \frac{n\pi c t}{L} \sin \frac{n\pi x}{L} \right).$$

The two parts correspond to the sine/cosine terms in t, which is easy to check directly.

4.4. **Eigenfunction expansion.** The solution via eigenfunctions is the same as for the heat equation (see HW); write $u_{tt} = -Lu$. The only difference is that

$$\langle u_{tt}, \phi_n \rangle = c_n''(t) \langle \phi_n, \phi_n \rangle$$

and both initial conditions are projected to get $c_n(0)$ and $c_n'(0)$.

4.5. **Smoothness.** Returning to the plucked string IBVP and solution, set $c = 1$ and $L = \pi$ and assume zero initial velocity:

$$\begin{aligned} u_{tt} &= u_{xx}, \quad x \in (0,\pi), \ t \in \mathbb{R} \\ u(0,t) = 0, &\quad u(\pi,t) = 0, \\ u(x,0) = f(x), &\quad u_t(x,0) = 0 \end{aligned} \tag{4.7}$$

$$f = \begin{cases} 2x/\pi & 0 \le x < \pi/2 \\ 2(\pi - x)/\pi & \pi/2 < x < \pi \end{cases}.$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(nt) \sin nx, \quad a_n = \frac{8}{\pi^2 n^2} \sin \frac{n\pi}{2}.$$

From the solution (Figure 3) it is evident that the series should converge to a function whose x-derivative is piecewise continuous (it has corners due to the 'trapezoidal' shape). Indeed, we have that

$$a_n \sim C/n^2 \implies \text{the series can be differentiated once.}$$

This is true for both t and x derivatives (both create factors of n), e.g.

$$u_t = A \sum_{n=1}^{\infty} n a_n \cos(nt) \sin nx, \qquad n a_n \sim C/n$$

so the series for u_t (and similarly u_x) will exhibit Gibbs' phenomenon. The solution to the wave equation has **at best the same smoothness as its initial condition** $u(x,0)$ (i.e. same number of derivatives).[2] That is, the wave equation does not smooth out solutions at all, unlike the heat equation.

[2] The result for the initial velocity $u_t(x,0)$ is slightly different (see HW) and also changes in higher dimensions; in even dimensions, u has one less derivative than its initial condition (see Huygen's principle).

Note (what is a solution?): We cannot differentiate again at all to get u_{tt} and u_{xx} as series, despite the fact that u **was obtained as a solution to** $u_{tt} = u_{xx}$. How is this possible? The eigenfunction series method avoids differentiating series; instead we solve

$$\langle u_{tt}, \phi_n \rangle = -\langle u, L\phi_n \rangle$$

which moves the derivatives on u_{xx} (not smooth) to ϕ_n (smooth). The framework here is that of a **weak solution**. Such solutions are the basis, for instance, of the finite element method in numerical computation and for most analysis of PDEs.
As a benefit, the weak solution $u(x,t)$ can be a non-differentiable function like the propagating triangle for the example here!

4.6. **Example: Dispersive waves.** Consider the wave equation with an added term:

$$\frac{1}{a^2}u_{tt} + \gamma^2 u = u_{xx}, \quad x \in [0,1]$$
$$u(0,t) = 0, \quad u(1,t) = 1$$
$$u(x,0) = f(x), \quad u_t(x,0) = 0.$$

We will proceed directly here.[3] Define the (self-adjoint) operator

$$Lu = -u_{xx} + \gamma^2 u,$$

(note that $Lu = -u_{xx}$ could also work). Setting $\tilde{\lambda} = \lambda - \gamma^2$, the eigenvalue problem is

$$-\phi'' = \tilde{\lambda}\phi, \qquad \phi(0) = \phi(1) = 0,$$
$$\implies \phi_n = \sin n\pi x, \quad \lambda = \pi^2 n^2 + \gamma^2, \qquad n \geq 1.$$

We seek a solution as an eigenfunction series $u(x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x)$. With $k_n = \langle \phi_n, \phi_n \rangle$,

$$\begin{aligned}
(k_n/a^2)c_n''(t) &= -\langle Lu, \phi_n \rangle \\
&= (u_x\phi_n - u\phi_n')\Big|_0^1 - \langle u, L\phi_n \rangle \\
&= -u(1,t)\phi_n'(1) - \lambda_n \langle u, \phi_n \rangle \\
&= -n\pi - \lambda_n k_n c_n
\end{aligned}$$
$$\implies c_n''(t) + a^2\lambda_n c_n(t) = -2a^2 n\pi.$$

(noting that $k_n = 1/2$ for all n). Imposing the initial condition, we find that

$$c_n(0) = b_n := 2\langle f, \phi_n \rangle, \qquad c_n'(0) = 0.$$

After solving for c_n we find that the solution is

$$u = \sum_{n=1}^{\infty} c_n(t)\phi_n(x), \qquad c_n(t) = b_n \cos(a\sqrt{\lambda_n}t) - \frac{2n\pi}{\lambda_n}(1 - \cos(a\sqrt{\lambda_n}t)).$$

By writing the solution as a sum of waves $h(x \pm c(\cdots)t)$ we can show that this solution is the sum of propagating waves; however, unlike the wave equation with $\gamma = 0$, the speed for each term will depend on n (this is 'dispersion': the speed depends on frequency).

[3]A time-independent solution $w(x)$ can be found, satisfying $\gamma^2 w = w''$ and $w(0) = 0, w(1) = 1$; then $v = u - w$ has homogeneous BCs and SoV can be used. Note that w is not a steady state as it was with the heat equation, but the procedure works the same.

5. Laplace's equation

The **Laplacian** of a function $u : \mathbb{R}^n \to \mathbb{R}$ is

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}, \tag{5.1}$$

sometimes written as $\nabla^2 u$ since it equals $\nabla \cdot (\nabla u)$ (divergence of the gradient).

The Laplacian shows up in the n-dimensional variants of the three essential linear PDEs (two of which we have already seen):

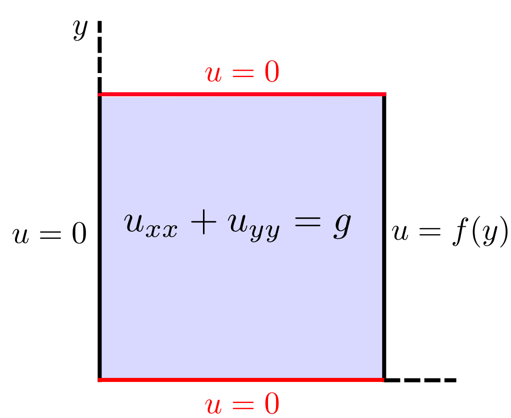
- The **heat equation** $u_t = k\Delta u$
- The **wave equation** $u_{tt} = c^2 \Delta u$
- **Laplace's equation** $\Delta u = 0$
- (and **Poisson's equation** $\Delta u = f$)

Laplace's equation is the 'steady-state' version of the heat equation ($u_t = 0$), so it describes the equilibrium state of a diffusing system. It therefore arises naturally in physics to describe systems that have 'settled' into their equilibrium. This equation describes a wide variety of phenomena: inviscid fluid flow (e.g. flow past an airfoil), stress in a solid, electric fields, wavefunctions (time independent) in quantum mechanics, and more.

What is the difference? Unlike the heat and wave equation, Laplace's equation is a 'boundary value problem' as there is no initial condition. There is also a difference in sign from the wave equation, which will change the solution.

5.1. **Solution in a rectangle.** Consider the following boundary value problem for Laplace's equation with a source term $g(x, y)$ in a square:

$$\begin{aligned}
-(u_{xx} + u_{yy}) &= g(x, y), \quad x \in (0, 1), \quad y \in (0, 1)\\
u(x, 0) = 0, \quad u(x, 1) &= 0, \quad x \in (0, 1)\\
u(0, y) = 0, \quad u(1, y) &= f(y), \quad y \in (0, 1).
\end{aligned}$$



The BCs in the y-direction are homogeneous (red) but inhomogeneous in the x-direction (at $x = 1$). This suggests expanding with eigenfunctions $\phi(y)$, as $u = \sum g_n(x)\phi_n(y)$.
Note: separation of variables could be used here, in which case this choice is required.

Solution with eigenfunctions: Due to homogeneous BCs in y, it is easiest to write

$$u_{xx} = Lu - g, \qquad Lu := -u_{yy}.$$

The eigenvalue problem, using the BCs at the top/bottom face, is

$$\phi''(y) + \lambda\phi(y) = 0, \quad \phi(0) = \phi(1) = 0 \tag{5.2}$$

$$\implies \phi_n(y) = \sin n\pi y, \quad \lambda_n = n^2\pi^2, \qquad n \geq 1.$$

Let $\langle f_1, f_2 \rangle$ denote the L^2 inner product in y, $\int_0^1 f_1 f_2 \, dy$. The solution is

$$u(x,y) = \sum_{n=1}^{\infty} h_n(x)\phi_n(y). \tag{5.3}$$

By the usual calculations, we get (note that the BCs are homogeneous)

$$h_n''\langle \phi_n, \phi_n\rangle = 0 + \langle Lu, \phi_n\rangle - \langle g, \phi_n \rangle$$

$$\implies h_n'' - \lambda_n h_n = r_n, \qquad r_n := \langle g, \phi_n\rangle / \langle \phi_n, \phi_n\rangle. \tag{5.4}$$

(**Remark:** Separation of variables would yield the same equations (5.2) and (5.4) if $g = 0$.)

The general solution to (5.4) is $a_n e^{n\pi x} + b_n e^{-n\pi x} + r_n/\lambda_n$ or, equivalently,

$$h_n(x) = a_n \sinh n\pi x + b_n \cosh n\pi x + r_n/\lambda_n$$

Finally, project the BCs at $x = 0$ and $x = 1$ onto ϕ_n to get boundary conditions for h_n:

$$0 = u(0,y) \implies h_n(0) = 0, \quad f = u(1,y) \implies h_n(1) = \langle f, \phi_n\rangle / \langle \phi_n, \phi_n \rangle.$$

Solution (no source) If there is no source ($g = 0$) then $h_n(0) = 0$ gives

$$h_n(x) = a_n \sin hn\pi x$$

and the BC at $x = 1$ gives

$$f(y) = u(1,y) = \sum_{n=1}^{\infty} a_n \sinh n\pi \phi_n(y) \implies a_n \sinh n\pi = \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}.$$

Explicitly, the coefficients are

$$a_n = \frac{1}{\sinh n\pi} \frac{\langle f, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle} = \frac{2}{\sinh n\pi} \int_0^1 f(y)\phi_n(y)\, dy, \qquad n \geq 1.$$

Since sinh grows exponentially, we may worry that (5.3) will not converge due to the $\sinh n\pi x$, but this is not the case (see below).

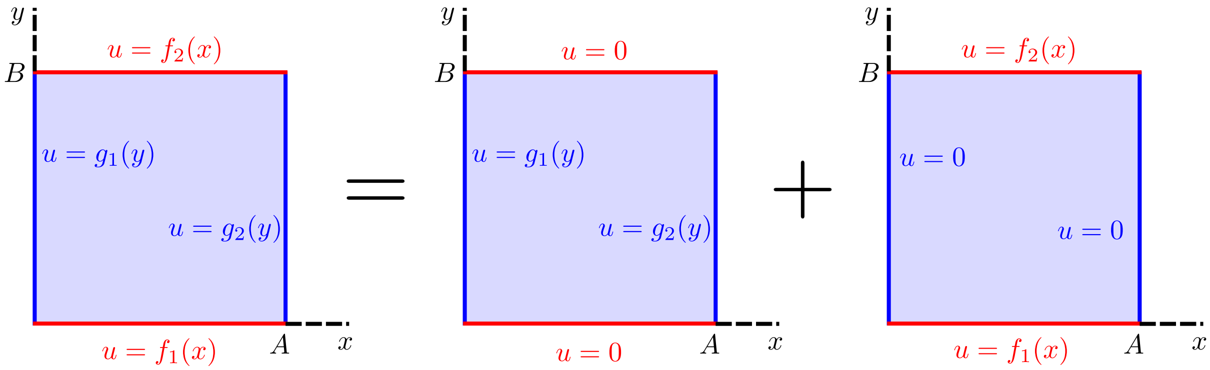
Smoothness: We can deduce the smoothness of the solution from the series

$$u(x,y) = \sum_{n=1}^{\infty} g_n(x)\phi_n(y), \qquad g_n = (\cdots)\frac{\sinh n\pi x}{\sinh n\pi}.$$

It is not too hard to show that $\sinh n\pi \sim Ce^{n\pi}$ and $\sinh n\pi x \sim Ce^{n\pi|x|}$ so g_n decays exponentially for x in the domain (since $|x| < 1$). It follows that the solution is smooth (the series has infinitely many derivatives!), **even if the boundary data ($f(y)$) is not**.

5.2. **Rectangle, with more boundary conditions.** Let's return to the rectangle example and consider how to solve the problem when there are inhomogeneous boundary conditions applied at all the sides for Laplace's equation in a rectangle of width A and height B:

$$\begin{aligned} 0 &= u_{xx} + u_{yy}, \quad x \in (0,a), \quad y \in (0,b) \\ u(x,0) = f_1(x), &\quad u(x,1) = f_2(x), \quad x \in (0,A) \\ u(0,y) = g_1(y), &\quad u(1,y) = g_2(y), \quad y \in (0,B). \end{aligned} \tag{5.5}$$



Both pairs of opposite sides (in blue and red above) could have non-homogeneous BCs. We could solve this directly using eigenfunctions in either direction. However, a better solution (smoother) can be obtained using superposition.

To do so, we break the problem up into two parts, each with **homogeneous BCs on one pairs of sides**. To be precise, we find v and w solving

$$\begin{aligned} 0 &= v_{xx} + v_{yy}, \quad x \in (0,A), \quad y \in (0,B) \\ v(x,0) = 0, &\quad v(x,B) = 0, \quad x \in (0,A) \\ v(0,y) = g_1(y), &\quad v(A,y) = g_2(y), \quad y \in (0,B). \end{aligned} \tag{5.6}$$

$$\begin{aligned} 0 &= w_{xx} + w_{yy}, \quad x \in (0,A), \quad y \in (0,B) \\ w(x,0) = f_1(x), &\quad w(x,B) = f_2(x), \quad x \in (0,A) \\ w(0,y) = 0, &\quad w(A,y) = 0, \quad y \in (0,B). \end{aligned} \tag{5.7}$$

The sum $u = v + w$ is then the solution to (5.5) (example plotted in Figure 4).

Solving for v: To solve (5.6), solve as in the previous example. For the eigenfunctions:

$$\phi'' + \lambda\phi = 0, \qquad \phi(0) = \phi(b) = 0,$$

$$\implies \phi_n(y) = \sin(n\pi y/B), \quad \lambda_n = n^2\pi^2/B^2.$$

There are no BCs to apply for h (both inhomogeneous), so we find the general solution to

$$h_n''(x) - \lambda_n h_n(x) = 0.$$

Set $\mu_n = n\pi/a$. It is convenient to use sinh centered at $x = 0$ and $x = a$ as the basis:

$$h_n(x) = a_n \sinh(\mu_n(A - x)) + b_n \sinh \mu_n x.$$

This is permitted since $e^{\mu_n x}$ and $e^{-\mu_n x}$ are solutions. It follows that

$$v(x,y) = \sum_{n=1}^{\infty} \Big[a_n \sinh(\mu_n(A - x)) + b_n \sinh(\mu_n x)\Big]\phi_n(y).$$

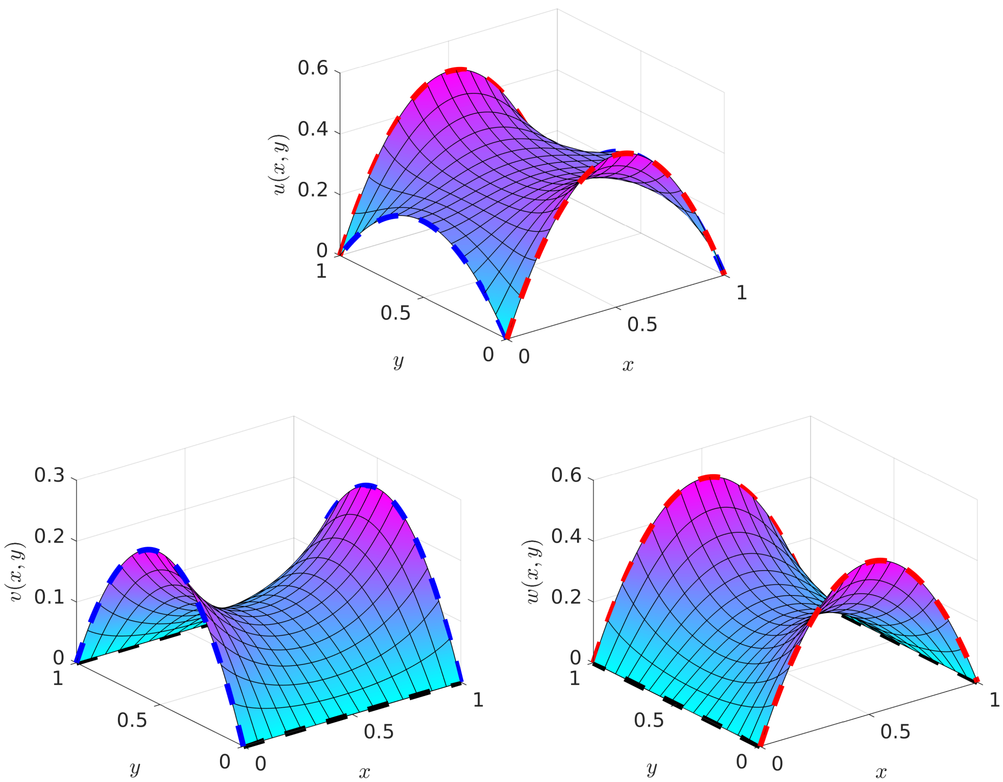


FIGURE 4. Solution $u = v + w$ to (5.5) (top) and parts v, w (bottom) solving (5.6) and (5.7) for $f_1 = f_2 = x(1-x)$ and $g_1 = g_2 = y(1-y)$.

Now use the BCs at $x = 0$ and $x = A$ (this is where the choice of basis is useful):

$$\text{at } x = 0: \quad g_1(y) = v(0, y) = \textstyle\sum_{n=1}^{\infty} a_n \sinh(\mu_n)\phi_n(y)$$
$$\text{at } x = A: \quad g_2(y) = v(A, y) = \textstyle\sum_{n=1}^{\infty} b_n \sinh(\mu_n A)\phi_n(y).$$

Let $\langle f, g\rangle = \int_0^B f(y)g(y)\,dy$ denote the L^2 inner product in $[0, B]$ (the y-direction). Then

$$a_n \sinh \mu_n = \frac{\langle g_1, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle} = \frac{2}{B}\int_0^B g_1(y) \sin\frac{n\pi y}{B}\,dy$$
$$b_n \sinh(\mu_n A) = \frac{\langle g_2, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle} = \frac{2}{B}\int_0^B g_2(y) \sin\frac{n\pi y}{B}\,dy.$$

Solving for w: The process for w is the same, but with eigenfunctions $\psi_n(x)$ in $[0, a]$ and coefficients $q_n(y)$ instead (left as an exercise).

The full solution: Finally, the solution to the original problem (5.5) is

$$u = v + w = \sum_{n=1}^{\infty} h_n(x)\phi_n(y) + \sum_{n=1}^{\infty} q_n(y)\psi_n(x).$$

Both v and w have homogeneous BCs that match the eigenfunctions, so they will have better convergence than the series for u obtained directly.

5.3. **Separable boundary conditions.** When applying the eigenfunction method, one must pick a direction for the eigenfunctions, either

$$u = \sum c_n(x)\phi_n(y) \quad \text{or} \quad u = \sum c_n(y)\phi_n(x).$$

If there is a direction where the BCs are homogeneous, this is typically a good choice (and required for using SoV). However, this relies on the ability to 'separate' the BCs into independent directions. For example, consider the triangle

$$\{(x, y) : x, y \geq 0,\ x + y \leq 1\}.$$

There is no single eigenvalue problem for $\phi(x)$ or $\phi(y)$; it would have to change in the other variable. The boundary for this problem is **non-separable**.

To deal with non-separable boundary conditions, one needs other techniques (which we'll see, if not study in detail, later).

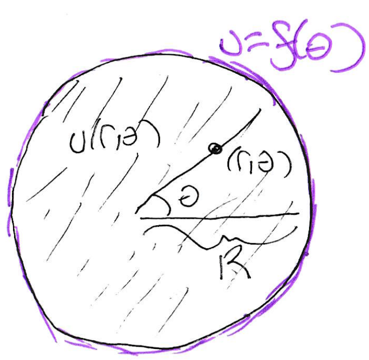
5.4. **In a circle.** Here, the boundary conditions can be separated using polar coordinates. Laplace's equation for $u(r, \theta)$ in a disk with a prescribed value $f(\theta)$ on the boundary is

$$\begin{gathered} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r \in (0, R), \quad \theta \in [0, 2\pi] \\ u(R, \theta) = f(\theta), \quad \theta \in [0, 2\pi] \end{gathered}$$

We also need **periodic boundary conditions** in θ and a boundedness condition:

$$u(r, 0) = u(r, 2\pi),\ u_r(r, 0) = u_r(0, 2\pi) \tag{5.8}$$

$$u(r, \theta) \text{is bounded for } r \in [0, R] \tag{5.9}$$



The correct way to write the problem in operator terms is

$$u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}Lu = 0, \quad Lu = -u_{\theta\theta}.$$

This is not obvious! To 'derive it', **we can use separation of variables.** Look for solutions

$$u = g(r)h(\theta).$$

Substituting into the PDE we get

$$g''(r)h(\theta) + \frac{1}{r}g'(r)h(\theta) + \frac{1}{r^2}g(r)h''(\theta) = 0$$

$$\implies \frac{r^2 g''(r) + r g'(r)}{g(r)} = -\lambda \frac{h''(\theta)}{h(\theta)} \tag{5.10}$$

With the periodic boundary conditions (5.8), we get a familiar eigenvalue problem:

$$h''(\theta) + \lambda h(\theta) = 0, \qquad h(0) = h(2\pi), \quad h'(0) = h'(2\pi)$$

$$\implies h_0 = a_0, \lambda_0 = 0, \quad h_n(\theta) = a_n \cos n\theta + b_n \sin n\theta, \quad \lambda_n = n^2, \quad n \geq 1 \tag{5.11}$$

where a_n, b_n are arbitrary.

Caution: As a warning, if the PDE is not homogeneous, separation of variables stops being useful here. At this point, we take the eigenfunctions and eigenvalues and proceed with the **eigenfunction expansion method** (see subsection 5.5.

In what follows, the PDE is homogeneous, so we can just find separated solutions and use superposition.

We now solve for g_n from (5.10):

$$r^2 g_n''(r) + r g_n'(r) - n^2 g_n(r) = 0.$$

The ODE is a Cauchy-Euler equation with roots $\pm n$; the solution is

$$g_n = c_n r^n + d_n r^{-n}.$$

By the boundedness condition (5.9), $d_n = 0$ so $g_n = c_n r^n$. The separated solutions are then

$$u_0 = a_0, \quad u_n = r^n(a_n \cos n\theta + b_n \sin n\theta), \quad n \geq 1$$

for arbitrary constants a_n and b_n (note that $1/2$ is not chosen here for simplicity).

Since the problem is homogeneous, the full solution is the superposition of the separated solutions. The full solution is the Fourier series (with $\phi_n = \cos n\theta$ for $n \geq 0$ and $\psi_n = \sin n\theta$)

$$u(r,\theta) = a_0\phi_0 + \sum_{n=1}^{\infty} r^n(a_n\phi_n + b_n\psi_n). \tag{5.12}$$

Imposing the boundary condition at $r = R$:

$$f(\theta) = u(R,\theta) = a_0\phi_0 + \sum_{n=1}^{\infty} R^n(a_n\phi_n + b_n\psi_n)$$

so by the usual calculation[4] for the coefficients (with $\langle f, g\rangle = \int_0^{2\pi} f(\theta)g(\theta)\,d\theta$)

$$\begin{aligned} a_n &= \frac{\langle f,\phi_n\rangle}{\langle \phi_n,\phi_n\rangle} = \begin{cases} \frac{1}{\pi R^n}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta & \text{for } n \geq 1 \\ \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta & \text{for } n = 0 \end{cases}, \\ b_n &= \frac{\langle f,\psi_n\rangle}{\langle \psi_n,\psi_n\rangle} = \frac{1}{\pi R^n}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta. \end{aligned} \tag{5.13}$$

Now we are done; the solution is the Fourier series (5.12) with coefficients (5.13).

[4]Note that this is the Fourier series, except with $\phi_0 = 1$ instead of $\phi_0 = 1/2$, chosen to match the heat equation examples from before. The choice of constant does not matter as long as you proceed from the orthogonality formula $\langle f, \phi_n\rangle / \langle \phi_n, \phi_n\rangle$.

5.5. Inhomogenous case. Suppose instead the PDE is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = g(r,\theta).$$

To solve, we assume **homogeneous** BCs, PDE etc. ($g = 0$ here) and can use separation of variables to obtain the appropriate L, σ and eigenfunctions/values.

Here the eigenfunctions are ϕ_n, ψ_n as before with $Lu = -u_{\theta\theta}$. But there is a source, so we **cannot continue** with separation of variables.

Instead, write the solution as an eigenfunction expansion to start,

$$u(x,t) = a_0\phi_0 + \sum_{n=1}^{\infty} a_n(r)\phi_n(\theta) + b_n(r)\psi_n(\theta)$$

Take inner product of the PDE (note that $\sigma = 1$) with ϕ_n and ψ_n to get coefficient ODEs. For ϕ_n's,

$$(c_n''(r) + \frac{1}{r}c_n'(r))\langle\phi_n,\phi_n\rangle - \frac{1}{r^2}\langle Lu,\phi_n\rangle = \langle g,\phi_n\rangle$$

and then $\langle Lu,\phi_n\rangle = \langle u, L\phi_n\rangle$ to get

$$a_n''(r) + \frac{1}{r}a_n'(r) - \lambda_n a_n(r) = \frac{\langle g,\phi_n\rangle}{\langle\phi_n,\phi_n\rangle}$$

and the same for ψ_n to get

$$b_n''(r) + \frac{1}{r}b_n'(r) - \lambda_n b_n(r) = \frac{\langle g,\psi_n\rangle}{\langle\psi_n,\psi_n\rangle}$$

Here, since the only inhomogeneous term is a source, it is the same as the previous example, but with an inhomogeneous term in the ODEs for a_n, b_n.

5.6. Solvability conditions. As we saw with BVPs in 1d, there can be solvability conditions on f. For instance, suppose in the above example, we instead have

$$u_r(R,\theta) = f(\theta).$$

That is, the flux into the disk at θ is $f(\theta)$. The solution still have the form (5.12), but now

$$f(\theta) = u_r(R,\theta) = \sum_{n=1}^{\infty} nR^{n-1}(a_n\phi_n + b_n\psi_n).$$

This determines a_n, b_n for $n \geq 1$ but not a_0. To determine the constraint on f, take the inner product of this formula with ϕ_0. Since ϕ_0 is orthogonal to all the eigenfunctions in the sum,

$$\langle f,\phi_0\rangle = 0 \implies \int_0^{2\pi} f(\theta)\,d\theta = 0$$

which says the net flux into the disk must be zero. Note that this can be obtained (as we'll see later in studying 2d problems generally) by integrating the PDE over the domain:

$$0 = \int_0^{2\pi}\int_0^r \left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) r\,dr\,d\theta = 2\pi\int_0^{2\pi}\left(\int_0^r (ru_r)_r\,dr\right)d\theta = 2\pi R\int_0^{2\pi} f(\theta)\,d\theta$$

using periodic boundary conditions (the $u_{\theta\theta}$ term is integrated out) and $ru_{rr} + u_r = (ru_r)_r$.

5.7. **Physics example: fluid flow.** In fluid dynamics, the steady (i.e. time-independent) flow of an inviscid fluid (e.g. water) can be described by a *streamfunction* ψ. The curves $\psi = \text{const.}$ give the path of the fluid flow.

Suppose the fluid flows in the x-direction at a constant speed U everywhere. Then a circle of radius R (really a cylinder) is placed in the flow at $r = 0$. The streamfunction $\psi(r, \theta)$, defined *outside* the circle, can be shown to be the solution to Laplace's equation,

$$\frac{1}{r}(r\psi_r)_r + \frac{1}{r^2}\psi_{\theta\theta} = 0, \qquad r \in (R, \infty), \quad \theta \in [0, 2\pi], \tag{5.14}$$

$$\psi \sim \text{`}Ur\sin\theta \text{ as } r \to \infty, \quad \psi(R, \theta) = 0. \tag{5.15}$$

Note the BC is ∞ is *not* that ψ is bounded. We can use a similar process to the above. The eigenfunctions/values and coefficient ODE are the same; the solution has the form

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} c_n(r)(a_n \cos n\theta + b_n \sin n\theta), \tag{5.16}$$

$$c_n(r) = (\cdots)r^n + (\cdots)r^{-n}.$$

We could solve the equation this way, but as a shortcut, note that the only 'inhomogeneous' part is the BC $\psi \to r\sin\theta$. Thus only the $\sin\theta$ term is non-zero in the solution:

$$\psi(r, \theta) = c_1(r)\sin\theta.$$

From the ODE for c_1 and the BCs we get

$$c_1 = ar + b/r, \;\; c_1(R) = 0, \;\; c_1 \sim Ur \text{ as } r \to \infty$$

so the solution is

$$\psi(r, \theta) = U\left(r - \frac{R^2}{r}\right)\sin\theta.$$

A contour plot of ψ shows the streamlines (the fluid flow past the cylinder).

6. Overview

We have now introduced the three canonical examples of the three main classes of (linear) PDEs. They represent, in a sense, the typical behavior of each type:

- **Parabolic:** e.g. $u_t = u_{xx}$ (heat equation). Solutions want to decay towards an equilibrium; diffusion spreads out the solution. The solution becomes smooth. Irreversible - only well-posed forward in time.
- **Hyperbolic:** e.g. $u_{tt} = c^2 u_{xx}$ (wave equation). Solutions propagate at a speed c (the wave speed). Does not smooth solutions; discontinuities remain.
- **Elliptic:** e.g. $u_{xx} + u_{yy} = 0$. Smooths initial data (like the heat equation); describes systems in equilibrium.

Each type has unique properties and captures a certain kind of qualitative behavior. Together, the three categories describe a vast array of physical phenomena, and are the building blocks for most PDE models of physical problems.

Note that the non-smoothing nature of the wave equation makes it awkward to solve with eigenfunctions. The main approach for hyperbolic equations, which better handles discontinuities, is different (starting with the **method of characteristics**). We will not study this method here (but is important to know if you intend to deal with hyperbolic PDEs).

Terminology: The terms 'parabolic' etc. only relate indirectly to the properties of the equation; they have a specific technical meaning that does not give much intuition.

Smoothness results: For a solution $u = \sum c_n(t)\phi_n(x)$,

$$u_{tt} = u_{xx},\ \text{hom BCs}, u(x,0) = f(x) \implies u \text{ is smooth for } t > 0$$

$$u_t = u_{xx} + h,\ \text{hom BCs} \implies u \text{ has two more derivs. than } h$$

$$u_{tt} = u_{xx},\ \text{hom BCs}, u(x,0) = f(x),\ u(x,0) = g(x) \implies u \text{ has derivs. equal to } f\text{, one more than } g$$

$$u_{xx} + u_{yy} = h,\ \text{hom BCs} \implies u \text{ has two more derivs. than } h$$

The heat equation smooths initial conditions completely. Both the heat equation and Laplace's equation smooth a source by two derivatives (so u_{xx} is defined for the series!). The wave equation does not smooth the initial condition.

7. Review

7.1. **More on modes and forcing.** Let us revisit the wave equation for a physical interpration of superposition. One term of the eigenfunction expansion, $h_n(t)\phi_n(x)$, is called an **eigenmode** (or just **mode**, or **Fourier mode** when relevant) of the solution. Consider

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + f(x,t) \quad x \in (0,\pi), \\ u(0,t) &= 0, \quad u(\pi,t) = 0 \\ u(x,0) &= 0, \quad u_t(x,0) = 0. \end{aligned}$$

That is, a string with fixed ends is driven by a forcing $h(x,t)$. Physically, the n-th term in the eigenfunction series, $h_n(t)\phi_n(x)$, is the response of the n-th 'mode of vibration' of the system to the forcing (the string wants to vibrate as standing waves at natural wavelengths).

One mode: Suppose the forcing is in the shape of one of the modes (a standing wave),

$$f(x,t) = f_N(t)\sin Nx.$$

The forcing is orthogonal to all the modes except the N-th ($\langle f, \phi_n\rangle = 0$ for $n \neq N$), so only the N-th mode of the system has a response:

$$u(x,t) = h_N(t)\phi_N(x), \quad h_N''(t) + \lambda_N c^2 h_N(t) = f_N(t).$$

This is the case since the system starts at rest so for all the other modes

$$h_n''(t) + k^2 h_n(t) = 0, \; h_n(0) = h_n'(0) = 0 \implies h_n(t) = 0.$$

Superposition: More generally, suppose

$$f(x,t) = \sum f_n(t)\phi_n(x).$$

Each mode evolves independently, driven by the n-th mode of the forcing $f(x,t)$:

$$h_n''(t) + \lambda_n c^2 h_n(t) = f_n(t).$$

The solution is the superposition of these modes:

$$u(x,t) = \sum_n h_n(t)\phi_n(x),$$

The same is true for initial conditions. Orthogonality says that distinct eigenmodes do not interact with each other - the system is, in effect, a superposition of independent one-dimensional systems for each mode.

Boundary forcing? The above is **not** quite true for inhomogeneous BCs. Suppose the string is instead forced by moving the point at $x = 0$:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \quad x \in (0,\pi), \\ u(0,t) &= f(t), \quad u(\pi,t) = 0 \\ u(x,0) &= 0, \quad u_t(x,0) = 0. \end{aligned}$$

Now we know from the eigenfunction expansion method that, after projecting with $\langle \cdot, \phi_n\rangle$,

$$h_n''(t) + \lambda_n c^2 h_n(t) = \frac{1}{\langle \phi_n, \phi_n\rangle}(\phi_n u_x - \phi_n' u)\Big|_0^\pi.$$

The boundary term enters into all modes (in general). Thus, if forced by oscillating one end, the response of the system will involve all modes.

7.2. **The heat equation, from simplest to most complicated.** Here the tools introduced for solving PDEs are reviewed by solving various forms of the heat equation, along with the associated procedure. It's important to note that the (direct) 'eigenfunction expansion method' works in all cases, and provides all the underlying structure.

Unless otherwise noted, the domain is $[0, \pi]$ and the PDE holds for $t > 0$. For the most part, the operator $Lu = -u_{xx}$ could be replaced by some other SL operator and the methods still apply (see last case).

Homogeneous: Separation of variables can be used here.

$$\begin{aligned} u_t &= u_{xx} \\ u(0,t) = 0, &\quad u(\pi,t) = 0 \\ u(x,0) &= f(x) \end{aligned} \tag{W0}$$

Let $f = \sum f_n \phi_n$ where ϕ_n's are the eigenfunctions; then

$$u(x,t) = \sum_n c_n(t)\phi_n(x), \quad c_n'(t) + \lambda_n c_n(t) = 0,\, c_n(0) = f_n.$$

For SoV: look for solutions $c(t)\phi(x) \implies c'(t)/c(t) = \phi''/\phi = -\lambda$.

One mode: Easily solved by recognizing that the solution is one term.

$$\begin{aligned} u_t &= u_{xx} + g(t)\sin Nx, \\ u(0,t) = 0, &\quad u(\pi,t) = 0 \\ u(x,0) &= A\sin(Nx) \end{aligned} \tag{W1}$$

$$\begin{aligned} \text{Solution:} \quad u(x,t) &= c_N(t)\phi_N(x), \\ c_N'(t) + \lambda_N c_N(t) &= g(t),\ c_N(0) = A \end{aligned}$$

Explicitly, $c_N(t) = Ae^{-\lambda_N t} + e^{-\lambda_N t}\int_0^t e^{\lambda_N s} g(s)\, ds$. If there is no source, this is a one term version of (W0).

Source term; homogeneous BCs: Superposition of solutions to (W1); solve directly (project, solve for n-th term). Can use separation of variables if there is no source ($g = 0$), or superposition into problems of the form (W1) if it is convenient.

$$\begin{aligned} u_t &= u_{xx} + g(x,t), \\ u(0,t) = 0, &\quad u(\pi,t) = 0 \\ u(x,0) &= f(x) \end{aligned} \tag{W2}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x),$$
$$c_n'(t) + \lambda_n c_n(t) = g_n(t),\ \ c_n(0) = f_n,$$
$$f = \sum_{n=1}^{\infty} f_n\phi_n, \quad g(x,t) = \sum_{n=1}^{\infty} g_n(t)\phi_n$$

Each mode $u_n = c_n(t)\phi_n(x)$ solves 'projected' PDE $u_t = u_{xx} + g_n(t)\phi_n,\ u(x,0) = f_n\phi_n$.

Steady state: Reduce to (W2) (with no source) by subtracting a time-independent solution.

$$\begin{aligned} u_t =& u_{xx} + g(x), \\ u(0,t) = A, &\quad u(\pi,t) = B \\ u(x,0) =& f(x) \end{aligned} \tag{W3}$$

Solution is $u = \overline{u}(x) + v(x,t)$ where

$$\begin{aligned} \overline{u}'' + g = 0 \\ \overline{u}(0) = A, \quad \overline{u}(\pi) = B, \end{aligned} \qquad \begin{aligned} v_t =& v_{xx}, \\ v(0,t) = 0, &\quad u(\pi,t) = 0 \\ v(x,0) =& f(x) - \overline{u}(x) \end{aligned}$$

$$\text{Solution:}\ \ u(x,t) = \overline{u}(x) + \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\phi_n(x), \quad a_n = \langle f - \overline{u}, \phi_n\rangle / \langle \phi_n, \phi_n\rangle.$$

Note that since the PDE/BCs are homogeneous for v, SoV can always be used on the v part.

Inhomogeneous, cheap way: Use a 'boundary function' to reduce to source case (W2).

$$\begin{aligned} u_t =& u_{xx} + g(x,t), \\ u(0,t) = A(t), &\quad u(\pi,t) = B(t) \\ u(x,0) =& f(x) \end{aligned} \tag{W4}$$

Solution is $u = v + w$ where w (**not a solution!**) satisfies BCs:

$$w(0,t) = A(t), \quad w(\pi,t) = B, \qquad \begin{aligned} v_t = v_{xx} + & g(x,t) - \widetilde{g}(x,t), \\ v(0,t) = 0, &\quad u(\pi,t) = 0 \\ v(x,0) = & f(x) - w(x,0) \end{aligned}$$

where $\widetilde{g}(x,t) = w_t - w_{xx}$ (so w solves the PDE $w_t = w_{xx} + \tilde{g}$).
Note that SoV cannot be used here due to the source (but superposition plus (W1)) could solve the problem for v).[5]

Inhomogeneous, direct: Use a direct eigenfunction expansion. For (W4), solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x)$$

[5] There is a trick to also move the source into an IC, to turn this problem into a homogeneous one where SoV applies, but the form of the solution can be inconvenient. This is **Duhamel's principle**.

where c_n's are obtained by projecting PDE:

$$u_t = -Lu + g \xRightarrow{\langle\cdot,\phi_n\rangle} c_n'(t) = -\lambda_n c_n(t) + \langle g, \phi_n\rangle + B_n.$$

The B_n's are boundary terms from $-\langle Lu, \phi_n\rangle = B_n - \langle u, L\phi_n\rangle$.

Non-uniform (convert to self-adjoint): Methods above apply (different eigenfunctions):

$$\begin{aligned} u_t &= -\,Lu + g(x,t), \\ u(0,t) = A(t), &\quad u(\pi,t) = B(t) \\ u(x,0) &= f(x) \end{aligned} \tag{W5}$$

Regardless of method, obtain σ by putting in self-adjoint form, $Lu = (1/\sigma)\tilde{L}u$. Eigenfunctions ϕ_n are orthogonal in σ inner product, and solve the eigenvalue problem

$$L\phi = \lambda\phi \text{ (+ BCs) }, \iff \tilde{L}\phi = \lambda\sigma\phi \text{ (+ BCs) }.$$

For eigenfunction expansion, project PDE in self-adjoint form using σ inner product,

$$u_t = -\frac{1}{\sigma}\tilde{L}u + g \xRightarrow{\langle\cdot,\phi_n\rangle_\sigma} \langle u_t, \phi_n\rangle_\sigma = -\langle \tilde{L}u, \phi_n\rangle + \langle g, \phi_n\rangle_\sigma$$

Solution has the form

$$u(x,t) = \sum_n c_n(t)\phi_n(x).$$

Variation: If the PDE has the form

$$\sigma u_t = -Lu + g$$

where L is **self-adjoint** then this is already in the right form, now multiplied by σ. Either divide by σ to get previous form or project with L^2 inner product:

$$\sigma u_t = -\tilde{L}u + \sigma g \xRightarrow{\langle\cdot,\phi_n\rangle} \langle \sigma u_t, \phi_n\rangle = -\langle \tilde{L}u, \phi_n\rangle + \langle \sigma g, \phi_n\rangle$$

Non self-adjoint approach: Consider (W5) (repeated here for convenience)

$$\begin{aligned} u_t &= -\,Lu + g(x,t), \\ u(0,t) = A(t), &\quad u(\pi,t) = B(t) \\ u(x,0) &= f(x) \end{aligned}$$

where L has adjoint L^*. Let ϕ_n, ψ_n denote eigenfunctions for L and L^* with eigenvalues λ_n and γ_n respectively. Then

$$u = \sum c_n(t)\phi_n, \quad c_n(t)\langle \phi_n, \psi_n\rangle = \langle u, \phi_n\rangle, \cdots$$

Take inner product of PDE with ψ_n to get ($k_n = \langle \phi_n, \psi_n\rangle$)

$$c_n'(t)k_n = -\langle Lu, \psi_n\rangle + g_n k_n$$

and with IC to get $c_n(0) = \langle f, \psi_n\rangle / k_n$. Then

$$c_n'(t)k_n = B_n - \langle u, L^*\psi_n\rangle + g_n k_n, \quad c_n(0) = \langle f, \psi_n\rangle / k_n.$$

Finally $\langle u, L^*\psi_n\rangle = \gamma_n c_n k_n$ (note: adjoint eigenvalue is not guaranteed to be the same).

For a BVP $Lu = f$, as above, but without the u_t part.

7.3. **Important details.** Things to watch out for when finding solutions.

Exceptional eigenvalues (different): Some (finitely many) eigenvalues/functions may come from 'the other case' (e.g. $\lambda < 0$ vs. $\lambda > 0$). For example,

$$\begin{aligned} u_t =& u_{xx}, \\ u(0,t) = 0, &\quad u_x(\pi,t) = 2u(\pi,t) \\ u(x,0) =& f(x) \end{aligned}$$

has positive eigenfunctions/values $\phi_n = \sin\sqrt{\lambda_n}x$ and λ_n solving $\sqrt{\lambda_n}\pi = \tan\sqrt{\lambda_n}\pi$ for $n \geq 1$. There is one negative eigenvalue[6] $\lambda_0 < 0$ with $\phi_0(x) = \sinh\sqrt{\lambda_n}x$. Solution:

$$u(x,t) = e^{-\lambda_0 t}\sinh\sqrt{\lambda_n}x + \sum_{n=1}^{\infty} c_n(t)\phi_n(x)$$

where $c_n(t) \to 0$ as $t \to \infty$ (but the first term $\to \infty$!).

There may also be more than one eigenfunction per eigenvalue, leading to terms like

$$c_n(t)(a_n\phi_n + b_n\psi_n)$$

instead of just $c_n(t)\phi_n(x)$. This appears most often with **periodic BCs**.

Exceptional eigenvalues (zero): A zero eigenvalue changes the limit as $t \to \infty$. Often, it also changes the form of the solution $c_n(t)$. Example:

$$\begin{aligned} u_t =& u_{xx} + g(x), \\ u_x(0,t) = 0, &\quad u_x(\pi,t) = 0 \\ u(x,0) =& f(x) \end{aligned}$$

where $\phi_n = \cos nx$ and $f = \sum f_n\phi_n$ and $g = \sum g_n\phi_n$. ODEs:

$$c_n'(t) + \lambda_n c_n = g_n, \quad c_n(0) = f_n.$$

$$\begin{aligned} \implies\ & c_n(t) = f_n e^{-\lambda_n t} + g_n(1 - e^{-\lambda_n t}),\ n \geq 1, \\ & c_0(t) = f_0 + g_0 t. \end{aligned}$$

A steady state exists iff $g_0 = 0$, i.e. source orthogonal to the zero eigenfunction ($\langle g, \phi_0\rangle = 0$).

Exceptional modes (ODE): As above; sometimes there is a case where the ODE must be solved differently. See resonance example ($c_n'' + \lambda_n c_n = \sin\omega t$ different if $\lambda_n = \omega^2$).

Leaving coefficients out: Some coefficients in the PDE can be left outside of the operator and weight function. For instance, we can write

$$e^{-t}\rho(x)u_t = ktxu_{xx} + g(x,t) \implies e^{-t}\sigma(x)u_t = -\frac{1}{\rho/x}(kt)Lu + \frac{1}{\rho}g(x,t)$$

[6]The choice of indexing is just to have the positive eigenvalues start at $n = 1$, not to be confused with the cases where λ_0 is zero.

where $Lu = -u_{xx}$. The weight function is $\sigma(x) = \rho/x$. The e^{-t} and kt are left out since they can be factored out of the inner product:

$$e^{-t}\langle u_t, \phi_n\rangle_\sigma = -kt\langle Lu, \phi_n\rangle + \langle g/x, \phi_n\rangle \implies e^{-t}c_n'(t) = -kt\lambda_n c_n(t) + \cdots .$$

This has the benefit that we can more easily use 'standard' results, e.g. $u_t = ku_{xx}$ has the same eigenvalues/functions regardless of k.

Practical note: If uncertain, separation of variables ($u = c(t)\phi(x)$) will provide the right eigenvalue problem (possibly not in self-adjoint form) and ODE for $c(t)$ without the boundary parts.

Implied constraints: The two most common are:

Boundedness: u is bounded in the domain

Periodic: u is periodic in one direction

e.g. $u(0,t) = u(2\pi, t)$ and $u_\theta(0,t) = u_\theta(2\pi, t)$ for the heat equation in a ring:

$$u_t = u_{\theta\theta}, \quad \theta \in [0, 2\pi], \ t > 0.$$

A variation is a condition in a limit, e.g.

$$\lim_{x\to\infty} u(x,y) = 0, \quad u(x,y) \text{ bounded as } x \to \infty, \text{ etc.}$$

Typically, these conditions **replace missing BCs** for non-regular operators.

7.4. **The toolbox.** A list of the various tools we have for accomplish various steps.

Orthogonal basis, projection: Let $\{\phi_n\}$ be a basis for L^2 functions in an interval $[a, b]$. Then we can write any such function $f(x)$ in the form

$$f(x) = \sum_n c_n \phi_n(x).$$

If f depends on other variables, the coefficients do as well:

$$f(x,t) = \sum_n c_n(t)\phi_n(x).$$

The basis is **orthogonal** with respect to an inner product $\langle f, g\rangle_\sigma = \int_a^b f(x)g(x)\sigma(x)\,dx$ if

$$\langle \phi_m, \phi_n\rangle_\sigma = 0 \text{ for } m \neq n.$$

When the basis is orthogonal with respect to this weighted inner product, the 'projection'

$$f \to \langle f, \phi_n\rangle_\sigma$$

zeros out all the terms other than the n-th:

$$\langle \sum_m c_m\phi_m, \phi_n\rangle_\sigma = \sum c_m\langle \phi_m, \phi_n\rangle_\sigma = c_n\langle \phi_n, \phi_n\rangle_\sigma.$$

Note that to get the coefficient itself, divide by the inner product of ϕ_n with itself:

$$f \to \frac{\langle f, \phi_n\rangle_\sigma}{\langle \phi_n, \phi_n\rangle_\sigma} \text{ gives the coeff. of } \phi_n \text{ in the expansion } f = \sum_n f_n\phi_n.$$

This projection ($\langle \cdot, \phi_n \rangle_\sigma$) is the **main tool for extracting the n-th term of the series**.

Eigenfunction basis: Assume $\sigma = 1$ for convenience here. Some expressions are 'easy' to project, such as:

$$\langle u_t, \phi_n \rangle = \langle \sum c_n'(t)\phi_n, \phi_n \rangle = c_n'(t)\langle \phi_n, \phi_n \rangle.$$

The same is true of forcing functions $f(x,t)$, u_{tt}, tu and so on. However to project

$$\langle Lu, \phi_n \rangle.$$

But Lu is not necessarily in the nice series form:

$$Lu = \sum_n c_n(t) L\phi_n.$$

For projection to work, we need $L\phi_n$ to be a **multiple of** ϕ_n. This means, exactly, that ϕ_n must be an eigenfunction of L. Then things work out nicely:

$$Lu = \sum_n c_n(t) L\phi_n = \sum_n (\lambda_n c_n(t))\phi_n.$$

This is why we need the ϕ's to not only be an orthogonal basis, but to also be eigenfunctions of the operator L.

Conversion to self-adjoint form: If L is second-order but not self-adjoint, we can convert it to self-adjoint form at the cost of a weight function $\sigma(x)$:

$$L = \frac{1}{\sigma}\tilde{L}.$$

The eigenvalue problem for L becomes a **weighted** eigenvalue problem for the self-adjoint operator $\tilde{L}$:

$$L\phi = \lambda\phi \iff \tilde{L}\phi = \lambda\sigma\phi.$$

The σ is important because **the eigenfunctions are orthogonal in the weigheed inner product**. We need the σ to proceed! This converts problems into 'self-adjoint' form, e.g.

$$u_t = -Lu + f \implies u_t = \frac{1}{\sigma}\tilde{L}u + f.$$

Then we can use that $\tilde{L}$ is self-adjoint in the L^2 inner product:

$$\langle (1/\sigma)\tilde{L}u, v \rangle_\sigma = \langle \tilde{L}u, v \rangle = \cdots + \langle u, \tilde{L}v \rangle$$

Caution: Note that if a problem is 'already self-adjoint' then this is not needed, e.g.

$$\sigma(x)u_t = -Lu + f$$

where L is self-adjoint. This is as above, but with σ multiplied out. The weight function is σ, the self-adjoint operator is L, the eigenvalue problem is $L\phi = \lambda\sigma\phi$. Here, either divide by σ and proceed as before, or take the L^2 inner product of both sides and note that $\langle \sigma u_t, \phi_n \rangle = \langle u_t, \phi_n \rangle_\sigma$.

Separation of variables: This method finds all 'separated' (or 'single term' or 'single mode') solutions like $c(t)\phi(x)$ for **homogeneous** problems (PDE and BCs). When this is all you need, SoV is enough. It also finds the eigenvalue problem automatically, since it always uses homogeneous BCs. It finds the coefficient ODEs for the **homogeneous** problem.

SoV cannot handle inhomgoeneous BCs or source terms. Sometimes, SoV can be used as a tool for finding 'special' separated solutions (not all solutions) to a complicated PDE.

Superposition: A linear problem for u can always be broken up into a superposition of parts, including inhomogeneous terms in the PDE, BCs or ICs. This is useful because it lets us **solve sub-problems that are (more) homogeneous and/or simpler to solve.**. This is useful when there is a simple solution that gives you one part, e.g.

$$\begin{aligned} u_t =& u_{xx} + t\sin(x), \\ u(0,t) = \sin(t), &\quad u(\pi,t) = \cos(t) \\ u(x,0) =& x^2 \end{aligned}$$

is complicated to solve, but we can deal with the source term by solving

$$\begin{aligned} v_t =& v_{xx} + t\sin x \\ v(0,t) = 0, &\quad v(\pi,t) = 0 \\ v(x,0) =& 0 \end{aligned}$$

to get $v = c_1(t)\sin x$ where $c_1'(t) + \lambda_1 c_1 = t$ (one term), then solve the problem with no source and add this to v. Some other uses include:

- Move inhom. BCs to a source ($u = v + w$ where w that satisfies the BCs) as in (W4)
- Splitting a Laplace BVP into two problems with hom. BCs one one pair of sides (or four problems where each one only has one inhom. BC!)
- Splitting a problem into one with zero ICs and one that is hom. except for ICs (what is the effect of the initial conditions on the solution?)
- Justifying the summation of separated solutions in SoV

Pointwise convergence: We find solutions in the form of eigenfunction series, e.g. something like

$$u(x) = \sum_n c_n\phi_n(x) := S_\infty(x).$$

The RHS (what we compute, the pointwise limit of the partial sums) is **not exactly equal to the solution** (LHS). Sturm-Liouville theory tells us what equals means:

- If u is continuous at a point x, then $u(x)$ and $S_\infty(x)$ agree (pointwise convergence).
- If u has a jump at a point x inside (a,b), the series converges to the average of the left/right limits at x. Near x, there is Gibbs'phenomenon (but the series does converge pointwise around the jump, just slowly).
- At the endpoints, if the BCs for u are **inhomogeneous**, then the theory 'sees' a discontinuity; there is Gibbs' phenomenon. The 'series' $S_\infty(x)$ has homogeneous BCs, but agrees with $u(x)$ arbitrarily close to the boundary. In particular,

$$\lim_{x\searrow a} S_\infty(x) = u(a), \quad \lim_{x\nearrow b} S_\infty(x) = u(b).$$

Self-adjointness and IBP: Integration by parts **moves derivatives from one function to another** at the cost of **creating some boundary terms**.

This is also how the adjoint works. If L with some hom. BCs is self-adjoint in the L^2 inner product (true of SL operators), then

$$\langle Lu, v\rangle = \langle u, Lv\rangle \text{ for all } u, v \text{ with hom. BCs.}$$

If u and v do not have hom. BCs, there are boundary terms from integration by parts:

$$\langle Lu, v\rangle = \text{boundary terms} + \langle u, Lv\rangle.$$

More generally, if L^* is the adjoint operator with BC^* (adjoint BCs) then

$$\langle Lu, v\rangle = \langle u, L^*v\rangle \text{ for all } u \text{ with hom. } BC \text{ and } v \text{ with hom. } BC^*.$$

The adjoint property lets us **move an operator from one function in an inner product onto the other** (at the cost of creating boundary terms if not homogeneous).

Bi-orthogonality: The adjoint eigenfunctions ψ_n and eigenfunctions ψ of L^* and L are **bi-orthogonal** ($\langle \phi_m, \psi_n\rangle = 0$ for $m \neq n$). Projection for the basis $\{\phi_n\}$ uses $\langle \cdot, \psi_n\rangle$ (inner product with adjoint eigenfunction):

$$f = \sum c_n\phi_n \implies \langle f, \psi_n\rangle = c_n\langle \phi_n, \psi_n\rangle.$$

The eigenfunction property plus adjoint property allows calculations like

$$\langle Lu, \psi_n\rangle = B + \langle u, L^*\psi_n\rangle = B + \langle u, \gamma_n\psi_n\rangle = B + \gamma_n\langle u, \psi_n\rangle$$

where $L^*\psi_n = \gamma_n\psi_n$ (γ_n an eigenvalue for the adjoint). This is the way to do the eigenfunction expansion without converting to self-adjoint form.

Differentiating series: An eigenfunction series

$$f = \sum c_n\phi_n$$

can be differentiated term-by-term **so long as the result makes sense** (converges). This can be used, for instance, to calculate

$$u_t = \frac{\partial}{\partial t}\sum_n c_n(t)\phi_n = \sum_n c_n'(t)\phi_n$$

or to calculate

$$u_{xx} = \sum_n c_n(t)\phi_n''(x) \text{ if this series converges .}$$

Note that if ϕ_n'' gives factors like n^2 we need $c_n \sim C/n^3$ for the above to work.

Smoothness from coefficients: If coefficients for $u = \sum c_n\phi_n$ decay like $c_n \sim C/n^{k+1}$ then u is (at least) k times differentiable, with a jump in the k-th. We can use this to deduce the smoothness of the series from the computed coefficients, and to know when Gibbs' phenomenon appears ($c_n \sim 1/n$; piecewise continuous but has jumps).

Eigenvalue problems, explicitly: See procedure for solving eigenvalue problems. For LCC eigenvalue problems (or any problem where the general solution can be found), we can always solve it by:

- Finding the general solution $\phi = \sum_{j=1}^n c_jy_j$ (where y_j's are solutions)
- Applying BCs to get constraints on the coefficients + λ

- Solve for λ (or approximate, or find numerically) or show there are no solutions
- Plug back into general solution to get ϕ's

Rayleigh quotient: Prove eigenvalue are positive by considering

$$L\phi = \lambda\sigma\phi$$

and taking the L^2 inner product with ϕ (multiply by ϕ, integrate over domain):

$$\langle \phi, L\phi\rangle = \lambda\langle \phi, \phi\rangle_\sigma.$$

Then expand the LHS by integrating by parts[7] to look for squared terms; show that $\langle \phi, L\phi\rangle \geq 0$. Then check $\lambda = 0$ case, either by solving $L\phi = 0$ directly or showing that ϕ must be zero from the Rayleigh quotient.

Solvability conditions: Some 'coefficient' equations may only have solutions if certain conditions (solvability conditions) hold. Check by attempting to solve; be on the lookout for 'degenerate' cases where the general formula fails (usually $\lambda = 0$). [8]

[7]For second order: IBP once. For other eqs: may need to IBP more than once. Note that if you integrate by parts twice for the second order equation, you get $\langle L\phi, \phi\rangle$, which is not useful (too far!).

[8]The general principle here is the **Fredholm alternative**; this abstract version is not on the first exam, but the 'direct' version by solving the problem you should know.