Chapter 10: Infinite domain problems and Fourier transform solutions of PDEs

Until now we have discussed and solved several PDEs (heat equation, wave equation, Laplace's equation) that were posed on a finite spatial domain (interval, rectangle, disk, ...). In particular, the solutions we obtained depended on conditions at the boundaries of these domains. We now turn to solving PDEs posed on domains that extend indefinitely in at least one direction.

Heat equation on the whole line

We start by considering the heat equation defined on the whole line

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}, \qquad \begin{array}{c} -\infty < x < +\infty \\ t > 0 \end{array}$$

with constant thermal properties ($k \in \mathbb{R}$ constant) and an initial condition

$$u(x,0) = f(x).$$

The initial condition describes the temperature distribution at time $t=0$. Note that since the domain $-\infty < x < +\infty$ does not have any boundaries, we are not imposing any boundary conditions!

However, usually one requires that the initial condition $f(x)$ (and also the solution $u(x,t)$ at any later time $t>0$) approaches 0 as $x \to \pm\infty$.

Recall that the heat equation on a finite interval $0 \leq x \leq L$

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}, \qquad \begin{matrix} 0 \leq x \leq L \\ t > 0 \end{matrix}$$

has the fundamental product solutions

$$\sin\left(\frac{n\pi x}{L}\right) e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 t} \quad \text{and} \quad \cos\left(\frac{n\pi x}{L}\right) e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 t}$$

(which ones are used depends on the boundary conditions) or in combined form

$$e^{-i \frac{n\pi}{L} \cdot x} \, e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 t}, \quad n \in \mathbb{Z}$$

(using Euler's formula

$$e^{i\theta} = \cos(\theta) + i \cdot \sin(\theta) \quad \text{for } \theta \in \mathbb{R} \;)$$

Here, $\omega = \frac{n\pi}{L}$, $n \in \mathbb{Z}$, are the allowed <u>frequencies</u> or <u>wave numbers</u>. Observe that as $L \to \infty$, the difference between two consecutive wave numbers

$$\frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L} \longrightarrow 0 \quad (\text{as } L \to \infty).$$

Hence, we expect that on the whole line all frequencies <u>$\omega \in \mathbb{R}$</u> should be allowable!

Indeed, we can check easily that for any $\omega \in \mathbb{R}$

$$u(x,t) := e^{-i\omega x} e^{-k\omega^2 t}$$

solves the heat equation (on the whole line)

$$\left(\frac{\partial}{\partial t} - k \cdot \frac{\partial^2}{\partial x^2}\right)\left(e^{-i\omega x} e^{-k\omega^2 t}\right)$$

$$= -k\omega^2 \cdot e^{-i\omega x} e^{-k\omega^2 t} \underbrace{- k \cdot (-i\omega)^2}_{= +k\omega^2} \cdot e^{-i\omega x} e^{-k\omega^2 t}$$

$$= (-k\omega^2 + k\omega^2)\, e^{-i\omega x} e^{-k\omega^2 t}$$

$$= 0$$

Thus, $\left\{ e^{-i\omega x} e^{-k\omega^2 t} \right\}_{\omega \in \mathbb{R}}$ is a <u>continuous</u> family of "product solutions" to the heat equation on the whole line.

Then we expect that by a (continuous version of) the superposition principle

$$u(x,t) = \int_{-\infty}^{+\infty} c(\omega)\, e^{-i\omega x} e^{-k\omega^2 t}\, d\omega$$

should also be a solution to the heat equation on the whole line for suitable coefficients $c(\omega)$.

Our initial condition $u(x,0) = f(x)$ then has to satisfy

$$f(x) = \int_{-\infty}^{+\infty} c(\omega)\, e^{-i\omega x}\, d\omega \; !$$

Correspondingly, in order to solve the initial value problem for the heat equation on the whole line, we need to understand if any "reasonably nice" initial condition $f(x)$ can be written as $\int_{-\infty}^{+\infty} c(\omega)\, e^{-i\omega x}\, d\omega$ with suitable coefficients depending on $f(x)$.

Fourier Transform

Motivation from the Fourier series identity

Recall that by Fourier's theorem, given a function $f(x)$ on an interval $-L \leq x \leq +L$ that is <u>periodic</u> ($f(-L) = f(+L)$) and <u>smooth</u>, we have that

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n \cdot e^{-i \frac{n\pi}{L} x}$$

with the complex Fourier coefficients

$$c_n = \frac{1}{2L} \int_{-L}^{+L} f(x)\, e^{+i \frac{n\pi}{L} x}\, dx, \qquad n \in \mathbb{Z}.$$

[Please note that the sign convention for $e^{\pm i \frac{n\pi}{L} x}$ in our textbook is different from the one in many other textbooks]

Combining the two expressions we obtain the <u>Fourier series identity</u>

$$f(x) = \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2L} \int_{-L}^{+L} f(\bar{x})\, e^{+i \frac{n\pi}{L} \bar{x}}\, d\bar{x} \right) e^{-i \frac{n\pi}{L} x}.$$

As before the allowable frequencies / wave numbers here are

$$\omega = \frac{n\pi}{L}, \quad n \in \mathbb{Z},$$

and the distance between consecutive frequencies is

$$\Delta\omega = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$$

Thus, we may also write

$$f(x) = \sum_{n=-\infty}^{+\infty} \frac{\Delta\omega}{2\pi} \int_{-L}^{+L} f(\bar{x}) e^{i\omega\bar{x}} \, d\bar{x} \; e^{-i\omega x}$$

Note: $\frac{1}{2L} = \frac{1}{2\pi} \cdot \frac{\pi}{L} = \frac{1}{2\pi} \cdot \Delta\omega$

<u>Fourier transform</u>

Letting $L \to \infty$ we note that $\Delta\omega \to 0$, i.e. the allowable frequencies become a <u>continuous spectrum</u> $\omega \in \mathbb{R}$. Moreover, heuristically as $L \to +\infty$, $\sum_{n=-\infty}^{+\infty} \frac{\Delta\omega}{2\pi} \left(\int_{-L}^{+L} f(\bar{x}) e^{i\omega\bar{x}} d\bar{x} \right) e^{-i\omega x}$ becomes an integral $\frac{1}{2\pi} \int_{-\infty}^{+\infty} \dots \, d\omega$.

Thus, we expect to have ("for sufficiently nice") functions $f(x)$ on the whole line that

$$(*1) \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(\bar{x}) e^{i\omega\bar{x}} \, d\bar{x} \right) e^{-i\omega x} \, d\omega$$

<u>Fourier integral identity</u>

Remarks:

- The Fourier integral identity (*1) of course has to be proved rigorously for a suitable class of functions, which is beyond the scope of this course. Here we motivated it heuristically from the results for Fourier series.

- A natural class of functions for which (*1) is true is the so-called Schwartz space of functions, the class of smooth and rapidly decaying functions $f(x)$ on $-\infty < \infty < +\infty$, for which

$$\sup_{x \in \mathbb{R}} \left| x^k \frac{\partial^l f}{\partial x^l}(x) \right| < \infty$$

for all $k, l \in \mathbb{N}$.

We next define the Fourier transform $F(\omega)$ of a function $f(x)$ on $-\infty < x < +\infty$ by

$$(*2) \quad \boxed{F(\omega) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\bar{x})\, e^{i\omega \bar{x}}\, d\bar{x}} \quad \omega \in \mathbb{R}$$

By the Fourier integral identity (*1) we then have for "sufficiently nice" functions (smooth and decaying enough as $x \to \pm\infty$) that

$$(*3) \quad \boxed{f(x) = \int_{-\infty}^{+\infty} F(\omega)\, e^{-i\omega x}\, d\omega}\ , \quad x \in \mathbb{R}$$

<u>inverse Fourier transform</u>

Remarks:

- (*2) and (*3) are sometimes called the <u>Fourier transform pair</u>
- Note the opposite signs in the exponents of $e^{+i\omega x}$ and $e^{-i\omega x}$ in (*2) and (*3).
- We heuristically "derived" the Fourier transform from our results for Fourier series. Generally speaking, one has the correspondence

  finite interval $\Rightarrow$ Fourier series (a <u>sum</u>)

  infinite interval $\Rightarrow$ Fourier transform (an <u>integral</u>)
- Notation: Sometimes the Fourier transform $F(\omega)$ of $f(x)$ is denoted by $\mathcal{F}[f(x)]$. Similarly, the inverse Fourier transform of $F(\omega)$ is given the notation $\mathcal{F}^{-1}[F(\omega)]$.

Interlude:

- The Fourier transform $F(\omega) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\bar{x})\, e^{i\omega\bar{x}}\, d\bar{x}$ is well-defined for any absolutely integrable function $f(x)$, i.e. $f(x)$ satisfies $\int_{-\infty}^{+\infty} |f(x)|\,dx < \infty$. However, to recover $f(x)$ from its Fourier transform $F(\omega)$ via the inverse Fourier transform (*I) one needs stronger assumptions on $f(x)$. One has the following

Pointwise convergence theorem:

If $f(x)$ is piecewise smooth (that is, on any finite interval $f(x)$ has finitely many jump discontinuities and is smooth in between them) and absolutely integrable

$$\int_{-\infty}^{+\infty} |f(x)|\,dx < \infty,$$

then

(a) $F(\omega)$ is continuous

(b) $F(\omega) \to 0$ as $\omega \to \pm\infty$

(c) The inverse Fourier transform

$$\int_{-\infty}^{+\infty} F(\omega)\, e^{-i\omega x}\, d\omega$$

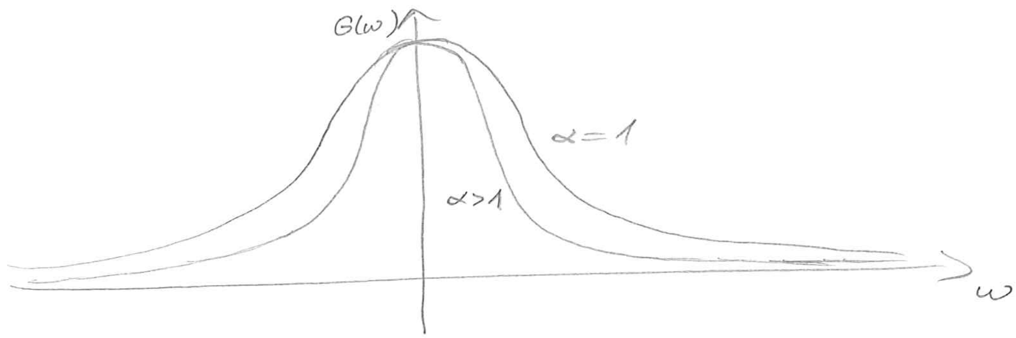
converges pointwise to

$$\frac{1}{2}\left(f(x+) + f(x-) \right)$$

Inverse Fourier transform of a Gaussian

In order to compute the solution to the heat equation on the whole line, we will need the inverse Fourier transform of a "bell-shaped" Gaussian

$$G(\omega) = e^{-\alpha\omega^2} \quad \text{for } \alpha > 0$$



We will now show that

$$g(x) = \int_{-\infty}^{+\infty} G(\omega)\, e^{-i\omega x}\, d\omega = \int_{-\infty}^{+\infty} e^{-\alpha\omega^2} e^{-i\omega x}\, d\omega$$

(*4)

$$= \sqrt{\frac{\pi}{\alpha}} \cdot e^{-\frac{x^2}{4\alpha}}.$$

Thus, the inverse Fourier transform of a Gaussian is itself a Gaussian!

This result also implies readily that the Fourier transform of a Gaussian is again a Gaussian:

Let $f(x) = e^{-\beta x^2}$ for some $\beta > 0$. Then

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\beta \bar{x}^2} e^{+i\omega \bar{x}} \, d\bar{x}$$

change of variables $x := -\bar{x}$, $dx = -d\bar{x}$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\beta x^2} e^{-i\omega x} \, dx$$

interchange roles of ω and x in (*4)

$$= \frac{1}{2\pi} \sqrt{\frac{\pi}{\beta}} \cdot e^{-\frac{\omega^2}{4\beta}} = \frac{1}{\sqrt{4\pi\beta}} \, e^{-\frac{\omega^2}{4\beta}}.$$

<u>Proof of (*4):</u>

We first observe that

$$g(x) = \int_{-\infty}^{+\infty} e^{-\alpha\omega^2} e^{-i\omega x} \, d\omega$$

solves an elementary ODE:

$$g'(x) = \frac{d}{dx} \int_{-\infty}^{+\infty} e^{-\alpha\omega^2} e^{-i\omega x} \, d\omega$$

$\frac{d}{dx}\left(e^{-i\omega x}\right) = -i\omega e^{-i\omega x}$

$$= \int_{-\infty}^{+\infty} (-i\omega) \, e^{-\alpha\omega^2} e^{-i\omega x} \, d\omega$$

$\frac{d}{d\omega}\left(e^{-\alpha\omega^2}\right) = -2\alpha\omega e^{-\alpha\omega^2}$

$$= i \cdot \int_{-\infty}^{+\infty} \frac{1}{2\alpha} \frac{d}{d\omega}\left(e^{-\alpha\omega^2}\right) e^{-i\omega x} \, d\omega$$

integrate by parts

$$= \frac{-i}{2\alpha} \int_{-\infty}^{+\infty} e^{-\alpha\omega^2} \underbrace{\frac{d}{d\omega}\left(e^{-i\omega x}\right)}_{= -ix\, e^{-i\omega x}} \, d\omega$$

$$\Rightarrow \quad g'(x) = -\frac{x}{2\alpha} \int_{-\infty}^{+\infty} e^{-\alpha\omega^2} e^{-i\omega x} \, d\omega = -\frac{x}{2\alpha} \cdot g(x)$$

Using the method of separation of variables (for ODEs)., we find that this ODE has the solution

$$g(x) = g(0) \, e^{-\frac{x^2}{2\alpha}}$$

with

$$g(0) = \int_{-\infty}^{+\infty} e^{-\alpha\omega^2} d\omega = \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{+\infty} e^{-z^2} dz$$

substitution:
$z := \sqrt{\alpha}\,\omega$
$\Rightarrow dz = \sqrt{\alpha}\, d\omega$

It therefore remains to compute the integral

$$I := \int_{-\infty}^{+\infty} e^{-z^2} dz$$

The result is $\sqrt{\pi}$ (which then yields (*4)) and it can be obtained via a beautiful trick:

It turns out that it is easier to compute the square of this integral ...

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-z^2} dz \right)^2$$

$$= \left(\int_{-\infty}^{+\infty} e^{-x^2} dx \right) \cdot \left(\int_{-\infty}^{+\infty} e^{-y^2} dy \right)$$

Fubini

$$\overset{\text{Fubini}}{=} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\, dx\, dy$$

$$= \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dx\, dy$$

$$= \int_0^{2\pi} \int_0^{\infty} e^{-r^2}\, r\, d\theta\, dr$$

polar coordinates:

$$x = r \cdot \cos(\theta)$$
$$y = r \cdot \sin(\theta) \qquad \Rightarrow \quad x^2 + y^2 = r^2$$
$$dx\, dy = r\, d\theta\, dr$$

$$\Rightarrow \quad I^2 = 2\pi \cdot \int_0^{\infty} \underbrace{e^{-r^2}\, r}_{= -\frac{1}{2} \frac{d}{dr}\left(e^{-r^2}\right)}\, dr$$

$$= 2\pi \left(-\frac{1}{2} e^{-r^2} \right) \Big|_{r=0}^{r=\infty}$$

$$= \pi$$

$$\Rightarrow \quad I = \sqrt{\pi}$$

Remark:

Alternatively, one can compute the integral

$$g(x) = \int_{-\infty}^{+\infty} e^{-\alpha\omega^2} e^{-i\omega x}\, d\omega = \int_{-\infty}^{+\infty} e^{-\alpha\omega^2 - i\omega x}\, d\omega$$

by completing the square in the exponent

$$-\alpha\omega^2 - i\omega x = -\alpha\left(\omega + i\frac{x}{2\alpha}\right)^2 - \frac{x^2}{4\alpha}$$

and using contour integration from complex analysis, see Section 10.3 in our textbook.
[But this requires that you have taken a course in complex analysis.]

Fourier transform and the heat equation

Using the Fourier transform, we can now finish the computation of the solution to the heat equation on the whole line

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < +\infty, \; t > 0$$

with initial condition

$$u(x,0) = f(x).$$

Earlier we showed that this initial value problem is solved by

$$u(x,t) = \int_{-\infty}^{+\infty} c(\omega)\, e^{-i\omega x}\, e^{-k\omega^2 t}\, d\omega$$

<u>if</u> the initial condition $f(x)$ can be written as

$$(\#1) \quad f(x) = \int_{-\infty}^{+\infty} c(\omega)\, e^{-i\omega x}\, d\omega$$

for suitable coefficients $c(\omega)$ depending on $f(x)$.
(Recall that we came to this conclusion using the analogy with the method of separation of variables on intervals $[-L, L]$ as $L \to \infty$ and the generalized superposition principle.)

Now we observe that (#1) is just a Fourier integral representation of $f(x)$, thus $c(\omega)$ is the Fourier transform of $f(x)$,

$$c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{i\omega x} dx.$$

Substituting $c(\omega)$ into the formula for the solution $u(x,t)$ we find that

$$u(x,t) = \int_{-\infty}^{+\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\bar{x}) e^{i\omega \bar{x}} d\bar{x} \right) e^{-i\omega x} e^{-k\omega^2 t} d\omega$$

This is a complete solution formula for $u(t,x)$ in terms of the initial condition; however, the formula is still quite complicated and we now try to simplify it:

First, interchange the orders of integration to get that

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\bar{x}) \left(\int_{-\infty}^{+\infty} e^{-k\omega^2 t} e^{-i\omega(x-\bar{x})} d\omega \right) d\bar{x}$$

Now recall that we computed the inverse Fourier transform of $e^{-\alpha\omega^2}$, $\alpha > 0$, to be

$$\mathcal{F}^{-1}(e^{-\alpha\omega^2})(x) = \int_{-\infty}^{+\infty} e^{-\alpha\omega^2} e^{-i\omega x}\, d\omega$$

$$= \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{x^2}{4\alpha}}$$

and observe that the integral

$$\int_{-\infty}^{+\infty} e^{-k\omega^2 t} e^{-i\omega(x-\bar{x})}\, d\omega = \mathcal{F}^{-1}\left(e^{-kt\omega^2}\right)(x-\bar{x})$$

is just the inverse Fourier transform of $e^{-kt\omega^2}$ evaluated at $x-\bar{x}$. Hence (with $\alpha = kt$) we obtain that

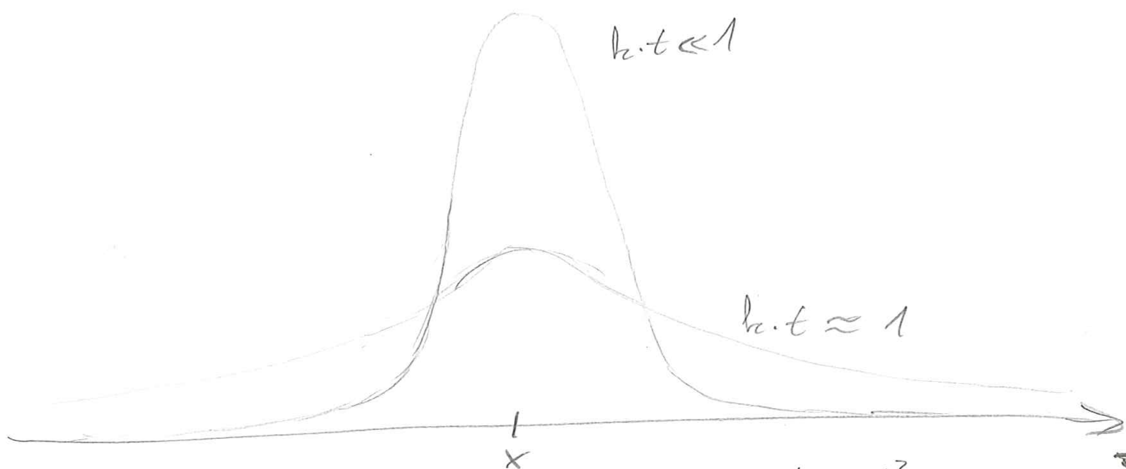
$$\int_{-\infty}^{+\infty} e^{-k\omega^2 t} e^{-i\omega(x-\bar{x})}\, d\omega = \sqrt{\frac{\pi}{kt}}\, e^{-\frac{(x-\bar{x})^2}{4kt}}.$$

It follows that the solution to the IVP for the heat equation on the whole line can be written as

$$(\#7)\quad u(x,t) = \int_{-\infty}^{+\infty} f(\bar{x}) \cdot \frac{1}{\sqrt{4\pi k t}} \cdot e^{-\frac{(x-\bar{x})^2}{4kt}}\, d\bar{x}$$

$\rightarrow$ We can read off this beautiful solution formula how the initial temperature $f(x)$ "influences" the temperature at time t.

The function $\frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-\bar{x})^2}{4kt}}$ is sometimes called the influence function



sketch of graph of $\bar{x} \mapsto \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-\bar{x})^2}{4kt}}$

- The influence function has the property that

$$\lim_{t \to 0^+} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-\bar{x})^2}{4kt}} = \delta(x-\bar{x}),$$

where $\delta(x-\bar{x})$ is the "Dirac delta function", which is defined by the property that

$$g(x) = \int_{-\infty}^{+\infty} g(\bar{x})\, \delta(x-\bar{x})\, d\bar{x}$$

for any function $g(x)$.

→ One can expect this behavior of the influence function since as $t \to 0^+$, its graph becomes more and more peaked and narrower.

Thus, the solution formula (#2) satisfies the initial condition

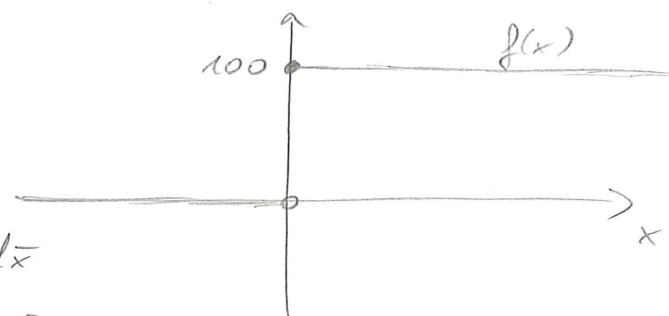
$$\lim_{t \to 0^+} u(x,t) = f(x).$$

- Moreover, we note that in the derivation of (#2) we had to assume that $f(x)$ is "quite nice" (smooth and decaying at infinity), however (#2) holds for much larger classes of functions (the integral in (#2) has to converge, roughly speaking)

Example:

Let's investigate how the following discontinuous initial conditions propagate:

$$u(x,0) = f(x) = \begin{cases} 0, & x < 0 \\ 100, & x \geq 0 \end{cases}$$



Then we have

$$u(x,t) = \int_{-\infty}^{+\infty} f(\bar{x}) \frac{1}{\sqrt{4\pi k t}} e^{-\frac{(x-\bar{x})^2}{4kt}} d\bar{x}$$

$$= \frac{100}{\sqrt{4\pi k t}} \int_0^\infty e^{-\frac{(x-\bar{x})^2}{4kt}} d\bar{x}$$

change of variables:

$$z := \frac{\bar{x} - x}{\sqrt{4kt}}$$

$$\Rightarrow dz = \frac{d\bar{x}}{\sqrt{4kt}}$$

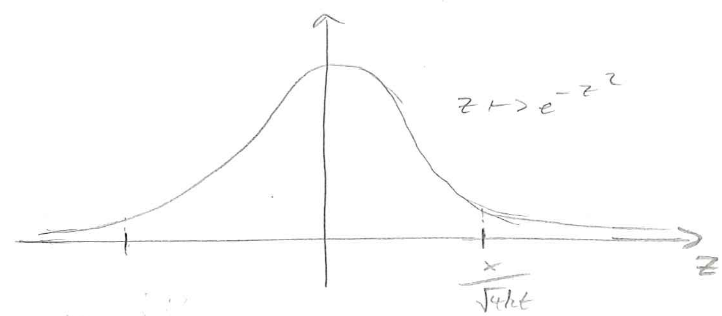
$$\bar{x} = 0 \mapsto -\frac{x}{\sqrt{4kt}}$$

$$= \frac{100}{\sqrt{\pi}} \int_{-\frac{x}{\sqrt{4kt}}}^{\infty} e^{-z^2} dz$$

The integrand no longer depends on any parameters and represents the area under a Gaussian curve.

Due to the evenness of $z \mapsto e^{-z^2}$ we may write

$$\int_{-\frac{x}{\sqrt{4kt}}}^{\infty} e^{-z^2}\,dz = \underbrace{\int_0^\infty e^{-z^2}\,dz}_{=\frac{\sqrt{\pi}}{2}\text{ (from before)}} + \int_0^{\frac{x}{\sqrt{4kt}}} e^{-z^2}\,dz$$

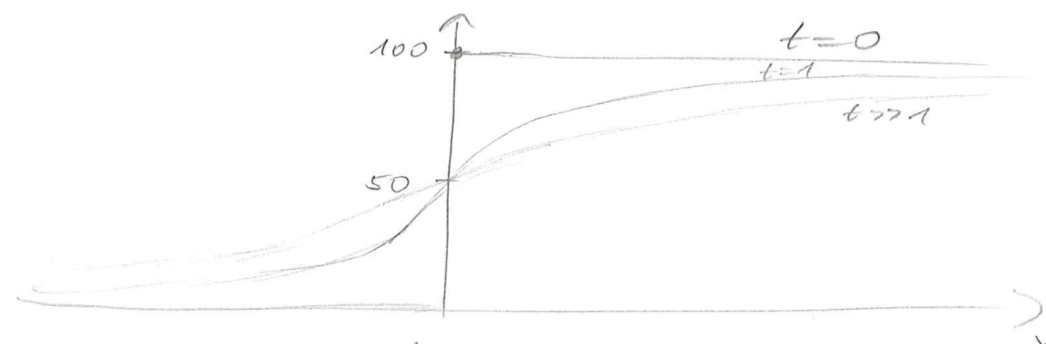


Thus,

$$u(x,t) = 50 + \frac{100}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-z^2}\,dz$$

We observe

- ⊛ the temperature $u(x,t)$ is non-zero <u>at all x</u> (!) for any positive $t>0$, even though $u=0$ for $x<0$ at $t=0$
  $\rightsquigarrow$ the thermal energy spreads at an <u>infinite</u> propagation speed!



graphs of $u(x,t)$

⊛ As $t \to \infty$, we have

$$\lim_{t \to \infty} u(x,t) = 50 \quad \text{for all } x \in \mathbb{R}.$$

⊛ The temperature $u(x,t)$ is constant whenever $\frac{x}{\sqrt{4kt}}$ is constant;

$\frac{x}{\sqrt{4kt}}$ is called the similarity variable.

Fourier transforming the heat equation

In the previous lectures we have solved the heat equation on the whole line

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < +\infty, \quad t > 0$$

$$u(x,0) = f(x)$$

by making analogies to the method of separation of variables (for linear PDEs on finite domains). Along the way we had to introduce the Fourier transform.

Having the Fourier transform at our disposal, we now develop a simpler strategy to derive the solution formula for the heat equation on the whole line that avoids making any analogies to the method of separation of variables.

This strategy will turn out to be quite general and powerful.

We begin by taking the (spatial) Fourier transform $\mathcal{F}$ of the entire heat equation (posed on the line)

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \quad \mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \mathcal{F}\left[k \cdot \frac{\partial^2 u}{\partial x^2}\right] = k \cdot \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right]$$

Thus, we need to compute the Fourier transforms of (time and spatial) derivatives of $u(x,t)$.

In the following we denote the (spatial) Fourier transform of $u(x,t)$ by

$$\overline{U}(\omega,t) := \mathcal{F}[u] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u(x,t)\, e^{+i\omega x}\, dx .$$

Note that $\overline{U}(\omega,t)$ is a function of the frequency ω <u>and</u> of time t.

Then we find for the (spatial) Fourier transform of the time derivative $\frac{\partial u}{\partial t}$:

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\partial u}{\partial t}(x,t)\, e^{+i\omega x}\, dx$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} u(x,t)\, e^{+i\omega x}\, dx\right)$$

$$\Rightarrow \left\| \mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{\partial}{\partial t}\overline{U}(\omega,t). \right\|$$

Hence:

The (spatial) Fourier transform of a time derivative equals the time derivative of the Fourier transform.

. Let's now consider the (spatial) Fourier transform of spatial derivatives:

$$\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\partial u}{\partial x}(x,t)\, e^{i\omega x}\, dx$$

integration by parts

$$= \frac{1}{2\pi}\Bigg(\underbrace{u(x,t)\, e^{i\omega x}\Big|_{x=-\infty}^{x=+\infty}}_{=0} - \int_{-\infty}^{+\infty} u(x,t) \underbrace{\frac{\partial}{\partial x}(e^{i\omega x})}_{=i\omega} dx \Bigg)$$

assuming $u(x,t) \to 0$ as $x \to \pm\infty$

$$= -\frac{i\omega}{2\pi}\int_{-\infty}^{+\infty} u(x,t)\, e^{i\omega x}\, d\omega$$

$$= -i\omega \cdot \mathcal{F}[u] = -i\omega\, U(\omega,t).$$

Iterating this computation, we find

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = -i\omega\, \mathcal{F}\left[\frac{\partial u}{\partial x}\right] = (-i\omega)^2\, \mathcal{F}[u] = (-i\omega)^2 \cdot U(\omega,t)$$

and more generally

$$\mathcal{F}\left[\frac{\partial^n u}{\partial x^n}\right] = (-i\omega)^n\, \mathcal{F}[u]$$

for any positive integer n.

(assuming that $u(x,t) \to 0$ as $x \to \pm\infty$ and the same for its derivatives)

Hence, the Fourier transform of the n-th derivative of a function with respect to x equals $(-i\omega)^n$ times the Fourier transform of the function.

$\rightsquigarrow$ The spatial Fourier transform turns spatial derivatives into multiplication on the frequency side!

· Going back to

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = k \cdot \mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right],$$

with $\overline{u}(\omega,t) := \mathcal{F}[u]$ we find that

$$(*) \quad \frac{\partial}{\partial t}\overline{u}(\omega,t) = k\cdot(-i\omega)^2\,\overline{u}(\omega,t) = -k\omega^2\cdot\overline{u}(\omega,t)$$

Observe that for every fixed frequency $\omega \in \mathbb{R}$, the equation $(*)$ is an ODE in time for the unknown $\overline{u}(\omega,t)$.

The general solution of $(*)$ is

$$(**) \quad \overline{u}(\omega,t) = \overline{u}(\omega,\underline{0})\cdot e^{-k\omega^2 t}$$

for every fixed $\omega \in \mathbb{R}$, where

$$\overline{u}(\omega,0) = \mathcal{F}[u(x,t=0)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{u(x,0)}_{=f(x)}\, e^{+i\omega x}\,dx = \mathcal{F}[f]$$

is the Fourier transform of the initial condition $f(x)$!

To obtain the nice solution formula for $u(x,t)$, we now want to take the inverse Fourier transform of (**):

$$u(x,t) = \mathcal{F}^{-1}\left[\bar{u}(\omega,t)\right] = \mathcal{F}^{-1}\left[\bar{u}(\omega,0)\, e^{-k\omega^2 t}\right]$$
$$= \mathcal{F}^{-1}\left[\mathcal{F}(f)(\omega)\cdot \underbrace{e^{-k\omega^2 t}}_{=\mathcal{F}\left[\sqrt{\frac{\pi}{kt}}\cdot e^{-\frac{x^2}{4kt}}\right]}\right],$$

in other words we have to compute the inverse Fourier transform $\mathcal{F}^{-1}$ of the <u>product</u> of two Fourier transforms, namely $\mathcal{F}[f](\omega)$ and $e^{-k\omega^2 t}$ (Gaussian).

To this end we use the

<u>Convolution Theorem</u>

<u>Q</u>: Suppose that $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(x)$ and $g(x)$, respectively. What is the function $h(x)$ whose Fourier transform $H(\omega)$ equals the product of $F(\omega)$ and $G(\omega)$:

$$H(\omega) = F(\omega)\cdot G(\omega) \;?$$

$$\left(\text{or} \quad h(x) = \mathcal{F}^{-1}\left[F(\omega)\cdot G(\omega)\right](x) \;?\right)$$

We have

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\bar{x}) e^{i\omega\bar{x}} d\bar{x} \qquad G(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\bar{x}) e^{i\omega\bar{x}} d\bar{x}$$

$$f(x) = \int_{-\infty}^{+\infty} F(\omega) e^{-i\omega x} d\omega \qquad g(x) = \int_{-\infty}^{+\infty} G(\omega) e^{-i\omega x} dx$$

and set

$$H(\omega) := F(\omega) \cdot G(\omega)$$

Then

$$h(x) := \mathcal{F}^{-1}\left[H(\omega)\right]$$

$$= \int_{-\infty}^{+\infty} H(\omega) e^{-i\omega x} d\omega$$

$$= \int_{-\infty}^{+\infty} F(\omega) \cdot \underbrace{G(\omega)} \cdot e^{-i\omega x} d\omega$$

inserting identity for $G(\omega)$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) \left(\int_{-\infty}^{+\infty} g(\bar{x}) e^{i\omega\bar{x}} d\bar{x} \right) e^{-i\omega x} d\omega.$$

interchanging order of integration

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\bar{x}) \underbrace{\left(\int_{-\infty}^{+\infty} F(\omega) e^{-i\omega(x-\bar{x})} d\omega \right)}_{= \mathcal{F}^{-1}[F](x-\bar{x}) = f(x-\bar{x})} d\bar{x}$$

just inverse Fourier transform of $F(\omega)$ evaluated at $x-\bar{x}$

$$\Rightarrow \boxed{h(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\bar{x}) f(x-\bar{x}) d\bar{x}}$$

this integral is called the convolution of $g(x)$ and $f(x)$ and denoted by $(g * f)$.

By a change of variables

$$w := x - \bar{x} \quad , \quad dw = -d\bar{x}$$
$$\Rightarrow \bar{x} = x - w$$

we see that also

$$h(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(x-w)\, f(w)\, dw.$$

In summary, the inverse Fourier transform of the product of two Fourier transforms is $\frac{1}{2\pi}$ times the convolution of the two functions.

Back to our heat equation:

We had derived that the solution $u(x,t)$ to the heat equation on the whole line

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2} \quad , \quad -\infty < x < \infty \ , \ t > 0$$
$$u(x,0) = f(x)$$

is given by

$$u(x,t) = \mathcal{F}^{-1}\left[U(\omega,t)\right] = \mathcal{F}^{-1}\left[\mathcal{F}[f](\omega) \cdot e^{-k\omega^2 t}\right].$$

Here, $\mathcal{F}[f](\omega)$ is the Fourier transform of the initial condition $f(x)$ and $e^{-k\omega^2 t} = e^{-kt\omega^2}$ (Gaussian) is the Fourier transform of $\sqrt{\frac{\pi}{kt}} \cdot e^{-\frac{x^2}{4kt}}$

(using that as derived earlier $\mathcal{F}\left(\sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{x^2}{4\alpha}}\right) = e^{-\alpha\omega^2}$ with $\alpha = kt$)

Thus, by the convolution theorem

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\bar{x}) \sqrt{\frac{\pi}{kt}}\, e^{-\frac{(x-\bar{x})^2}{4kt}}\, d\bar{x}$$

$$= \int_{-\infty}^{+\infty} f(\bar{x}) \frac{1}{\sqrt{4\pi kt}}\, e^{-\frac{(x-\bar{x})^2}{4kt}}\, d\bar{x}$$

(as we had derived earlier).

We summarize the (general) strategy used here to solve the linear heat equation:

1. Take the Fourier transform of the (linear) PDE (with constant coefficients)
2. Solve the resulting ODE (in time) for the Fourier transform (of the solution) $\bar{u}(\omega,t)$ for each fixed frequency ω.
3. Apply the initial conditions to determine $\bar{u}(\omega, t=0)$
4. Use the convolution theorem. [usually]

Examples: Using the Fourier transform to solve linear PDEs

Example 1: 1D wave equation on an infinite interval

We consider the one-dimensional wave equation on the whole line $-\infty < x < +\infty$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \cdot \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < +\infty \\ u(x,0) = f(x) \\ \dfrac{\partial u}{\partial t}(x,0) = \underline{\underline{0}} \end{cases}$$

Note: that to simplify things a bit here, we assume that the initial condition for $\frac{\partial u}{\partial t}(x,0)$ is zero.

We define the Fourier transform of the (unknown) solution $u(x,t)$, which we assume to be sufficiently smooth and decaying,

$$\overline{U}(\omega,t) := \frac{1}{2\pi} \int_{-\infty}^{+\infty} u(x,t)\, e^{i\omega x}\, dx.$$

$$\Rightarrow u(x,t) = \int_{-\infty}^{+\infty} \overline{U}(\omega,t)\, e^{-i\omega x}\, d\omega$$

by the Fourier integral identity

Now we take the Fourier transform of the 1D wave equation

$$\mathcal{F}\left(\frac{\partial^2 u}{\partial t^2}\right) = c^2 \cdot \mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$\Rightarrow \quad \frac{\partial^2}{\partial t^2}\mathcal{F}(u) = c^2 (-i\omega)^2 \mathcal{F}(u)$$

$$\Rightarrow \quad \boxed{\frac{\partial^2 \overline{u}}{\partial t^2} = -c^2 \omega^2 \cdot \overline{u}}$$

(*) ⇝ ODE in time for each fixed frequency!

and for the initial conditions we have

$$\overline{u}(\omega, 0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)\, e^{i\omega x}\, dx$$

$$\frac{\partial \overline{u}}{\partial t}(\omega, 0) = 0$$

· The general solution of the ODE (*) for each fixed frequency $\omega \in \mathbb{R}$ is

$$\overline{u}(\omega, t) = A(\omega) \cos(c\omega t) + B(\omega) \cdot \sin(c\omega t)$$

The initial conditions imply

$$A(\omega) = \overline{u}(\omega, 0)$$

$$B(\omega) = 0$$

$$\Rightarrow \overline{u}(\omega, t) = \overline{u}(\omega, 0) \cdot \cos(c\omega t)$$

$$\Rightarrow \left\| u(x,t) = \int_{-\infty}^{+\infty} \overline{u}(\omega, 0) \underbrace{\cos(c\omega t)}_{= \frac{1}{2}\left(e^{+i\omega ct} + e^{-i\omega ct}\right)} e^{-i\omega x}\, d\omega \right\|$$

Thus,

$$u(x,t) = \frac{1}{2}\int_{-\infty}^{+\infty} \overline{u}(\omega,0)\left(e^{-i\omega(x-ct)} + e^{-i\omega(x+ct)}\right) d\omega$$

Now recall that $\overline{u}(\omega,0)$ is the Fourier transform of $f(x)$, therefore by the Fourier integral identity

$$f(x) = \int_{-\infty}^{+\infty} \overline{u}(\omega,0)\, e^{-i\omega x}\, d\omega .$$

Comparing the last two identities, we conclude

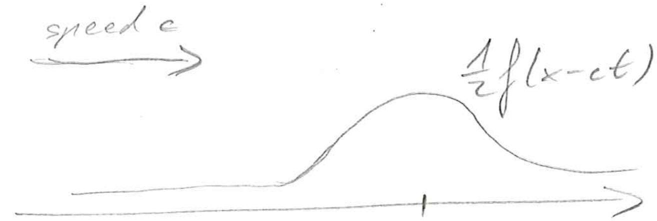
$$\Big\| \; u(x,t) = \frac{1}{2}\big(f(x-ct) + f(x+ct)\big) \; \Big\|$$

Thus, $u(x,t)$ is the sum of two terms $\frac{1}{2} f(x-ct)$ and $\frac{1}{2} f(x+ct)$.

Note that the wave form of $\frac{1}{2} f(x-ct)$ is of fixed shape, it is a travelling wave moving at constant speed c.

Say $c > 0$, then:

At $t=0$:



$\rightarrow$ If $c > 0$, $\frac{1}{2} f(x-ct)$ moves with speed c to the right; similarly $\frac{1}{2} f(x+ct)$ moves with speed c to the left.

Interpretation:

The initial position of the string ($f(x)$) breaks in two, if started at rest ($\frac{\partial u}{\partial t}(x,0)=0$), half moving to the left and half moving to the right at equal speeds c.
The solution is the sum of these two travelling waves.

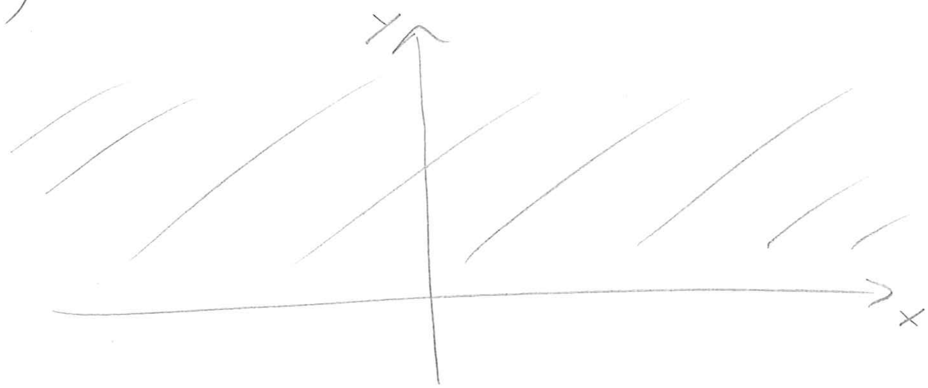
Example 2: Laplace's equation in a half-plane

We consider Laplace's equation in the half-plane $\{(x,y) \in \mathbb{R}^2 \mid y>0\}$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 , \quad \begin{matrix} -\infty < x < +\infty \\ y > 0 \end{matrix}$$

subject to the boundary condition

$$u(x,0) = f(x)$$



If $f(x) \to 0$ as $x \to \pm\infty$, it is reasonable to assume the following three other boundary conditions for the (unknown) solution

$$\lim_{x \to +\infty} u(x,y) = 0 \qquad \text{for all fixed } y \geq 0,$$

$$\lim_{x \to -\infty} u(x,y) = 0 \qquad \text{for all fixed } y \geq 0,$$

$$\lim_{y \to +\infty} u(x,y) = 0 \qquad \text{for all fixed } x \in \mathbb{R}.$$

Since for every $y > 0$, $u(x,y)$ is defined for all $-\infty < x < \infty$, and decays as $x \to \pm\infty$, for every $y > 0$ we can take the Fourier transform in x of $u(x,y)$:

$$\overline{U}(\omega,y) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} u(x,y)\, e^{i\omega x}\, dx$$

$$u(x,y) = \int_{-\infty}^{+\infty} \overline{U}(\omega,y)\, e^{-i\omega x}\, d\omega$$

Taking the Fourier transform in x of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

we obtain

$$F\left(\frac{\partial^2 u}{\partial x^2}\right) + F\left(\frac{\partial^2 u}{\partial y^2}\right) = 0$$

$$\Rightarrow \underbrace{(-i\omega)^2}_{=-\omega^2} \overline{U} + \frac{\partial^2 \overline{U}}{\partial y^2} = 0$$

$$\Rightarrow \left\| \frac{\partial^2 \overline{U}}{\partial y^2} = \omega^2 \overline{U} \right\|$$

- Since $u(x,y) \to 0$ as $y \to +\infty$, the Fourier transform in x, $\overline{U}(\omega,y)$, also vanishes as $y \to +\infty$,

$$\overline{U}(\omega,y) \to 0 \quad \text{as } y \to +\infty.$$

- Moreover, at $y=0$, $\overline{U}(\omega,0)$ is the Fourier transform in x of the boundary condition $f(x)$,

$$\overline{U}(\omega,0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)\, e^{i\omega x}\, dx$$

· The general solution of the ODE (for any $\omega \in \mathbb{R}$)

$$\frac{\partial^2 \overline{u}}{\partial y^2} = \omega^2 \overline{u}$$

is

$$\overline{u}(\omega, y) = a(\omega) \cdot e^{+\omega y} + b(\omega) \cdot e^{-\omega y}.$$

We now determine the coefficients $a(\omega)$ and $b(\omega)$ from the boundary conditions

(1) $\overline{u}(\omega, y) \longrightarrow 0$ as $y \longrightarrow +\infty$,

(2) $\overline{u}(\omega, 0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)\, e^{i\omega x}\, dx$

We have to be careful with (1); note that when $\omega > 0$, $e^{+\omega y} \to \infty$ as $y \to \infty$ while if $\omega < 0$, $e^{\omega y} \to 0$ as $y \to \infty$.

Thus, we need to have

$$\overline{u}(\omega, y) = \begin{cases} b(\omega)\, e^{-\omega y} & , \omega > 0 \\ a(\omega)\, e^{+\omega y} & , \omega < 0. \end{cases}$$

It is therefore more convenient to write

$$\overline{u}(\omega, y) = c(\omega) \cdot e^{-|\omega| y}.$$

From the boundary condition (2) we then get that $c(\omega) = \overline{u}(\omega, 0)$ is just the Fourier transform of $f(x)$.

$$\Rightarrow \quad \overline{u}(\omega, y) = \mathcal{F}(f)(\omega) \cdot e^{-|\omega| y}$$

Hence, by the convolution theorem,

$$u(x,y) = \mathcal{F}^{-1}(\overline{U}(\omega,y)) = \frac{1}{2\pi} f * \mathcal{F}^{-1}(e^{-|\omega| y})$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\bar{x})\, g(x-\bar{x},y)\, dy,$$

where $g(x,y)$ is the inverse Fourier transform of $e^{-|\omega| y}$:

$$\mathcal{F}^{-1}(e^{-|\omega| y})(x) = \int_{-\infty}^{+\infty} e^{-|\omega| y}\, e^{-i\omega x}\, d\omega$$

$$= \int_{-\infty}^{0} \underbrace{e^{\omega y}\, e^{-i\omega x}}_{=e^{(y-ix)\omega}}\, d\omega + \int_{0}^{\infty} \underbrace{e^{-\omega y} \cdot e^{-i\omega x}}_{=e^{(-y-ix)\omega}}\, d\omega$$

$$= \left[\frac{1}{y-ix} e^{(y-ix)\omega}\right]_{-\infty}^{0} + \left[\frac{1}{-y-ix} e^{(-y-ix)\omega}\right]_{0}^{\infty}$$

note that $y>0$

$$= +\frac{1}{y-ix} - \frac{1}{-y-ix}$$

$$= \frac{1}{y-ix} + \frac{1}{y+ix}$$

$$= \frac{2y}{x^2+y^2}$$

Thus, the solution $u(x,y)$ is

$$\boxed{u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\bar{x})\, \frac{2y}{(x-\bar{x})^2+y^2}\, d\bar{x}}$$

__Note:__ We derived this solution formula under the assumption that $f(x) \to 0$ as $x \to \pm\infty$; but in fact, this solution formula is valid under much weaker assumptions, the integral just has to be convergent (for which it suffices that $f(x)$ is bounded).

<u>Example of a boundary condition:</u>

Consider the simple boundary condition

$$f(x) = \begin{cases} 0 & , \quad x < 0 \\ 1 & , \quad x > 0 \end{cases}$$

Then the solution to the Laplace equation on the half-plane with BC $f(x)$ is

$$u(x,y) = \frac{1}{2\pi} \int_0^\infty \frac{2y}{\underbrace{(x-\bar{x})^2}_{=(\bar{x}-x)^2}+y^2} \, d\bar{x}$$

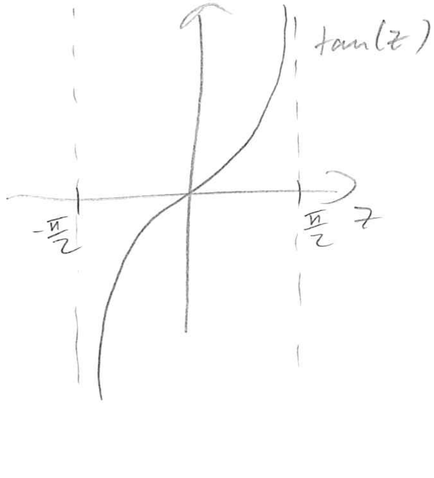
$$= \frac{1}{\pi} \int_0^\infty \frac{1}{y} \cdot \frac{1}{1+\left(\frac{\bar{x}-x}{y}\right)^2} \, d\bar{x}$$

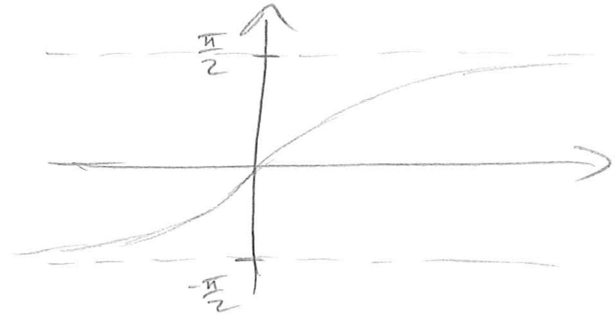
$$= \frac{1}{\pi} \left[\arctan\left(\frac{\bar{x}-x}{y}\right) \right]_{\bar{x}=0}^{\bar{x}=\infty}$$

$$= \frac{1}{\pi} \left(\frac{\pi}{2} - \left(\arctan\left(-\frac{x}{y}\right)\right)\right)$$

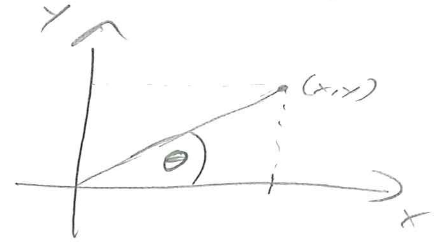
$$= \frac{1}{\pi} \left(\frac{\pi}{2} + \arctan\left(\frac{x}{y}\right)\right)$$

$$\frac{d}{d\bar{x}} \arctan(\bar{x}) = \frac{1}{1+\bar{x}^2}$$
$$\Rightarrow \frac{d}{d\bar{x}}\left(\arctan\left(\frac{\bar{x}-x}{y}\right)\right) = \frac{1}{y} \frac{1}{1+\left(\frac{\bar{x}-x}{y}\right)^2}$$





If we let θ denote the angle that the point (x,y) makes with the x-axis, then



$$\tan(\theta) = \frac{y}{x}$$

and

$$\tan\left(\frac{\pi}{2}-\theta\right) = \frac{x}{y}$$

$$\Rightarrow \quad \frac{\pi}{2}-\theta = \arctan\left(\frac{x}{y}\right)$$

Thus, we may write the solution $u(x,y)$ more succinctly as

$$u(x,y) = \frac{1}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}-\theta\right) = 1-\frac{\theta}{\pi}$$

Note: For such a simple BC we can check the answer using polar coordinates (r,θ):

$\Delta u = 0$ becomes $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

with BCs $u(r,0) = 1$ and $u(r,\pi) = 0$

Since the BCs only depend on the angle, the solution will only depend on the angle $u = u(\theta)$.

$\Rightarrow \frac{d^2 u}{d\theta^2} = 0$ has the solution $u(\theta) = 1-\frac{\theta}{\pi}$ satisfying $u(0) = 1$, $u(\pi) = 0$.

We summarize some important properties of the Fourier transform that we have discussed: (see also table 10.4.1 in our textbook)

$f(x) = \int_{-\infty}^{+\infty} F(\omega) e^{-i\omega x} d\omega$	$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\bar{x}) e^{+i\omega \bar{x}} d\bar{x}$
$\frac{\partial f}{\partial x}$	$(-i\omega) F(\omega)$
$\frac{\partial^2 f}{\partial x^2}$	$(-i\omega)^2 F(\omega)$
$\frac{\partial f}{\partial t}$	$\frac{\partial F}{\partial t}$
$\frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\bar{x})\, g(x-\bar{x})\, d\bar{x}$	$F(\omega) \cdot G(\omega)$
$f(x-\beta)$	$e^{+i\omega\beta} F(\omega)$
$x \cdot f(x)$	$-i \frac{dF}{d\omega}$