

THE FREDHOLM ALTERNATIVE (AN IMPORTANT GENERAL PRINCIPLE)

Recall that that some boundary value problems have unique solutions, while others only have solutions if certain solvability conditions hold. For example, consider

$$\begin{aligned} Lu &= f \\ u_x(0) &= u_x(\pi) = 0 \end{aligned}$$

where L with the BCs is self-adjoint with eigenfunctions ϕ_n . Then, if $\lambda_n \neq 0$ for all n ,

$$u = \sum_n \frac{f_n}{\lambda_n} \phi_n \text{ is the unique solution (where } f = \sum f_n \phi_n \text{).}$$

On the other hand, if there is a zero eigenvalue $\lambda_0 = 0$ then

$$\langle f, \phi_0 \rangle \neq 0 \implies \text{there is no solution}$$

$$\langle f, \phi_0 \rangle = 0 \implies u = a_0 \phi_0 + \sum_{n \neq 0} \frac{f_n}{\lambda_n} \phi_n \text{ is a sol. for any } a_0.$$

This result hints at an important general principle for linear operators:¹

Fredholm Alternative theorem (FAT); general principle:

Let L be a linear operator with adjoint L^* . Then **exactly one** of the following is true:

A) The **inhomogeneous problem**

$$Lu = f \tag{1}$$

has a unique solution u .

B) The **homogeneous** adjoint problem

$$L^*u = 0 \tag{2}$$

has a non-trivial solution.

That is,

- If (1) has a unique solution, then $\lambda = 0$ is not an eigenvalue of the adjoint.
- If $\lambda = 0$ is an eigenvalue of L^* , then (1) has either no solutions or infinitely many.

This applies to a broad array of ‘linear’ problems (we’ll see this for BVPs, PDEs and integral equations). Note that the precise statement depends on the type of problem. The FAT provides key intuition for when **unique solutions** exist and when solutions only exist under certain conditions.

¹The result is typically called the ‘Fredholm alternative’ without the word ‘theorem’.

For boundary value problems: The BCs are part of the 'problems' (1) and (2). The FAT says that exactly one of the following is true:

A) The inhomogeneous problem

$$Lu = f, \quad BCu = 0 \quad (3)$$

has a unique solution.

B) The homogeneous problem for L^* ,

$$L^*\psi_0 = 0, \quad BC^*\psi_0 = 0 \quad (4)$$

has a non-trivial solution ψ_0 (that is, $\lambda = 0$ is an eigenvalue for L^* with adjoint BCs).

Case B has two parts: If the adjoint operator L^* has a zero eigenvalue, then either

B1) The problem (3) has no solution

B2) The problem (3) has infinitely many solutions.

A **solvability condition** determines which of (B1) or (B2) is true. As we saw in the example, the terms for the zero eigenvalue are where to look: take $\langle \cdot, \phi_0 \rangle$ of the BVP for any eigenfunctions ψ_0 to find the conditions.²

Note: The FAT doesn't immediately transfer to equations with time like

$$u_t = -Lu + f.$$

We have observed that it tends to apply; if L is self-adjoint with a zero eigenvalue λ , then any multiple of $\phi_0(x)$ can be added to the solution.

Use in practice: Suppose we want to solve (with L not necessarily self-adjoint)

$$Lu = f, \quad Bu = 0.$$

First, find the adjoint and BCS L^* and B^* and check if $\lambda = 0$ is an eigenvalue for L^* .

If $\lambda = 0$ is not an eigenvalue, the BVP has a unique solution. Solve the equation (e.g. with eigenfunctions) without concern.

If $\lambda = 0$ is an eigenvalue, take $\langle \cdot, \psi_0 \rangle$ of the BVP for any eigenfunction ψ_0 for $\lambda = 0$ to find solvability conditions. The other terms can be solved (uniquely) by the standard procedure.

Example 1 (self-adjoint): Consider the BVP

$$-u_{xx} = f, \quad u_x(0) = au(0), \quad u_x(1) = u(1).$$

The operator $Lu = -u_{xx}$ is self-adjoint. Hence to apply the FAT, we check for a zero eigenvalue of L (same as L^*):

$$\phi'' = 0, \quad \phi'(0) = a\phi(0), \quad \phi'(1) = 2\phi(1).$$

²The examples for BVP have a single eigenfunction for $\lambda = 0$ which gives one solvability condition; we'll shortly see an example with more than one in the context of integral equations.

The general solution is $\phi(x) = c_1 + c_2x$ and the BCs give

$$c_2 = ac_1, \quad c_2 = 2(c_1 + c_2) \implies c_2 = -2c_1$$

No solution for ϕ_0 exists if $a \neq -2$ but $\phi = 1 - 2x$ is a solution if $a = -2$. From the FAT, it follows that there is a unique solution if and only if $a \neq -2$.

To get the solvability condition when $a = 1$, look at the ϕ_0 term:

$$\langle f, \phi_0 \rangle = \langle Lu, \phi_0 \rangle = \langle u, L\phi_0 \rangle = 0 \implies \int_0^1 f(x)(1-x) dx = 0.$$

When this condition holds, $u = c_0\phi_0 + \sum_{n \geq 1} c_n\phi_n$ is a solution for any c_0 .

Example 2 (not self adjoint): Consider the BVP for $u(x)$,

$$-xu_{xx} = f, \quad u_x(1) = u_x(2) = 0$$

Let $Lu = -xu_{xx}$ (not self-adjoint³). The adjoint and adjoint BCs are

$$L^*v = -(xv)_{xx}, \quad v(1) = -v_x(1), \quad v(2) = -2v_x(2).$$

from the calculation

$$\langle Lu, v \rangle = \langle u, L^*v \rangle + ((xv)_x u - xv u_x) \Big|_{x=1}^{x=2}$$

which requires v to have BCs $(xv)_x = 0 \implies v(1) + v_x(1) = 0$ and $v(2) + 2v_x(2) = 0$.

Now look for a zero eigenvalue for the adjoint:

$$L^*v = 0 \implies (xv)_{xx} = 0 \implies v = c_1 + c_2/x \implies v(x) = 1/x$$

after applying the boundary conditions. Thus $\lambda = 0$ is an eigenvalue of L^* with eigenfunction $\psi_0 = 1/x$, so this is FAT case (B): there is a solvability condition for f . To find it, try to compute the solution:

$$u = \sum c_n \phi_n, \quad (\text{recall } c_n = \langle u, \psi_n \rangle / \langle \phi_n, \psi_n \rangle),$$

and take the inner product of the BVP with ψ_n to get

$$\langle f, \psi_n \rangle = \langle Lu, \psi_n \rangle = \langle u, L^*\psi_n \rangle = \lambda_n c_n \langle \phi_n, \psi_n \rangle$$

where λ_n, ψ_n are the adjoint eigenvalues/functions. For $n = 0$,

$$\langle f, \psi_0 \rangle = 0 \implies \boxed{\int_1^2 f(x)/x dx = 0}$$

is the solvability condition. If this holds, then

$$u = c_0\phi_0 + \sum_{n=1}^{\infty} \frac{\langle f, \psi_n \rangle}{\lambda_n \langle \phi_n, \psi_n \rangle} \phi_n$$

is a solution for any c_0 (where ϕ_0 is the eigenfunction for L with $\lambda = 0$).

³You could convert trivially this to self-adjoint form and avoid the adjoint by dividing by x , but the point here is to illustrate the more general procedure.