Formula

## 1. Sturm-Liouville theory in 1d

$$L^2 \text{ inner product: } \langle f, g\rangle = \int_a^b f(x)g(x)\,dx, \quad L^2[a,b] = \{f : \langle f, f\rangle < \infty\}$$

$$\text{Weighted inner product: } \langle f, g\rangle_\sigma = \int_a^b f(x)g(x)\sigma(x)\,dx, \quad L^2_\sigma[a,b] = \{f : \langle f, f\rangle_\sigma < \infty\}$$

A **Sturm-Liouville (SL) operator** in $[a,b]$ (with $p(x) \geq 0$) has the form

$$Lu = -(p(x)u_x)_x + q(x)u. \tag{1}$$

A **Sturm-Liouville** problem (SLP) for $L$ and a 'weight function' $\sigma(x)$ has the form

$$Lu = \lambda\sigma(x)u, \qquad x \in (a,b) \text{ plus hom. BCs.} \tag{2}$$

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$$\textbf{Green's formula for (1): } \langle Lu, v\rangle = p(uv_x - vu_x)\Big|_a^b + \langle u, Lv\rangle \tag{3}$$

for all smooth $u, v$ in $[a,b]$. Moreover, $L$ regular $\implies$ self adjoint in the $L^2$ inner product:

$$\langle Lu, v\rangle = \langle u, Lv\rangle \text{ for all } u, v \text{ satisfying the BCs.}$$

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**Main theorem:** The SLP (2) for a **self-adjoint** $L$ has the following properties:

i) The eigenvalues and eigenfunctions are real; the eigenvalues are an infinite sequence with a smallest eigenvalue, tending to $\infty$:

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty.$$

ii) The eigenfunctions form a basis for $L^2_\sigma[a,b]$, and they are orthogonal in the weighted inner product. That is, the eigenfunctions satisfy

$$\langle \phi_m, \phi_n\rangle_\sigma = \int_a^b \phi_m(x)\phi_n(x)\sigma(x)\,dx = 0, \qquad m \neq n$$

Every $f \in L^2_\sigma[a,b]$ has a unique representation in the basis:

$$f = \sum_{n=1}^\infty a_n\phi_n(x), \qquad a_n = \frac{\langle f, \phi_n\rangle_\sigma}{\langle \phi_n, \phi_n\rangle_\sigma}. \tag{4}$$

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**Eigenvalue problems** (standard results, 1d):

$$-\phi'' = \lambda\phi,\ \phi(0) = \phi(L) = 0 \implies \phi_n = \sin(n\pi x/L),\ \lambda_n = (n\pi/L)^2,\ n \geq 1$$

$$-\phi'' = \lambda\phi,\ \phi'(0) = \phi'(L) = 0 \implies \phi_n = \cos(n\pi x/L),\ \lambda_n = (n\pi/L)^2,\ n \geq 0$$

$$-\phi'' = \lambda\phi,\ \phi(0) = \phi'(L) = 0 \implies \phi_n = \sin((n - \tfrac{1}{2})\pi x/L),\ \lambda_n = ((n - \tfrac{1}{2})\pi/L)^2,\ n \geq 1$$

$$-\phi'' = \lambda\phi,\ \phi'(0) = \phi(L) = 0 \implies \phi_n = \cos((n - \tfrac{1}{2})\pi x/L),\ \lambda_n = ((n - \tfrac{1}{2})\pi/L)^2,\ n \geq 1$$

$$-\phi'' = \lambda\phi, \quad \phi \text{ } 2\pi\text{-periodic} \implies \phi_n = \cos nx \text{ or } \sin nx,\ \lambda_n = n^2, \quad n \geq 0$$

## 2. EIGENVALUE/COEFFICIENT ODEs (SEPARABLE PROBLEMS)

**Bessel equation:** for $R(r)$ (disk, cylinder), $\lambda > 0$ (for $\lambda < 0$ use Modified Bessel)

$$R'' + \frac{1}{r}R' + (\lambda - \frac{\nu^2}{r^2})R = 0, \quad \lambda > 0 \implies y = c_1 J_\nu(x\sqrt{\lambda}) + c_2 Y_\nu(x\sqrt{\lambda})$$

- For $\lambda = 0$, use **Cauchy-Euler** procedure instead (bottom of page)
- Values at zero: $J_0(0) = 1$ and $J_\nu(0) = 0$ for $\nu > 0$, $\quad |Y_\nu(0)| = \infty$ (unbounded)
- Zeros: $J_\nu(z)$ has positive zeros $\gamma_{\nu,n}$: $0 < \gamma_{\nu,1} < \gamma_{\nu,2} < \cdots \to \infty$
- Zeros: $J'_\nu(z)$ has positive zeros $\gamma'_{\nu,n}$: $0 < \gamma'_{\nu,1} < \gamma'_{\nu,2} < \cdots \to \infty$

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**Spherical Bessel equation:** For $R(r)$ (sphere),

$$\frac{1}{r^2}(r^2 R')' + (\lambda - \frac{\nu(\nu+1)}{x^2})R = 0 \implies R = c_1 \frac{J_{\nu+1/2}(r\sqrt{\lambda})}{\sqrt{r}} + c_2 \frac{Y_{\nu+1/2}(r\sqrt{\lambda})}{\sqrt{r}}$$

$J$ term bounded in $[0, a]$ and $Y$ term unbounded in $[0, a]$ ($\infty$ at $r = 0$)

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**Spherical harmonics:** Eigenfunctions $Y_n^m(\theta, \phi)$ for $-\nabla_s^2 Y = \lambda Y$ (sphere surface), where

$$\nabla_s^2 Y = \frac{1}{\sin\phi}(\sin\phi\, Y_\phi)_\phi + \frac{1}{\sin^2\phi} Y_{\theta\theta}$$

- Related to full Laplacian $\nabla^2 u(r, \theta, \phi)$ by $\nabla^2 u = \frac{1}{r^2}(r^2 u_r)_r + \frac{1}{r^2}\nabla_s^2 u$.
- Separated, $Y = g(\theta)h(\phi)$ with $-g'' = m^2 g, \quad m \geq 0$.

**$\phi$-dir:** $$\frac{1}{\sin\phi}(\sin\phi\, h')' + \left(\lambda - \frac{m^2}{\sin^2\phi}\right) h = 0$$

**transformed:** $$((1-\xi^2)y')' + (\lambda - \frac{m^2}{1-\xi^2})y = 0, \qquad \xi = \cos\phi, y(\xi) = h(\phi)$$

**Solutions:** $$\begin{cases} Y_n^m = P_n^m(\cos\phi)(\cos m\theta \text{ or } \sin m\theta), \\ \lambda_n = n(n+1) \text{ (independent of } m) \end{cases} \qquad 0 \leq m \leq n$$

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**Modified Bessel functions:** (Bessel, opposite sign), $\eta > 0$

$$y'' + \frac{1}{x}y' - (\eta + \frac{\nu^2}{x^2})y = 0, \implies y = c_1 I_\nu(x\sqrt{\eta}) + c_2 K_\nu(x\sqrt{\eta})$$

- Zeros: None; both $K_\nu(z)$ and $I_\nu(z)$ are positive for $z > 0$
- Values at zero: $I_0(0) = 1$ and $I_\nu(0) = 0$ for $\nu > 0$, $\quad K_\nu(0) = \infty$ (unbounded)

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**Cauchy-Euler equations:** $p, q$ are **constants**. Three cases, depending on roots of $p(r)$.

$$x^2 y'' + xpy' + qy = 0,\ x > 0 \implies x^\alpha \text{ is a solution} \iff 0 = p(\alpha) = \alpha(\alpha - 1) + p\alpha + q.$$

$$\begin{aligned} \alpha_1 \neq \alpha_2, \text{ real} &\implies y = c_1 x^{\alpha_1} + c_2 x^{\alpha_2} \\ \alpha_1 = \alpha_2 &\implies y = c_1 x^{\alpha_1} + c_2 x^{\alpha_1}\log x \\ \alpha = s \pm \omega i &\implies y = c_1 x^s \cos(\omega \log x) + c_2 x^s \sin(\omega \log x). \end{aligned}$$

## 3. Calculus/complex variables

$\int \cdots dV$ = volume integral (over domain) and $\int \cdots dS$ = integral over boundary,
$\Omega$ = domain and $\partial\Omega$ = boundary, $\mathbf{n}$ = outward normal, $\partial u/\partial \mathbf{n} = \nabla u \cdot \mathbf{n}$ =normal derivative.

**Divergence thm:** $$\int_\Omega \nabla \cdot \mathbf{v}\, dV = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\, dS.$$

**Int. by parts:** $$\int_\Omega f\nabla^2 g\, dV = \int_{\partial\Omega} f\frac{\partial g}{\partial \mathbf{n}}\, dS - \int_\Omega \nabla f \cdot \nabla g\, dV$$

**Green's formula:** $$\int_\Omega (f\nabla^2 g - g\nabla^2 f)\, dV = \int_{\partial\Omega} f\frac{\partial g}{\partial \mathbf{n}} - g\frac{\partial f}{\partial \mathbf{n}}\, dS$$

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**Cylindrical:** radius $r$, angle $\theta$ (in $xy$ plane) and $z$

**Unit vectors:** $\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}, \quad \hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}, \quad \hat{z} = \hat{z}$

**volume:** $dV = r\,dr\,d\theta\,dz$, **surface (rad.** $a$**):** $dS = a\,dz\,d\theta$

**Laplacian:** $$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

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**Spherical:** radius $r$, azimuthal angle $\theta$, polar angle $\phi$

**Coordinates:** $$\begin{cases} x = r\sin\phi\cos\theta, \\ y = r\sin\phi\sin\theta, \\ z = r\cos\phi \end{cases} \qquad \theta \in [0, 2\pi],\ \phi \in [0,\pi], r \geq 0$$

**volume:** $dV = r^2\sin\phi\,dr\,d\theta\,d\phi$, **surface (rad.** $a$**):** $dS = a^2\sin\phi\,d\phi\,d\theta$

**Laplacian:** $$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial f}{\partial r}) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi\frac{\partial f}{\partial \phi})$$

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## 4. Complex variables

Cauchy-Riemann eqs.: $u_x = v_y, \quad u_y = -v_x \quad$ for $f(z) = u + iv$

Cauchy integral formula: $$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz, \quad n \geq 0.$$

Principal value: $$\text{PV}\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\,dx = \lim_{\epsilon \searrow 0}\left(\int_{-\infty}^{-\epsilon} \frac{f(x)}{x-x_0}\,dx + \int_{\epsilon}^{\infty} \frac{f(x)}{x-x_0}\,dx\right).$$

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**Common Taylor/Laurent series**

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{3!}z^3 + \cdots \qquad \sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \cdots$$

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots \qquad \cos z = 1 + \frac{1}{2}z^2 + \frac{1}{4!}z^4 + \cdots$$

$$\cot z = \frac{1}{z} - \frac{z}{3} - \frac{1}{45}z^3 - \cdots \qquad \tan z = z + \frac{z}{3} + \frac{2}{15}z^3 + \cdots$$

## 5. Transforms

**Fourier transform** $(f(x) \to F(k))$

**common functions:**

$$e^{-ax^2} \to \frac{1}{\sqrt{4\pi a}} e^{-k^2/(4a)},$$

$$\sqrt{\pi/b}\; e^{-x^2/(4b)} \to e^{-bk^2}$$

$$\delta(x - x_0) \to \frac{1}{2\pi} e^{ikx_0}$$

$$\frac{2a}{x^2 + a^2} \to e^{-a|k|}$$

**transform rules:**

$$\mathcal{F}^{-1}(F) = \int_{-\infty}^{\infty} F(k) e^{-ikx}\, dk$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) g(x - y)\, dy \to F(k)G(k)$$

$$f(x - a) \to e^{iak} F(k)$$

$$e^{-iax} f(x) \to F(k - a)$$

$$\partial f/\partial x \to -ikF(k)$$

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**Sine and cosine tranfsorm:**

**Sine transforms:**

$$S[f] = \frac{2}{\pi} \int_0^{\infty} f(x) \sin kx\, dx,$$

$$S^{-1}[F] = \int_0^{\infty} F(k) \sin kx\, dk$$

$$S[e^{-ax}] \to \frac{2}{\pi} \frac{k}{a^2 + k^2},$$

$$S[\frac{x}{x^2 + a^2}] \to e^{-ak}$$

$$S[1] \to \frac{2}{\pi k}$$

$$S[f'(x)] \to -kC[f(x)]$$

$$S[f''(x)] \to \frac{2}{\pi} k f(0) - k^2 F(k)$$

**Cosine transforms:**

$$C[f] = \frac{2}{\pi} \int_0^{\infty} f(x) \cos kx\, dx,$$

$$C^{-1}[F] = \int_0^{\infty} F(k) \cos kx\, dk$$

$$C[e^{-ax}] \to \frac{2}{\pi} \frac{a}{a^2 + k^2},$$

$$C[\frac{a}{x^2 + a^2}] \to e^{-ak}$$

$$C[e^{-ax^2}] \to \frac{1}{\pi a} e^{-k^2/(4a)}$$

$$C[f'(x)] \to -\frac{2}{\pi} f(0) + kS[f(x)]$$

$$C[f''(x)] \to -\frac{2}{\pi} f'(0) - k^2 F(k)$$

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**Laplace transform:** $(f(t) \to F(s))$

$$\frac{d^n f}{dt^n} = -f^{(n-1)}(0) - s f^{(n-2)}(0) - \cdots - s^{n-1} f(0) + s^n F(s)$$

**common functions:**

$$t^n \;(n > -1) \to n! s^{-(n+1)}$$

$$e^{at} \to 1/(s - a)$$

$$\sin at \to \frac{a}{s^2 + a^2}$$

$$\delta(t - a) \to e^{-as}$$

$$H(t - a) \to e^{-as}/s$$

$$e^{-a\sqrt{s}} \to \frac{a}{\sqrt{4\pi} t^{3/2}} e^{-a^2/4t}$$

**transform rules:**

$$\mathcal{L}^{-1}(F) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) e^{st}\, ds$$

$$df/dt \to -f(0) + sF(s)$$

$$-tf(t) \to dF/ds$$

$$\int_0^t f(t - t_0) g(t_0)\, dt_0 \to F(s)G(s)$$

$$H(t - a) f(t - a) \to e^{-as} F(s) \;(a > 0)$$