Decay of Fourier coefficients

To get a better sense of how good the series approximations tend to be, we return to Fourier series (this is just to get the general principle, although in some cases the eigenfunctions are in fact the Fourier ones exactly). Consider L^2 functions in $[-\pi,\pi]$ and the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

The convergence of the series and smoothness of the function f are related to the decay of the coefficients with n. Some intuition for how a_n, b_n depend on n is useful.

Worst case: Suppose f as a periodic function is piecewise continuous but has a jump (e.g. the square wave). Then

$$a_n, b_n = O(1/n).$$

This is enough to get norm convergence:

$$\|f - S_N\|^2 = \text{const.} \cdot \sum_{n=N+1}^{\infty} (a_n^2 + b_n^2) = O(1/N).$$

Pointwise, there are issues. In particular, plugging in a value of x gives an expression like

$$\sum_{n=1}^{\infty} \frac{1}{n}(\cdots)$$

which is on the edge of failing to converge. This is what leads to Gibbs' phenomenon.

Nicer case: However, if the function is smoother then we can do better. The result is

Basic coefficient bounds: If $f \in L^2[-\pi,\pi]$ and its derivatives up to $f^{(k-1)}$ are continuous **as periodic functions** and $f^{(k)}$ is continuous except at a set of jumps then

$$|a_n| \leq \frac{C}{n^k}, \quad |b_n| \leq \frac{C}{n^k}$$

for some constant C (see optional box below for details.)

Thus, in general,smoother $f \implies$ faster convergence of its Fourier series.

Informally, we get one factor of $1/n$ for each derivative (as a periodic function), starting with the 0-th and ending with the first one that has a jump. We saw this for the square/triangle:

$$\text{square} \implies \text{ jump in } f \implies c_n \sim 1/n$$
$$\text{tri.} \implies f \text{ cts.} + \text{ jump in } f' \implies c_n \sim 1/n^2.$$

The 2π-periodic function

$$f_c(x) = x(\pi - |x|), \qquad x \in [-\pi,\pi]$$

has a continuous first derivative (check: $f_c'(-\pi) = f_c'(\pi)$) and a jump in f_c'', so

$$f_c \implies f \text{ cts.} + f' \text{ cts.} + \text{ jump in } f'' \implies c_n \sim 1/n^3$$

and so on. Derivatives of all orders tends to lead to exponential decay (fast!).

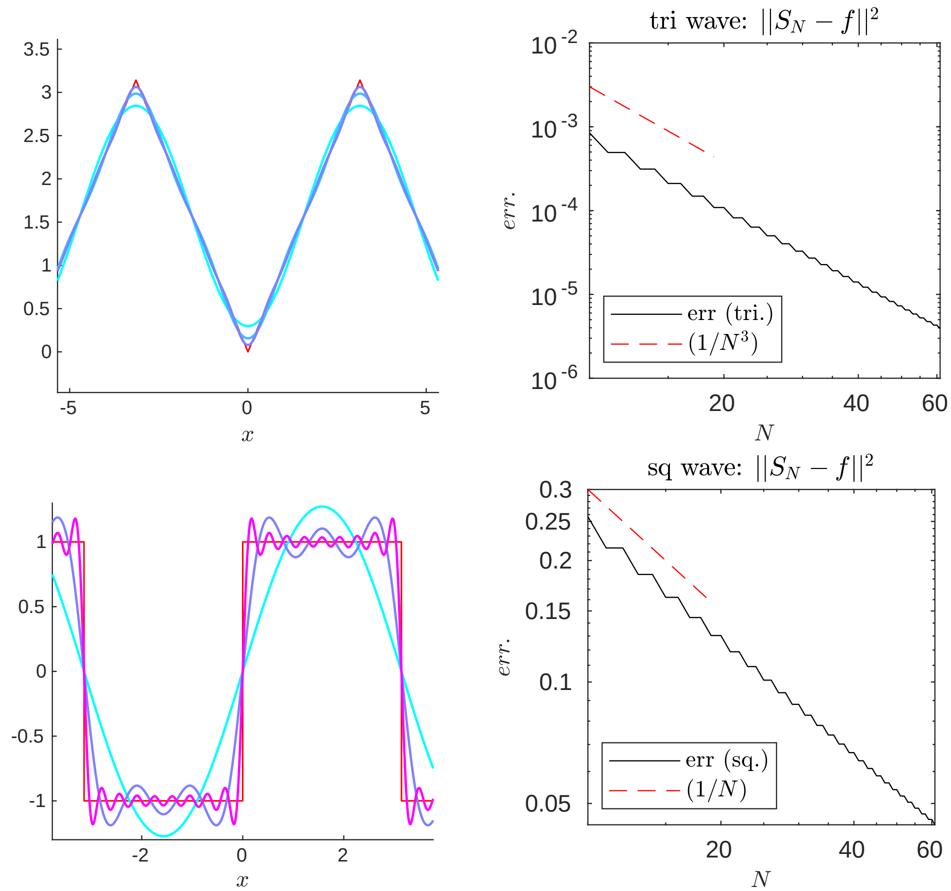

FIGURE 1. Partial sums for the triangle/square wave and log-log plot of the squared L^2 error $\int_{-\pi}^{\pi} |S_N(x) - f(x)|^2\,dx$. For the triangle, $c_n \sim 1/n^2$ and the error decreases like $\sum_{n=N}^{\infty}(1/n^2)^2 \sim 1/N^3$. For the square, $c_n \sim 1/n$, so the error decreases like $\sum_{n=N}^{\infty}(1/n)^2 \sim 1/N$.

Deriving the coefficient bounds: Suppose f is continuous as a periodic function and f' is piecewise continuous. Then

$$\begin{aligned} b_n &= \int_{-\pi}^{\pi} f(x)\sin n\pi x\,dx \\ &= -\frac{1}{n\pi} f(x)\cos n\pi x\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\cos n\pi x\,dx \\ &= \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\cos n\pi x\,dx. \end{aligned}$$

But if f' is continuous on $[-\pi,\pi]$ then $|f'|$ has a maximum value, say M, so

$$|b_n| \le \frac{1}{n\pi}\int_{-\pi}^{\pi} M\,dx = \frac{2}{n}.$$

A similar result holds for the a_n's. The process can be continued if f is smoother; each iteration gives a factor of $1/n$ and continuity of the derivative zeros the boundary terms.