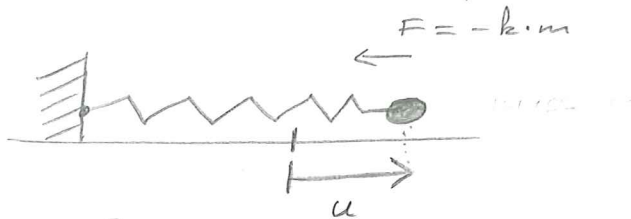


# Chapter 1

What is a partial differential equation (PDE)?

Recall: Ordinary differential equations (ODEs)

Example: Motion of a mass on a spring



spring constant  $k$   
mass  $m$

$$\frac{d^2 u}{dt^2} = -\frac{k}{m} \cdot u$$

$\rightarrow$  Newton's 2<sup>nd</sup> law of motion

$\rightarrow$  Hooke's law

General solution:

$$u(t) = c_1 \cdot \cos\left(\sqrt{\frac{k}{m}} t\right) + c_2 \cdot \sin\left(\sqrt{\frac{k}{m}} t\right)$$

Definition: PDE

Let  $t, x, y, \dots$  be independent variables and let  $u$  be an unknown function (dependent variable) of at least two of these.

A PDE is an equation that relates the independent variables, the dependent variable  $u$ , and the partial derivatives of  $u$ .

The order of a PDE is the highest derivative that appears.

PDEs are ubiquitous in the natural sciences and engineering. It is somehow a miracle that phenomena in nature and physical processes can be described so well in the language of mathematics, in particular in terms of ODEs and PDEs!

Examples of important 2<sup>nd</sup> order PDEs:

• Laplace's equation:

$$u = u(x, y)$$

independent variables  $x, y$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

→ electrostatics, steady-states in heat conduction

• Heat equation:

$$u = u(t, x)$$

independent variables:  $t, x$

↑ time      ↑ spatial coordinate

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

→ conduction of heat, diffusion processes

• Wave equation:

$$u = u(t, x)$$

independent variables:  $t, x$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

→ models wave propagation:

vibration of strings or membranes,  
propagation of light (Maxwell's equations  
of electromagnetism in vacuum)

• Schrödinger's equation

$u = u(t, x)$  complex-valued

$$i \cdot \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0$$

imaginary  
unit  $i = \sqrt{-1}$

→ fundamental evolution equation  
of quantum mechanics

In fact, these are the four fundamental  
types of 2<sup>nd</sup> order PDEs.

Goals of this course:

- learn how PDEs arise in physical problems
- develop solution techniques  
(separation of variables, Fourier series,  
Fourier transform, method of characteristics, ...)
- classification of PDEs and differences  
between properties of solutions of PDEs  
in various classes

## Derivation of the heat equation from physical principles

(more specifically here: derivation of the conduction of heat in a one-dimensional rod)

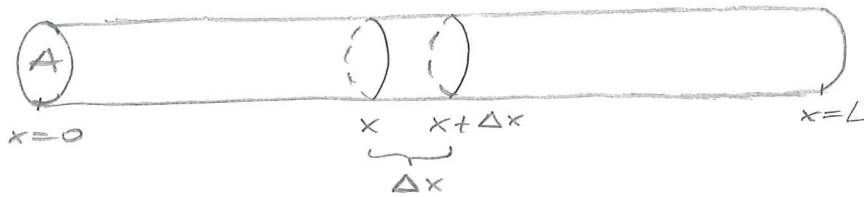
Without being too precise let us first try to define the physical quantities involved:

Temperature is a physical quantity expressing hot and cold. The temperature of a body is related to its internal energy or thermal energy (think of it as how strongly the molecules in say an iron rod swing or how fast the molecules in a gas move about).  
Heat is energy in transfer.

Two fundamental principles of physics:

- conservation of energy
- "heat flows from hotter to colder regions"  
(Fourier's law of heat conduction)

Consider a rod of constant cross-sectional area  $A$  oriented in the  $x$ -direction (from  $x=0$  to  $x=L$ ):



• Thermal energy density

$e(x,t) :=$  amount of thermal energy per unit volume

• Then the thermal energy (or heat energy) in the above thin slice of the rod is given by

$$\text{thermal energy} = e(x,t) \cdot A \cdot \Delta x$$

(assuming that the thermal density  $e(x,t)$  is approximately constant in the slice)

• By the conservation of energy principle, the thermal energy in the thin slice can only change in time due to heat flowing across the edges and being generated inside

(the lateral surfaces are assumed to be insulated)

$$(\neq 1) \quad \begin{array}{l} \text{rate of change} \\ \text{of thermal energy} \\ \text{in time} \end{array} = \begin{array}{l} \text{heat energy} \\ \text{flowing across} \\ \text{boundaries per} \\ \text{unit time} \end{array} + \begin{array}{l} \text{thermal energy} \\ \text{generated inside} \\ \text{per unit time} \end{array}$$

For the thin slice we have

$$\text{rate of change of thermal energy in time} = \frac{\partial}{\partial t} (e(x,t) A \Delta x)$$

• Heat flux:

$\phi(x,t) :=$  amount of thermal energy per unit time flowing to the right per unit surface area

→ Note: If  $\phi(x,t) < 0$ , then thermal energy is flowing to the left!

• Heat sources:

$Q(x,t) :=$  thermal energy per unit volume generated per unit time

• Thus, from (\*1) we obtain that

$$\frac{\partial}{\partial t} (e(x,t) \cdot A \cdot \Delta x) \approx \phi(x,t) \cdot A - \phi(x+\Delta x, t) \cdot A + Q(x,t) \cdot A \cdot \Delta x$$

Dividing by  $A \cdot \Delta x$  yields

$$\frac{\partial e}{\partial t}(x,t) \approx \underbrace{\frac{\phi(x,t) - \phi(x+\Delta x, t)}{\Delta x}}_{\xrightarrow{\Delta x \rightarrow 0} -\frac{\partial \phi}{\partial x}(x,t)} + Q(x,t)$$

In the limit  $\Delta x \rightarrow 0$  we find

$$(*2) \quad \frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q$$

- We usually describe how hot an object is by its temperature, not by its thermal energy density.

$$u(x,t) := \text{temperature}$$

- The precise relation between the thermal energy density and the temperature depends on the material:

$$e(x,t) = c(x) \cdot \rho(x) \cdot u(x,t)$$

specific heat

(thermal energy that must be supplied to a unit mass of a substance to raise its temperature by one unit)

- Thus, (\*2) becomes

$$(*3) \quad c(x) \cdot \rho(x) \cdot \frac{\partial u}{\partial t} = - \frac{\partial \phi}{\partial x} + Q$$

- How does the heat flux  $\phi(x,t)$  depend on the temperature  $u(x,t)$ ?

Fourier's law of heat conduction

$$\phi = - k_0 \cdot \frac{\partial u}{\partial x}$$

thermal conductivity

(again a material dependent quantity)

Observe that Fourier's law captures several qualitative properties of heat flow which are familiar from experiments and daily life:

(i) If the temperature is constant in a region, no heat energy flows.

(ii) If there are temperature differences, the heat energy flows from the hotter region to the colder region.

(iii) The greater the temperature differences (for the same material), the greater is the flow of heat energy.

(iv) The flow of heat energy will vary for different materials, even with the same temperature differences.

- Substituting Fourier's law into (\*3) yields a PDE for the temperature  $u(x,t)$ :

$$(*4) \quad c \cdot \rho \cdot \frac{\partial u}{\partial t} = - \frac{\partial}{\partial x} \left( -k_0 \cdot \frac{\partial u}{\partial x} \right) + Q$$

• In a uniform rod, the specific heat  $c$ , the mass density  $\rho$ , and the thermal conductivity  $k_0$  are just constants (and are not functions of the location  $x$ )

Then (\*4) becomes

$$c \cdot \rho \cdot \frac{\partial u}{\partial t} = k_0 \cdot \frac{\partial^2 u}{\partial x^2} + Q.$$



- If also  $Q = 0$  (i.e. no heat sources), we arrive at

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}$$

Heat equation

where

$$k = \frac{k_0}{c \cdot \rho} \quad \text{thermal diffusivity}$$

- In order to solve this PDE we need to be given an initial condition (IC):

$$u(x, t=0) = f(x)$$

↳ initial temperature distribution

Q: Is this enough information to predict the future temperature?

No, we need to know what happens at the two boundaries,  $x=0$  and  $x=L$ , i.e. we need two boundary conditions.

The appropriate boundary condition depends on the physical mechanism in effect at each end!

# Boundary Conditions

## Prescribed temperature:

The temperature of one end of the rod, for example at  $x=0$ , may be approximated by a prescribed temperature

$$u(0,t) = u_{II}(t),$$

where  $u_{II}(t)$  is the temperature of a fluid bath (or reservoir) with which the rod is in contact.

## Insulated boundary

Sometimes it is possible to describe the heat flow rather than the temperature

$$-K_0(0) \cdot \frac{\partial u}{\partial x}(0,t) = \phi(t), \quad \leftarrow \begin{array}{l} \text{(remember Fourier's} \\ \text{law of heat} \\ \text{conduction)} \end{array}$$

where  $\phi(t)$  is given.

When an end is perfectly insulated, there is no heat flow at the boundary and thus

$$\frac{\partial u}{\partial x}(0,t) = 0.$$

## Newton's law of cooling

This is a mixture of the previous two cases taking into account that (when the end is not insulated), the heat flow leaving the rod is proportional to the temperature difference between the end of the rod and the prescribed external temperature

$$-k_0 \cdot \frac{\partial u}{\partial x}(l, t) = -H \cdot (u(l, t) - u_{\infty}(t))$$

↑ heat transfer coefficient  
 $H > 0$

→ check that the signs make sense!

### Summary:

Three different kinds of boundary conditions.

For example, at  $x=0$ ,

$$u(0, t) = u_{\infty}(t)$$

prescribed temperature  
("Dirichlet BC")

$$-k_0(0) \cdot \frac{\partial u}{\partial x}(0, t) = \phi(t)$$

prescribed heat flux  
("Neumann BC")

$$-k_0(0) \cdot \frac{\partial u}{\partial x}(0, t) = -H \cdot (u(0, t) - u_{\infty}(t))$$

Newton's law of cooling  
("Robin BC")

### Note:

In the limit  $H \rightarrow 0$ , Newton's law of cooling approaches the insulated boundary condition.

In the limit  $H \rightarrow \infty$ , Newton's law of cooling approaches the prescribed temperature condition.

## Equilibrium temperature distribution

We now begin to determine solutions to the heat equation

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq L$$

$$(IC) \quad u(x, 0) = f(x)$$

for two types of boundary conditions by seeking special equilibrium or steady-state solutions

$$u(x, t) = u(x),$$

which are independent of time.

Thus,  $\frac{\partial u}{\partial t} = 0$  and the heat equation just becomes an ODE

$$\frac{d^2 u}{dx^2} = 0.$$

By integrating twice, we obtain its general solution

$$u(x) = C_1 \cdot x + C_2 \quad \rightarrow \text{graph: straight line}$$

for some constants  $C_1, C_2$ .

→ In doing steady state calculations, the initial conditions are usually ignored (to a certain extent).

## Steady prescribed temperature BCs

Suppose that the boundary conditions at  $x=0$  and  $x=L$  are steady

$$u(0,t) = T_1$$

$$u(L,t) = T_2$$

for fixed (temperatures)  $T_1, T_2$ .

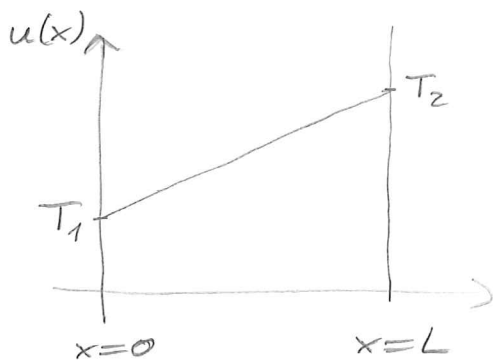
Then we have for an equilibrium solution

$$u(0) = T_1 \Rightarrow C_2 = T_1$$

$$u(L) = T_2 \Rightarrow C_1 \cdot L + \underbrace{C_2}_{=T_1} = T_2$$

Thus,  $C_1 = \frac{T_2 - T_1}{L}$  and

$$u(x) = T_1 + \frac{T_2 - T_1}{L} \cdot x$$



Expectation for the time-dependent case:

If we wait for a very long time, we would imagine that the influence of the steady temperatures at the two ends dominates.

It is therefore reasonable to expect that in the long run the initial temperature distribution is forgotten and the solution approaches the equilibrium solution

$$\lim_{t \rightarrow \infty} u(x,t) = u(x) = T_1 + \frac{T_2 - T_1}{L} \cdot x.$$

→ We will derive in the next weeks that this happens indeed.

### Insulated boundaries

Let's do another steady-state calculation in the case of insulated boundaries at  $x=0$  and at  $x=L$ :

$$\frac{\partial u}{\partial x}(0,t) = 0,$$

$$\frac{\partial u}{\partial x}(L,t) = 0.$$

The general solution for the equilibrium temperature distribution is (from above)

$$u(x) = C_1 \cdot x + C_2$$

for some constants  $C_1, C_2$

Here the <sup>insulated</sup> boundary conditions are satisfied if  $C_1 = 0$ .

Thus,

$$u(x) = C_2$$

is an equilibrium solution for any constant  $C_2$  (no uniqueness!).

For the time-dependent problem, we would again expect that in the long run the solution approaches an equilibrium state

$$\lim_{t \rightarrow \infty} u(x, t) = u(x) = C_2.$$

But physically it does not make sense that it should be an arbitrary equilibrium temperature  $C_2$ .

It should have something to do with the initial condition  $u(x, 0) = f(x)$ .

We should have conservation of the total thermal energy in the rod

(since no heat can flow in or out at the ends)!

$$\int_0^L \underbrace{e(x, t)}_{= c \cdot \rho \cdot u(x, t)} dx = \text{constant for all } t \geq 0$$

thermal  
energy  
density

Hence, for any  $t > 0$

$$\int_0^L c \cdot \rho \cdot u(x, t) dx = \int_0^L c \cdot \rho \cdot \underbrace{u(x, 0)}_{= f(x)} dx$$

$$\Rightarrow \int_0^L \underbrace{u(x, t)}_{\substack{t \rightarrow \infty \\ \rightarrow C_2}} dx = \int_0^L f(x) dx$$

$$\Rightarrow \underbrace{\int_0^L C_2 dx}_{= C_2 \cdot L} = \int_0^L f(x) dx$$

$$\Rightarrow \boxed{C_2 = \frac{1}{L} \int_0^L f(x) dx} \quad \text{average of the initial temperature distribution}$$

→ We will soon derive that this is indeed the limiting equilibrium temperature in the case of insulated boundary conditions.

### Heat equation in higher space dimensions

$$\frac{\partial u}{\partial t} = k \cdot \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (\text{two space dimensions})$$

$$\frac{\partial u}{\partial t} = k \cdot \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad (\text{three space dimensions})$$