Chapter 2 : Method of Separation of Variables

We develop a technique called the <u>method of separation of variables</u> to solve certain PDEs. We will do so at the example of the heat equation for a one-dimensional rod with zero boundary conditions

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2} \quad , \qquad \begin{array}{c} 0 < x < L \\ t > 0 \end{array}$$

$$(\mathrm{BC}) \quad \begin{array}{l} u(0,t) = 0 \\ u(L,t) = 0 \end{array}$$

$$(\mathrm{IC}) \quad u(x,0) = f(x)$$

Our strategy will be to first look for "special solutions" of the form

$$u(x,t) = \phi(x) \cdot G(t) \quad \rightarrow \text{"separation of variables"}$$

and to then "add these up to match the initial condition $f(x)$".

More generally, the method of separation of variables is used when the PDE and the boundary conditions are <u>linear</u> and <u>homogeneous</u>.

Linearity and homogeneity are fundamental concepts for the study of PDEs. We should therefore first carefully define and discuss these more abstract notions.

Linearity and Homogeneity

Before defining what a linear PDE and a linear operator are, let's look at some examples:

Algebraic equations:

linear	nonlinear
$x + 2y = 0$	$x^3 = 2x$
$3x - y = 1$	

ODEs:

linear	nonlinear
$\frac{dy}{dt} + t^2 y = \cos(3t)$	$\frac{dy}{dt} = t^2 + e^y$

PDEs:

linear	nonlinear
$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$	$\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2$

Heuristically, what distinguishes linear equations from nonlinear ones? How does one recognize by eye a linear equation:

- the unknown quantity only appears to the first power and does not
- appear inside (transcendental) functions like sin or log
- there may be terms that do not involve the unknown at all and it is OK for the independent variable to appear in nonlinear functions.

Let's now give a formal definition of linear:

Definition:

An operation L on functions (usually referred to as an operator) is linear if it satisfies

$$L(c_1 \cdot u_1 + c_2 \cdot u_2) = c_1 \cdot L(u_1) + c_2 \cdot L(u_2)$$

for any two functions u_1, u_2 and any constants c_1, c_2.

Examples of linear operators:

- differentiation of u: $L(u) := \frac{du}{dx}$
- multiplication of u by a given function: $L(u) = x^2 \cdot u(x)$
- integration: $L(u) = \int_0^1 u(x)\,dx$

Claim: The heat operator

$$L(u) = \frac{\partial u}{\partial t} - k \cdot \frac{\partial^2 u}{\partial x^2}$$

is linear

Proof:

We have

$$L(c_1 \cdot u_1 + c_2 \cdot u_2) = \frac{\partial}{\partial t}(c_1 \cdot u_1 + c_2 \cdot u_2) - k \cdot \frac{\partial^2}{\partial x^2}(c_1 \cdot u_1 + c_2 \cdot u_2)$$

$$= c_1 \cdot \frac{\partial u_1}{\partial t} + c_2 \cdot \frac{\partial u_2}{\partial t} - k \cdot c_1 \frac{\partial^2 u_1}{\partial x^2} - k \cdot c_2 \cdot \frac{\partial^2 u_2}{\partial x^2}$$

$$= c_1 \cdot \left(\frac{\partial u_1}{\partial t} - k \cdot \frac{\partial^2 u_1}{\partial x^2} \right) + c_2 \cdot \left(\frac{\partial u_2}{\partial t} - k \cdot \frac{\partial^2 u_2}{\partial x^2} \right)$$

$$= c_1 \cdot L(u_1) + c_2 \cdot L(u_2)$$

▨

Definition:

A linear equation is an equation of the form

$$L(u) = f,$$

where L is a linear operator, f is a "given" or "known" function and u is the unknown.

Thus, the heat equation is a linear PDE

$$L(u) = \frac{\partial u}{\partial t} - k \cdot \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

or in the more common form

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2} + f(x,t)$$

The u-independent terms enter the picture in the role of f.
This leads to an absolutely crucial distinction:

Definition:

A linear equation $L(u) = f$ is *homogeneous* if $f = 0$; it is *nonhomogeneous* if $f \neq 0$.

A fundamental property of solutions to linear *and* homogeneous equations is that they can be added together in the following sense:

Principle of Superposition

If u_1, u_2 satisfy a linear homogeneous equation, then any linear combination $c_1 \cdot u_1 + c_2 \cdot u_2$ for $c_1, c_2 \in \mathbb{R}$ also satisfies the same linear homogeneous equation.

Proof:

Let L be the linear operator and suppose $L(u_1) = L(u_2) = 0$.
Then by linearity of L,

$$L(c_1 \cdot u_1 + c_2 \cdot u_2) = c_1 \cdot \underbrace{L(u_1)}_{=0} + c_2 \cdot \underbrace{L(u_2)}_{=0} = 0.$$

The concepts of linearity and homogeneity also apply to boundary conditions.

Examples of linear boundary conditions

$$u(0,t) = f(t)$$

$$\frac{\partial u}{\partial x}(L,t) = g(t)$$

homogeneous if and only if $f(t) = 0$, $g(t) = 0$

$$\frac{\partial u}{\partial x}(0,t) = 0$$

A nonlinear boundary condition would for example be:

$$\frac{\partial u}{\partial x}(L,t) = u(L,t)^2.$$

⇝ The method of separation of variables can apply to determine the solution(s) of a PDE if the PDE and the boundary conditions are linear and homogeneous.

Heat equation with zero temperatures at finite ends

We now introduce the method of separation of variables in the context of solving the heat equation for a one-dimensional rod $(0 < x < L)$ with no sources and both ends immersed in a 0° temperature bath:

(PDE) $$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}, \qquad \begin{matrix} 0 < x < L \\ t > 0 \end{matrix}, \qquad k > 0$$

(BC) $$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}$$

(IC) $$u(x,0) = f(x)$$

We first seek to determine special solutions of the form

(*) $$u(x,t) = \underbrace{\phi(x)}_{\text{only a function of } x} \cdot \overbrace{G(t)}^{\text{only a function of } t}$$

Substituting (*) into the heat equation $\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}$, we find

$$\phi(x) \cdot \frac{dG}{dt} = k \cdot \frac{d^2\phi}{dx^2} \cdot G$$

Dividing by $k \cdot \phi(x) \cdot G(t)$ ("separating variables"), we find

$$\underbrace{\frac{1}{k \cdot G(t)} \cdot \frac{dG}{dt}}_{\substack{\text{function of} \\ t \text{ only!}}} = \underbrace{\frac{1}{\phi(x)} \cdot \frac{d^2\phi}{dx^2}}_{\substack{\text{function of} \\ x \text{ only!}}}$$

Key observation:

The LHS depends only on t, while the RHS depends only on x. Thus, the identity can only hold for all t and x, if in fact both sides are equal to the same constant!

Hence, we must have

$$\frac{1}{k \cdot G} \cdot \frac{dG}{dt} = \frac{1}{\phi} \cdot \frac{d^2\phi}{dx^2} = -\lambda$$

the minus sign is just for convenience in later computations!

for some arbitrary constant $\lambda \in \mathbb{R}$, known as the <u>separation constant</u>.

This gives rise to two <u>simpler</u> ODEs for $G(t)$ and $\phi(x)$:

$$\frac{dG}{dt} = -\lambda \cdot k \cdot G,$$

$$\frac{d^2\phi}{dx^2} = -\lambda \cdot \phi.$$

Now our product solution $u(x,t) = \phi(x) \cdot G(t)$ shall also satisfy the BCs

$$u(0,t) = 0 \quad \Rightarrow \quad \phi(0) \cdot G(t) = 0$$

So either $\phi(0) = 0$ or $G(t) = 0$ (for all t).
Note that $G(t) = 0$ would imply $u(x,t) = 0$, which is a trivial solution that we already know.
To get non-trivial solutions, we require

$$\phi(0) = 0$$

and analogously at the other end

$$\phi(L) = 0.$$

The time-dependent equation

We now have to solve the two ODEs separately. ~~Start~~ with the time-dependent one which does not come with additional conditions

$$\frac{dG}{dt} = -\lambda k \cdot G.$$

Making an exponential ansatz $G(t) = e^{r \cdot t}$ and substituting, we must have $r = -\lambda \cdot k$, hence the general solution is

$$G(t) = c \cdot e^{-\lambda k t}, c \in \mathbb{R}.$$

Now observe:

$\lambda > 0$: $G(t)$ decays exponentially as t increases (recall that $k > 0$)

$\lambda = 0$: $G(t)$ constant in time

$\lambda < 0$: $G(t)$ grows exponentially as t increases

⇝ We do not expect this for a heat equation without external sources!

⇒ Expect: $\lambda \geq 0$

(We introduced the artificial minus sign further above to have this expectation for non-negative λ)

⇝ We will soon see that only certain values of λ (≥ 0) are allowable.

Boundary Value Problem

We turn to the x-dependent ODE with two homogeneous boundary conditions

$$\begin{cases} \frac{d^2\phi}{dx^2} = -\lambda \cdot \phi \\ \phi(0) = 0 \\ \phi(L) = 0 \end{cases}$$

boundary value problem

Note that $\phi(x) = 0$ is a (so-called) trivial solution. We will now see that for certain special values of λ, called eigenvalues, this boundary value problem has non-trivial solutions, called eigenfunctions (associated with the eigenvalues λ).

Let's try to determine the non-trivial solutions, make the ansatz

$$\phi(x) = e^{rx}$$

Then we must have

$$r^2 = -\lambda .$$

Distinguish the following cases:

- $\lambda > 0 \rightsquigarrow r = \pm i\sqrt{\lambda}$
- $\lambda = 0 \rightsquigarrow r = 0$
- $\lambda < 0 \rightsquigarrow r = \pm\sqrt{-\lambda}$

(· λ has non-zero imaginary part) ← ignore this case

⤳ From considering the time-dependent ODE we expect that only $\lambda \geq 0$ is possible for physical reasons; we will see here that $\lambda < 0$ would lead only to the trivial solution anyway.

1st case: $\lambda > 0$

Then two independent solutions are $e^{\pm i\sqrt{\lambda}x}$. To have real-valued independent solutions we rather choose $\cos(\sqrt{\lambda}x)$ and $\sin(\sqrt{\lambda}x)$.

Thus, the general solution is

$$\phi(x) = c_1 \cdot \cos(\sqrt{\lambda}x) + c_2 \cdot \sin(\sqrt{\lambda}x)$$

for some constants c_1, c_2.

Now we have to match the BCs:

$$\phi(0) = 0 \Rightarrow c_1 = 0 \Rightarrow \phi(x) = c_2 \cdot \sin(\sqrt{\lambda}\cdot x)$$

$$\phi(L) = 0 \Rightarrow \sin(\sqrt{\lambda}\cdot L) = 0$$

Hence, $\sqrt{\lambda}\cdot L$ must be a zero of the sine function:

$$\sqrt{\lambda}\cdot L = n\cdot\pi \quad , \quad n = 1, 2, 3, \dots$$

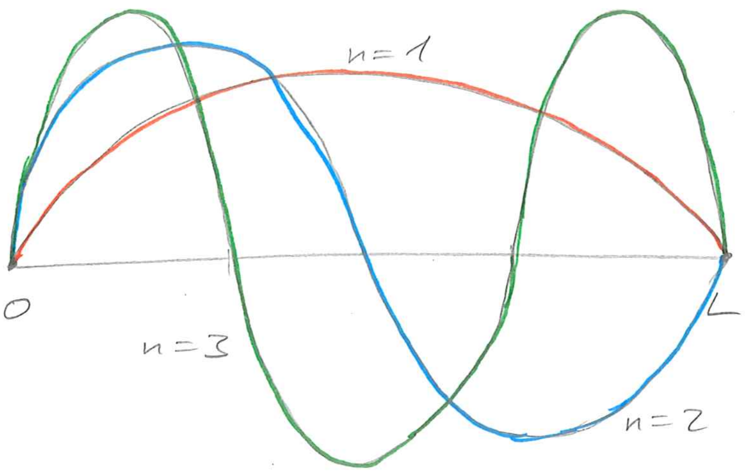
↖ $\sqrt{\lambda} > 0$, so only positive integers

The special eigenvalues λ are thus positive integers

$$\lambda = \left(\frac{n\cdot\pi}{L}\right)^2 \quad , \quad n = 1, 2, 3, \dots$$

with corresponding eigenfunctions

$$\phi(x) = c \cdot \sin\left(\frac{n\pi x}{L}\right) \quad , \quad c \in \mathbb{R}.$$



2^{nd} case: $\lambda = 0$

If $\lambda = 0$, we consider $\frac{d^2\phi}{dx^2} = 0$ with general solution

$$\phi(x) = c_1 \cdot x + c_2 .$$

Then the boundary condition $\phi(0) = 0$ implies $c_2 = 0$ and $\phi(L) = 0$ yields

$$0 = c_1 \cdot L \quad \Rightarrow \quad c_1 = 0 \quad (\text{since } L > 0).$$

Hence, for $\lambda = 0$ we only get the trivial solution $\phi(x) = 0$.

3^{rd} case: $\lambda < 0$

If $\lambda < 0$, then $\frac{d^2\phi}{dx^2} = \underbrace{-\lambda}_{>0} \cdot \phi$ has the two independent solutions $e^{+\sqrt{-\lambda} \cdot x}$ and $e^{-\sqrt{-\lambda} x}$, thus the general solution is

$$\phi(x) = c_1 \cdot e^{+\sqrt{-\lambda} \cdot x} + c_2 \cdot e^{-\sqrt{-\lambda} \cdot x} , \quad c_1, c_2 \in \mathbb{R}.$$

To match the BCs we must have

$$0 = c_1 + c_2 \quad \Rightarrow \quad c_2 = -c_1$$

$$0 = c_1 \cdot e^{+\sqrt{-\lambda}\cdot L} + \underbrace{c_2}_{=-c_1} \cdot e^{-\sqrt{-\lambda}\cdot L}$$

$$= c_1 \cdot \underbrace{\left(e^{+\sqrt{-\lambda}\cdot L} - e^{-\sqrt{-\lambda}\cdot L} \right)}_{\neq 0 \text{ for any } L>0!}$$

$$\Rightarrow \quad c_1 = 0 \quad \text{and} \quad c_2 = 0$$

Thus, for $\lambda < 0$ we only find the trivial solution $\phi(x) \equiv 0$.

Summary:

We have found that the boundary value problem

$$\frac{d^2\phi}{dx^2} + \lambda \cdot \phi = 0 ,$$

$$\phi(0) = 0 ,$$

$$\phi(L) = 0 ,$$

only has non-trivial solutions when $\lambda > 0$; more precisely for the eigenvalues

$$\lambda = \left(\frac{n\cdot\pi}{L}\right)^2 , \qquad n = 1, 2, 3, \ldots$$

with corresponding eigenfunctions

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right) , \qquad n = 1, 2, 3, \ldots$$

Product solutions

By putting together the solutions to the time-dependent problem and to the boundary value problem that we have found, we obtain the following product solutions to the heat equation

$$\boxed{u(x,t) = B \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 \cdot t}, \quad n = 1,2,3,\ldots}$$

⇝ Note that these special solutions are all exponentially decaying as $t \to \infty$.

Initial value problems (IVPs)

We now hope to be able to use these product solutions to solve IVPs for the heat equation

$$\frac{\partial u}{\partial t} - k \cdot \frac{\partial^2 u}{\partial x^2} = 0$$

$$(BC) \quad \begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}$$

$$(IC) \quad u(x,0) = f(x)$$

for arbitrary (!?) initial conditions $u(x,0) = f(x)$.

Observe that at time $t=0$, the special product solutions are of the form

$$u(x,0) = B \cdot \sin\left(\frac{n\pi x}{L}\right).$$

Thus, if the (IC) $f(x)$ happens to be a multiple of $\sin\left(\frac{n\pi x}{L}\right)$ for some $n = 1,2,3,\dots,$ say $4 \cdot \sin\left(\frac{3\pi x}{L}\right)$, then we already know that

$$u(x,t) = 4 \cdot \sin\left(\frac{3\pi x}{L}\right) \cdot e^{-k \cdot \left(\frac{3\pi}{L}\right)^2 t}$$

is the (unique) solution to the IVP for such special initial data.

Recall:

By the superposition principle, given any solutions $u_1, u_2, \dots, u_M$ of a linear homogeneous PDE, then any finite linear combination

$$c_1 \cdot u_1 + c_2 \cdot u_2 + \dots + c_M \cdot u_M = \sum_{n=1}^{M} c_n \cdot u_n$$

is also a solution.

Thus, for any $M \in \mathbb{N}$ and any $B_1, \dots, B_M \in \mathbb{R}$,

$$u(x,t) = \sum_{n=1}^{M} B_n \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 \cdot t}$$

is a solution to the heat equation (on $0 \le x \le L$) with zero boundary conditions.

Correspondingly, if the initial condition $f(x)$ is of the form

$$f(x) = \sum_{n=1}^{M} B_n \cdot \sin\left(\frac{n\pi x}{L}\right)$$

for some $M \in \mathbb{N}$ and constants $B_1, \dots, B_n$, then we also already have the (unique) solution to the corresponding IVP.

Q: What to do when the initial condition $f(x)$ is <u>not</u> a finite linear combination of such $\sin\left(\frac{n\pi x}{L}\right)$ functions?

We will soon discuss so-called Fourier series and we will see that "any reasonable" initial condition $f(x)$ can be written as an <u>infinite</u> linear combination of $\sin\left(\frac{n\pi x}{L}\right)$, known as a type of <u>Fourier series</u>

$$f(x) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right)$$

the coefficients $B_n = B_n(f)$ depend on $f(x)$ of course.

Then we expect that the corresponding (unique) solution to the heat equation with $f(x)$ as (IC) is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right) e^{-k\cdot\left(\frac{n\pi}{L}\right)^2 t}$$

with B_n from above

$\rightsquigarrow$ We will have to discuss the convergence of such infinite series and how such an infinite series can be a solution to our heat equation.

But first we find out how to determine the coefficients B_n for a given initial condition $f(x)$:

Orthogonality of Sines

We assume that it is possible to write an initial condition $f(x)$ as

$$(\#1) \quad f(x) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } 0 \leq x \leq L.$$

- To determine the coefficients B_n from $f(x)$, we will use the following important identity:

$$(\#2) \quad \int_0^L \sin\left(\frac{n\pi x}{L}\right) \cdot \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 \ , & \underline{\underline{m \neq n}} \\ \frac{L}{2} \ , & \underline{\underline{m = n}} \end{cases}$$

for any integers m, n

[$\rightsquigarrow$ Deriving (#2) will be part of HW2!]

- Now multiply (#1) by $\sin\left(\frac{m\pi x}{L}\right)$ for any $m \in \mathbb{N}$

$$f(x) \cdot \sin\left(\frac{m\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot \sin\left(\frac{m\pi x}{L}\right)$$

and integrate from $x = 0$ to $x = L$

$$\int_0^L f(x)\cdot \sin\left(\frac{m\pi x}{L}\right) dx = \sum_{n=1}^{\infty} B_n \cdot \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\cdot \sin\left(\frac{m\pi x}{L}\right) dx}_{\overset{\text{by}(\#2)}{=} \begin{cases} 0 & , n \neq m \\ \frac{L}{2} & , n = m \end{cases}}$$

Note: One has to justify, in principle, that integration and summation can be interchanged

the only non-zero term occurs when $n = m$!

$$\Rightarrow \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = B_m \cdot \frac{L}{2}$$

$$\Rightarrow \boxed{B_m = \frac{2}{L}\cdot \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx}$$

Orthogonality

Just like two vectors $\vec{v} = v_1 \hat{\imath} + v_2 \hat{\jmath} + v_3 \vec{k}$ and $\vec{w} = w_1 \hat{\imath} + w_2 \hat{\jmath} + w_3 \vec{k}$ are orthogonal (perpendicular) if

$$\vec{v}\cdot\vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = 0,$$

we can think of two functions $f(x), g(x)$ on an interval $0 \leq x \leq L$ to be orthogonal; we say that $f(x), g(x)$ are <u>orthogonal</u> if

$$\int_0^L f(x) g(x)\, dx = 0.$$

A set of functions each member of which is orthogonal to <u>every other</u> member is called an <u>orthogonal set of functions</u>, for instance the family of sine functions $\left\{ \sin\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty}$.

Brief summary

The method of separation of variables for <u>linear homogeneous</u> PDEs proceeds in two main steps:

1. Hunt for product solutions
   $$u(x,t) = \phi(x) \cdot G(t).$$
   During this step we only use the homogeneous conditions (PDE) as well as (BC), and ignore the initial condition.
2. Superpose the product solutions (form a linear combination or an infinite series of them) and solve for the coefficients to match the initial condition (IC).

→ Read Section 2.3.7 on physics interpretation of the obtained solutions to the heat equation

Worked example: Heat conduction in a rod with insulated ends

Let's determine the solutions to the following heat equation:

(PDE) $\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}$, $0 \le x \le L$, $t > 0$

(BC) $\frac{\partial u}{\partial x}(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = 0$ ⇝ Neumann boundary conditions

(IC) $u(x,0) = f(x)$

⇝ The main difference to the previous problem is that here we have Neumann boundary conditions (ends of the rod are thermally insulated), while before we had Dirichlet boundary conditions (ends of the rod at 0°).

The (PDE) and (BC) are linear and homogeneous, so we can use the method of separation of variables:

We make a product ansatz

$$u(x,t) = \phi(x) \cdot G(t).$$

As before we must then have

$$\frac{dG}{dt} = -\lambda k \cdot G$$

$$\frac{d^2\phi}{dx^2} = -\lambda \cdot \phi$$

for some separation constant $\lambda \in \mathbb{R}$.

Then

$$G(t) = e^{-\lambda k t}$$

and we are left to find all non-trivial solutions to the following Boundary value problem

$$\frac{d^2\phi}{dx^2} = -\lambda \cdot \phi$$

$$\left.\begin{aligned} \frac{d\phi}{dx}(0) &= 0 \\ \frac{d\phi}{dx}(L) &= 0 \end{aligned}\right\} \rightarrow \text{now with Neumann BCs!}$$

Again have to distinguish three cases for the sign of λ:

1^{st} case: $\lambda > 0$

General solution:

$$\phi(x) = c_1 \cdot \cos(\sqrt{\lambda}\, x) + c_2 \cdot \sin(\sqrt{\lambda}\, x), \quad c_1, c_2 \in \mathbb{R}$$

$$\Rightarrow \frac{d\phi}{dx} = \sqrt{\lambda} \cdot \left(-c_1 \cdot \sin(\sqrt{\lambda}\, x) + c_2 \cdot \cos(\sqrt{\lambda}\, x)\right)$$

From $\frac{d\phi}{dx}(0) = 0$ we infer $c_2 = 0$

and then $\frac{d\phi}{dx}(L) = 0$ leads to

$$0 = -\sqrt{\lambda} \cdot c_1 \cdot \sin(\sqrt{\lambda} \cdot L)$$

As before we conclude that

$$\sqrt{\lambda} \cdot L = n \cdot \pi, \qquad n = 1, 2, 3, \ldots$$

Hence, the eigenvalues are

$$\lambda_n = \left(\frac{n \cdot \pi}{L}\right)^2 \quad , \qquad n = 1, 2, 3, \dots$$

with associated eigenfunctions

$$\phi_n(x) = c_1 \cdot \cos\left(\frac{n\pi x}{L}\right) \quad , \quad n = 1, 2, 3, \dots$$

which are now cosines !

The resulting product solutions are

$$u(x,t) = A \cdot \cos\left(\frac{n\pi x}{L}\right) \cdot e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 t} \quad , n = 1, 2, 3, \dots$$

where $A \in \mathbb{R}$ is arbitrary.

2nd case: $\lambda = 0$

If $\lambda = 0$, then $\frac{d^2\phi}{dx^2} = 0$ has the general solution

$$\phi(x) = c_1 + c_2 \cdot x . \qquad \Rightarrow \quad \frac{d\phi}{dx}(x) = c_2 .$$

Then $\frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(L) = 0$ implies $c_2 = 0$.

Thus, for any constant $c_1 \in \mathbb{R}$, $\phi(x) = c_1$ is a solution to the boundary value problem.

Correspondingly, since $e^{-\lambda k t} = 1$ for $\lambda = 0$, in this case we obtain from the product solution ansatz that

$$u(x,t) = A_0$$

for any constant $A_0 \in \mathbb{R}$ is also a solution to (PDE) satisfying the (BC).

3rd case: $\lambda < 0$

⤳ You will show on HW 3 (Problem 2.4.4) that for $\lambda < 0$ the boundary value problem has no non-trivial solutions.

By the superposition principle we find that

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cdot \cos\left(\frac{n\pi x}{L}\right) e^{-k\cdot\left(\frac{n\pi}{L}\right)^2 t}$$

(*)

$$= \sum_{n=0}^{\infty} A_n \cdot \cos\left(\frac{n\pi x}{L}\right) e^{-k\cdot\left(\frac{n\pi}{L}\right)^2 t}$$

↖ note that $\cos\left(\frac{n\pi x}{L}\right) = 1$ and $e^{-k\left(\frac{n\pi}{L}\right)^2 t} = 1$ for $n=0$

is a solution to (PDE) satisfying (BC).

Correspondingly, by evaluating at $t=0$, it follows that we can solve the IVP for any initial condition $f(x)$ that can be written as

$$(**)\ f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cdot \cos\left(\frac{n\pi x}{L}\right), \quad 0 \leq x \leq L.$$

⤳ We will see that "any reasonable" initial condition $f(x)$ can be written as such a cosine series.

To determine the coefficients A_n for a given initial condition $f(x)$, we use the following <u>orthogonality relation for cosines</u>

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cdot\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & , \quad n \neq m \\ \frac{L}{2} & , \quad n = m \neq 0 \\ L & , \quad n = m = 0 \end{cases}$$

⤳ Check this! See also Problem 2.3.6.

As before, now multiply (**) by $\cos\left(\frac{m\pi x}{L}\right)$ and integrate $\int_0^L \ldots dx$ to get

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = \sum_{n=0}^{\infty} A_n \cdot \underbrace{\int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx}_{= \begin{cases} 0 & , n \neq m \\ \frac{L}{2} & , n=m\neq 0 \\ L & , n=m=0 \end{cases}}$$

By the orthogonality relations for cosines it follows that

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx,$$

$$A_m = \frac{2}{L}\int_0^L f(x)\cdot\cos\left(\frac{m\pi x}{L}\right)dx.$$

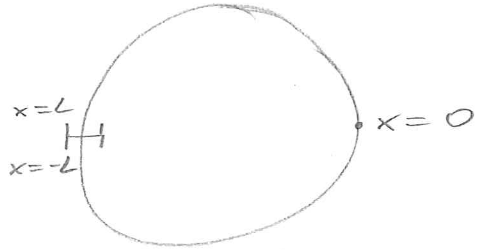
<u>Observe:</u> In (*) all terms with $n \geq 1$ are exponentially decaying as $t \to \infty$ due to the $e^{-k\cdot\left(\frac{n\pi}{L}\right)^2 t}$ factor. Thus,

$$\lim_{t\to\infty} u(x,t) = A_0 = \frac{1}{L}\int_0^L f(x)\,dx,$$

which is the average of the initial temperature distribution! Recall that we expected this to happen from physical considerations!

Worked example: Heat conduction in a thin circular ring

We consider the heat conduction in a thin wire (with lateral sides insulated) that is bent into the shape of a circle and we assume perfect thermal contact at its connected ends



⤳ For "technical convenience" it is helpful here to say that the wire has length 2L.

(PDE) $\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}$, $-L \leq x \leq L$, $t > 0$

(BC) $u(-L,t) = u(L,t)$
$\frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$ } We assume perfect thermal contact at the connected ends, thus, temperature and heat flux should be continuous there.

(IC) $u(x,0) = f(x)$

Here we speak of mixed boundary conditions or periodic boundary conditions. Note that they are linear and homogeneous!

We use the method of separation of variables and make the product ansatz

$$u(x,t) = \phi(x) \cdot G(t).$$

As before we obtain the solutions $G(t) = c \cdot e^{-\lambda k t}$, $\lambda \in \mathbb{R}$, for the time-dependent ODE, and the following boundary value problem

$$\begin{cases} \dfrac{d^2\phi}{dx^2} = -\lambda \cdot \phi \quad , \quad -L \leq x \leq L \\ \phi(-L) = \phi(L) \\ \dfrac{d\phi}{dx}(-L) = \dfrac{d\phi}{dx}(L) \end{cases}$$

<u>1st case:</u> $\lambda > 0$

General solution:

$$\phi(x) = c_1 \cdot \cos(\sqrt{\lambda}\, x) + c_2 \cdot \sin(\sqrt{\lambda} \cdot x) \, , \quad c_1, c_2 \in \mathbb{R}.$$

Thus, the BC $\phi(-L) = \phi(L)$ implies

$$c_1 \cdot \cos(\sqrt{\lambda} \cdot (-L)) + c_2 \cdot \sin(\sqrt{\lambda} \cdot (-L))$$

$$= c_1 \cdot \cos(\sqrt{\lambda} \cdot L) + c_2 \cdot \sin(\sqrt{\lambda} \cdot L).$$

Now cosine is even, i.e. $\cos(\sqrt{\lambda} \cdot (-L)) = \cos(\sqrt{\lambda} \cdot L)$, while sine is odd, i.e. $\sin(\sqrt{\lambda} \cdot (-L)) = -\sin(\sqrt{\lambda} \cdot L)$

$$\Longrightarrow \quad 2\, \underline{\underline{c_2}} \cdot \sin(\sqrt{\lambda} \cdot L) = 0$$

From

$$\frac{d\phi}{dx} = \sqrt{\lambda} \cdot \left(-c_1 \cdot \sin(\sqrt{\lambda}\, x) + c_2 \cdot \cos(\sqrt{\lambda}\, x) \right)$$

and the other BC $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$, again using that cosine is even, while sine is odd, we obtain

$$2\, \underline{\underline{c_1}} \cdot \sin(\sqrt{\lambda} \cdot L) = 0$$

Thus, if $\sin(\sqrt{\lambda}\cdot L) \neq 0$, then we must have $c_1 = c_2 = 0$, which is just the trivial solution.

For non-trivial solutions we therefore need

$$\sin(\sqrt{\lambda}\cdot L) = 0,$$

which again leads to the eigenvalues

$$\lambda = \left(\frac{n\cdot\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$

but now with corresponding eigenfunctions

$$\cos\left(\frac{n\pi x}{L}\right) \quad \underline{\text{and}} \quad \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots$$

⇝ We did not conclude that we must have $c_1 = 0$ or $c_2 = 0$

⇝ It was convenient to choose the length $2L$ for the wire to get the same eigenvalues here as in the previous examples

<u>2nd case:</u> $\lambda = 0$

General solution to $\frac{d^2\phi}{dx^2} = 0$:

$$\phi(x) = c_1 + c_2\cdot x, \quad c_1, c_2 \in \mathbb{R}$$

Then the BC $\phi(-L) = \phi(L)$ implies

$$c_1 - c_2\cdot L = c_1 + c_2\cdot L$$

$$\Rightarrow 2c_2\cdot L = 0 \Rightarrow c_2 = 0$$

$$\Rightarrow \phi(x) = c_1 \quad \text{and} \quad \frac{d\phi}{dx} = 0$$ which certainly satisfies the other BC $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$

Hence, any constant function

$$\phi(x) = c_1 , \quad c_1 \in \mathbb{R}$$

is an eigenfunction for the eigenvalue $\lambda = 0$.

$\underline{3^{rd} \text{ case}:}$ $\lambda < 0$

$\rightsquigarrow$ Exercise: Show that only the trivial solution $\phi(x) = 0$ is possible here.

Superposing all of the above product solutions, we obtain the following solution to the IVP

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) \cdot e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 t}$$
$$+ \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot e^{-k \left(\frac{n\pi}{L}\right)^2 t}$$

We will see that any "reasonable" initial condition $f(x)$, $-L \leq x \leq L$, satisfying periodic boundary conditions, can be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right), \quad -L \leq x \leq L.$$

To determine the coefficients a_0, a_n, b_n from $f(x)$ we proceed as in the previous examples using the following orthogonality properties for any non-negative integers n, m

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cdot \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \\ 2L, & n = m = 0 \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cdot \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & n \neq m \\ L, & n = m = 0 \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0$$

We obtain that (see page 64 in textbook for details)

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

$$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \cos\left(\frac{m\pi x}{L}\right) dx$$

$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

Summary:

In the previous three worked examples we have seen that the eigenvalues and eigenfunctions for the x-dependent ODE

$$\frac{d^2\phi}{dx^2} = -\lambda \cdot \phi$$

crucially depend on the types of boundary conditions.

(See also table 2.4.1 on page 65)

Laplace's equation: Solutions and qualitative properties

Physical significance:

- arises in electrostatics, gravitation, fluid dynamics,...
- it is the steady-state heat equation:

The heat equation for a 2-dimensional domain is

$$\frac{\partial u}{\partial t} = k \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

where $u = u(x,y,t)$ is the temperature function. Then steady-state solutions $u(x,y,t) = \underline{\underline{u(x,y)}}$ must satisfy

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(Laplace equation 2D)}$$

Notation:

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Laplace's equation inside a rectangle

We consider Laplace's equation in a rectangle $(0 \le x \le L,\ 0 \le y \le H)$ when the temperature is a prescribed function of position (independent of time) on the boundary:

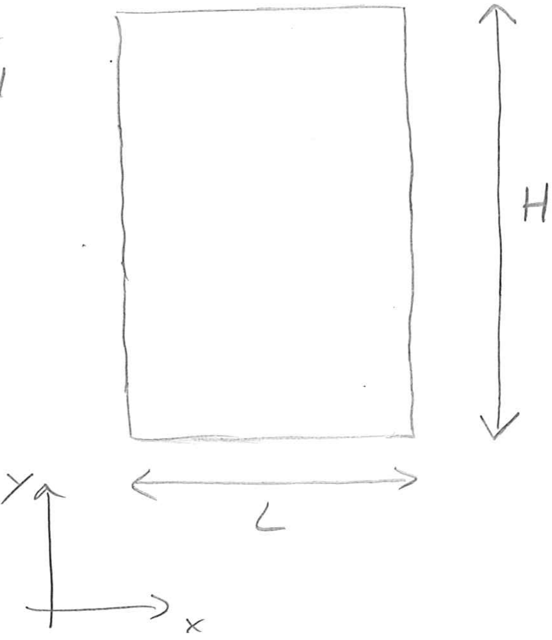
$$\text{(PDE)} \quad \underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}_{\Delta u} = 0, \quad \begin{matrix} 0 \le x \le L \\ 0 \le y \le H \end{matrix}$$

$$\text{(BC1)} \quad u(0,y) = g_1(y)$$

$$\text{(BC2)} \quad u(L,y) = g_2(y)$$

$$\text{(BC3)} \quad u(x,0) = f_1(x)$$

$$\text{(BC4)} \quad u(x,H) = f_2(x)$$



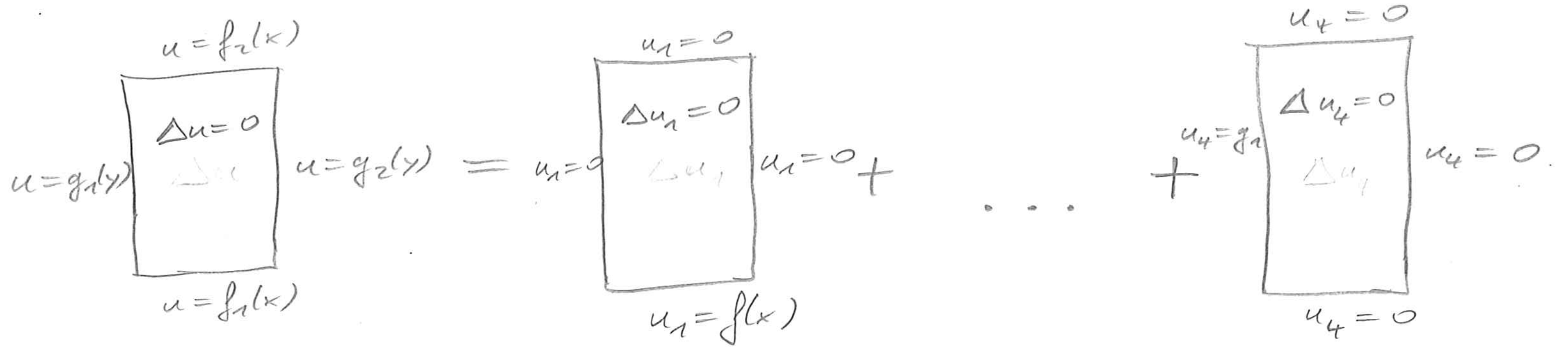
<u>Note:</u> The PDE is linear and homogeneous; the BCs are linear but <u>not</u> homogeneous!

Thus, we <u>cannot</u> apply the method of separation of variables directly (because when we separate variables, the boundary value problem determining the separation constant must have homogeneous boundary conditions)!

<u>Trick:</u> Use the superposition principle to break the problem into four problems, each having <u>only one</u> nonhomogeneous condition

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y),$$

where each $u_j(x,y)$, $j = 1,2,3,4$ satisfies:



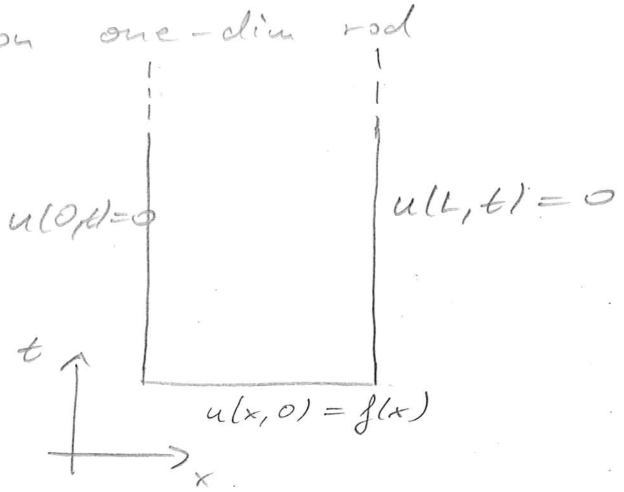
Now we can try to solve for each u_j separately using the method of separation of variables, where we will have to match only one nonhomogeneous BC.

Analogy to heat equation on one-dim rod

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L$$

(BC) $u(0,t) = 0$
$u(L,t) = 0$

(IC) $u(x,0) = f(x)$



Underlying principle:

Work with only one nonhomogeneous condition at a time, so that you can exploit the superposition principle correctly.

We treat the case of $u_4(x,y)$ in detail, all other cases work similarly:

$$(\text{PDE}) \qquad \frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0 \qquad \begin{array}{l} 0 \le x \le L \\ 0 \le y \le H \end{array}$$

$$(\text{BCs}) \qquad \begin{aligned} u_4(0,y) &= g_1(y) \\ u_4(L,y) &= 0 \\ u_4(x,0) &= 0 \\ u_4(x,H) &= 0 \end{aligned}$$

First ignore the nonhomogeneous BC $u_4(0,y) = g_1(y)$, and make product ansatz

$$u_4(x,y) = h(x) \cdot \phi(y).$$

The three homogeneous BCs lead to

$$h(L) = 0$$

$$\phi(0) = 0$$

$$\phi(H) = 0.$$

Inserting the ansatz into our (PDE):

$$\frac{d^2 h}{dx^2} \cdot \phi(y) + h(x) \cdot \frac{d^2 \phi}{dy^2} = 0$$

$$\Rightarrow \quad \underbrace{\frac{1}{h(x)} \frac{d^2 h}{dx^2}}_{\substack{\text{depends} \\ \text{only on } x}} = \underbrace{-\frac{1}{\phi} \frac{d^2 \phi}{dy^2}}_{\substack{\text{depends} \\ \text{only on } y}} = \lambda$$

for some separation constant $\lambda \in \mathbb{R}$.

We obtain two ODEs:

$$\frac{d^2h}{dx^2} = \lambda \cdot h \quad , \quad 0 \leq x \leq L$$

$$\frac{d^2\phi}{dy^2} = -\lambda \cdot \phi \quad , \quad 0 \leq y \leq H$$

Note: We have <u>two</u> BCs for ϕ, namely $\phi(0) = 0$ and $\phi(H) = y$. This gives a boundary value problem for ϕ, which will allow us to determine the eigenvalues λ!

Let's first solve the BVP for ϕ:

$$\frac{d^2\phi}{dy^2} = -\lambda \cdot \phi \quad , \quad 0 \leq y \leq H$$

$$\phi(0) = 0$$

$$\phi(H) = 0$$

We have already computed that the eigenvalues and eigenfunctions for this BVP are

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2 \quad , \quad n = 1, 2, \dots$$

$$\phi_n = \sin\left(\frac{n\pi y}{H}\right) \quad , \quad n = 1, 2, \dots$$

For these values of λ we now determine the solution(s) to the ODE for $h(x)$

$$\frac{d^2h}{dx^2} = \left(\frac{n\pi}{H}\right)^2 \cdot h \quad , \quad 0 \leq x \leq L,$$

satisfying

$$h(L) = 0$$

As fundamental systems of solutions for this ODE we can work with $\left\{ e^{+\left(\frac{n\pi}{H}\right)\cdot x}, e^{-\left(\frac{n\pi}{H}\right)\cdot x} \right\}$, $\left\{ \cosh\left(\frac{n\pi}{H}\cdot x\right), \sinh\left(\frac{n\pi}{H}\cdot x\right) \right\}$, but also (by translation invariance) $\left\{ \cosh\left(\frac{n\pi}{H}(x-L)\right), \sinh\left(\frac{n\pi}{H}(x-L)\right) \right\}$.

Any of these works, but the latter makes things particularly neat. As the general solution we then have

$$h(x) = a_1 \cdot \cosh\left(\frac{n\pi}{H}(x-L)\right) + a_2 \cdot \sinh\left(\frac{n\pi}{H}(x-L)\right).$$

To satisfy $h(L) = 0$, we must have $a_2 = 0$ (because $\cosh(0) = 1$ and $\sinh(0) = 0$).

Putting things together we have found the product solutions

$$u_4(x,y) = A \cdot \sinh\left(\frac{n\pi}{H}\cdot(x-L)\right) \cdot \sin\left(\frac{n\pi y}{H}\right), \quad n = 1, 2, \dots$$

Now we try to combine these to also satisfy the nonhomogeneous BC!

By the superposition principle, any infinite linear combination

$$(*) \quad u_4(x,y) = \sum_{n=1}^{\infty} A_n \cdot \sinh\left(\tfrac{n\pi}{H}(x-L)\right) \cdot \sin\left(\tfrac{n\pi y}{H}\right) \quad , \quad A_n \in \mathbb{R},$$

solves $\Delta u_4 = 0$ and satisfies the three homogeneous BCs.

Correspondingly, let's require at $x=0$ that

$$g_1(y) = u_4(0,y) = \sum_{n=1}^{\infty} \underbrace{\left(A_n \cdot \sinh\left(\tfrac{n\pi}{H}\cdot(-L)\right)\right)}_{\text{constant coefficient}} \cdot \sin\left(\tfrac{n\pi y}{H}\right)$$

This is a (Fourier) sine series! By orthogonality of the sines $\left\{\sin\left(\tfrac{n\pi y}{H}\right)\right\}_{n=1}^{\infty}$ we have

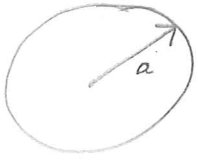
$$A_n \cdot \sinh\left(\tfrac{n\pi}{H}(-L)\right) = \frac{2}{H} \cdot \int_0^L g_1(y) \cdot \sin\left(\tfrac{n\pi y}{H}\right) dy$$

$$(**) \implies A_n = \frac{2}{H \cdot \sinh\left(\tfrac{n\pi}{H}(-L)\right)} \int_0^L g(y) \cdot \sin\left(\tfrac{n\pi y}{H}\right) dy$$

Thus, $(*)$ with $(**)$ is the desired solution to the problem for $u_4(x,y)$.

Laplace's equation for a circular disk

Consider Laplace's equation on a circular disk of radius a with prescribed temperature on the boundary of the disk:



Due to the geometry of the problem use polar coordinates (r, θ):

$$u = u(r,\theta) \quad , \quad \begin{array}{c} 0 \le r \le a \\ -\pi \le \theta \le \pi \end{array}$$

(PDE) $$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial u}{\partial r}\right) + \frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2} = 0$$

Laplace operator Δ in polar coordinates

(BC) $$u(a,\theta) = f(\theta)$$

The use of polar coordinates requires some additional compatibility conditions:

- polar coordinates are singular at $r=0$; for physical reasons require

$$|u(0,\theta)| < \infty$$

- periodicity condition: $\theta = -\pi$ and $\theta = \pi$ correspond to the same point!

$$u(r,-\pi) = u(r,\pi)$$

$$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi)$$

Note: The compatibility conditions are all linear and homogeneous.

We try to use the method of separation of variables and make the product ansatz:

$$u(r,\theta) = \phi(\theta)\cdot G(r),$$

where we need to satisfy the three compatibility conditions, but ignore the nonhomogeneous BC $u(a,\theta) = f(\theta)$ for the moment:

$$\phi(-\pi) = \phi(\pi)$$

$$\frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi)$$

Inserting the product ansatz into the PDE

$$\frac{1}{r}\frac{d}{dr}\left(r\cdot\frac{dG}{dr}\right)\cdot\phi + \frac{1}{r^2}G(r)\cdot\frac{d^2\phi}{d\theta^2} = 0$$

Multiply by r^2 and divide by $G(r)\cdot\phi(\theta)$ to get

$$\frac{r}{G}\cdot\frac{d}{dr}\left(r\cdot\frac{dG}{dr}\right) + \frac{1}{\phi}\cdot\frac{d^2\phi}{d\theta^2} = 0$$

$$\Rightarrow \underbrace{\frac{r}{G}\cdot\frac{d}{dr}\left(r\cdot\frac{dG}{dr}\right)}_{\text{only } r\text{-dependent}} = \underbrace{-\frac{1}{\phi}\cdot\frac{d^2\phi}{d\theta^2}}_{\text{only } \theta\text{-dependent}} = \lambda$$

↑ separation constant $\lambda\in\mathbb{R}$

We first solve the EVP for ϕ:

$$\frac{d^2\phi}{d\theta^2} = -\lambda \cdot \phi$$

$$\phi(-\pi) = \phi(\pi)$$

$$\frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi)$$

$\Rightarrow$ By analogy to the problem for the circular wire with $\underline{L = \pi}$:

Eigenvalues : $\lambda = \left(\frac{n\pi}{L}\right)^2 = n^2$, $n = \underline{\underline{0}}, 1, 2, \ldots$

Eigenfunctions : $\sin(n\theta)$, $\cos(n\theta)$

Now we turn to the ODE for $G(r)$:

$$\frac{r}{G} \cdot \underbrace{\frac{d}{dr}\left(r \cdot \frac{dG}{dr}\right)}_{= r \cdot \frac{d^2G}{dr^2} + \frac{dG}{dr}} = \lambda = n^2$$

$$\Rightarrow r^2 \cdot \frac{d^2G}{dr^2} + r \cdot \frac{dG}{dr} - n^2 \cdot G = 0$$

From $|u(0,\theta)| < \infty$ we inherit the requirement

$|G(0)| < \infty$.

This is a 2^{nd} order linear ODE with variable coefficients, which are in general very difficult to solve explicitly.

Here we are lucky though, because the linear operator $r^2 \frac{d^2}{dr^2} + r \cdot \frac{d}{dr} - n^2 \cdot$ has the property that any power $G(r) = r^n$ "reproduces" itself!

Substitute $G(r) = r^{\mu}$ to get

$$\mu\cdot(\mu-1)\cdot r^{\mu} + \mu\cdot r^{\mu} - n^2\cdot r^{\mu} = 0$$

$$\Rightarrow \quad (\mu(\mu-1) + \mu - n^2)\cdot r^{\mu} = 0$$

$$\Rightarrow \quad (\mu^2 - n^2)\cdot r^{\mu} = 0$$

$$\Rightarrow \quad \mu = \pm n$$

Thus, for $n \neq 0$ the general solution is

$$G(r) = c_1\cdot r^{n} + c_2\cdot r^{-n}, \qquad c_1, c_2 \in \mathbb{R}.$$

For $n = 0$, one solution is $r^0 = 1$ (constant sol.).
To find another independent solution, note that for $n = 0$

$$\frac{d}{dr}\left(r\cdot\frac{dG}{dr}\right) = 0$$

$$\Rightarrow \quad \frac{dG}{dr} = \frac{\text{const.}}{r}$$

$$\Rightarrow \quad G(r) = c\cdot \ln(r)$$

Hence, for $n = 0$ the general solution is

$$G(r) = \bar{c}_1 + \bar{c}_2\cdot \ln(r), \qquad c_1, c_2 \in \mathbb{R}.$$

The condition $|G(0)| < \infty$ rules out the $\ln(r)$ solution as well as the r^{-n} solution for $n \geq 1$. Thus,

$$G(r) = c\cdot r^{n}, \qquad n \geq 0.$$

Using the superposition principle we may put together all product solutions to arrive at the solution(s)

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n \cdot r^n \cdot \cos(n\theta) + \sum_{n=1}^{\infty} B_n \cdot r^n \cdot \sin(n\theta),$$

$$0 \leq r \leq a$$
$$-\pi \leq \theta \leq \pi$$

To satisfy the nonhomogeneous BC $u(a,\theta) = f(\theta)$ we must have

$$f(\theta) = \sum_{n=0}^{\infty} \underbrace{A_n \cdot a^n}_{\text{modified coefficient}} \cdot \cos(n\theta) + \sum_{n=1}^{\infty} \underbrace{B_n \cdot a^n}_{\text{modified coefficient}} \cdot \sin(n\theta) .$$

Using the orthogonality formulas for the family of $\cos(n\theta)$, $\sin(n\theta)$ (as in the treatment of the circular wire with $\underline{\underline{L=\pi}}$) we conclude that

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta ,$$

$$A_n \cdot a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta , \quad n \geq 1,$$

$$B_n \cdot a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta , \quad n \geq 1.$$

Qualitative properties of Laplace's equation

[Honors Material]

Mean-value theorem

If we evaluate the solution to Laplace's equation on a circular disk at the origin $r = 0$, we find that

$$u(0, \theta) = A_0 = \frac{1}{2\pi} \cdot \int_{-\pi}^{+\pi} f(\theta) \, d\theta$$

Thus, the temperature at the origin is the average value of the temperature on the boundary of the circle!

This mean-value property for Laplace's equation holds more generally:

Theorem:

Let $R \subset \mathbb{R}^2$ be a region and let u be a solution to $\Delta u = 0$ in R.

For any $x_0 \in R$ and $r > 0$ such that

$$\mathbb{B}(x_0, r) := \{ y \in \mathbb{R}^2 \mid |y - x_0| < r \} \subset R,$$

it holds that

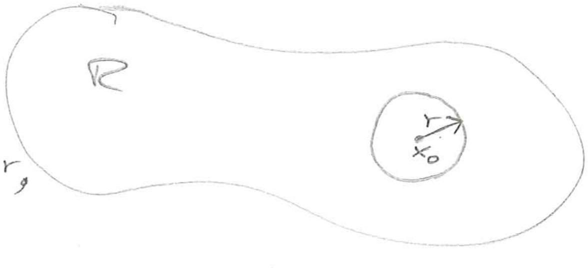
$$u(x_0) = \frac{1}{2\pi r} \underbrace{\int_{\partial \mathbb{B}(x_0,r)}}_{\text{boundary of disk } \mathbb{B}(x_0,r)} u(y) \, \underbrace{dS(y)}_{\text{surface measure on boundary } \partial\mathbb{B}(x_0,r)}$$

$$\partial\mathbb{B}(x_0, r) = \{ y \in \mathbb{R}^2 \mid |y - x_0| = r \}.$$

Proof:

⤳ We could just use our previous analysis for the disk $\mathbb{B}(x_0,r)$ via explicit computation, but we can also argue indirectly as follows:

Define



$$\phi(s) := \frac{1}{2\pi s}\int_{\partial\mathbb{B}(x_0,\underline{\underline{s}})} u(y)\,dS(y)\,,\quad 0<s\le r,$$

We first show that $\phi(s)$ is constant.
Once we know that, we just observe that then

$$\frac{1}{2\pi r}\int_{\partial\mathbb{B}(x_0,r)} u(y)\,dS(y) = \phi(r) \underset{\substack{\uparrow \\ \phi \text{ constant}}}{=} \lim_{s\to 0}\phi(s) = u(x_0)$$

and we are done!

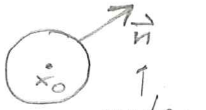
To conclude that $\phi(s)$ is constant, we compute

$$\phi'(s) = \frac{d\phi}{ds} = \frac{d}{ds}\left(\frac{1}{2\pi \cancel{s}}\int_{\partial\mathbb{B}(x_0,\underline{\underline{1}})} u(x_0+sz)\,\cancel{s}\,dS(z)\right)$$

surface measure on $\partial\mathbb{B}(x_0,\underline{\underline{1}})$

$$= \frac{1}{2\pi}\int_{\partial\mathbb{B}(x_0,\underline{\underline{1}})} (\nabla u)(x_0+sz)\cdot z\,dS(z)$$

$$= \frac{1}{2\pi\underline{\underline{s}}}\int_{\partial\mathbb{B}(x_0,s)} (\nabla u)(y)\cdot\vec{n}\,dS(y)$$



outer normal vector

divergence theorem

$$= \frac{1}{2\pi s}\int_{\mathbb{B}(x_0,s)} \underbrace{\nabla\cdot(\nabla u)(y)}_{=(\Delta u)(y)=0}\,dy$$

u solves Laplace's equation!

$$= \underline{\underline{0}}$$

Maximum principle

A solution to Laplace's equation in a region $R \subset \mathbb{R}^2$ cannot attain its maximum in the interior of the region R (unless the temperature is constant everywhere).

Proof: (by contradiction)

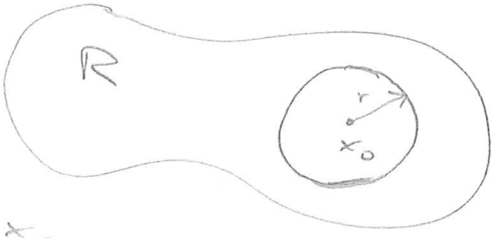
Let $R \subset \mathbb{R}^2$ be a region and let u be a solution to Laplace's equation

$$\Delta u = 0 \quad \text{in } R.$$

Let $x_0 \in R$ be a point in the interior of R and assume that

$$u(x_0) = M := \max_{x \in R} u(x),$$

i.e. u assumes its maximum at x_0.



Now for any

$0 < r < \text{dist}(x_0, \partial R)$ ↖ distance of x_0 to the boundary ∂R

we have by the mean-value property

$$M = u(x_0) = \frac{1}{2\pi r} \int_{\partial B(x_0,r)} \underbrace{u(y)}_{\le M\,!} \, dS(y) \le M$$

We arrive at a contradiction; thus the statement must be true. ⇐ not possible unless u is constant everywhere!

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Analogously, we can show that the minimum cannot be attained in the interior of R. (Minimum principle)

Thus, a solution to Laplace's equation $\Delta u = 0$ in R must assume its maximum and its minimum on the boundary of the region R.

Well-posedness and uniqueness

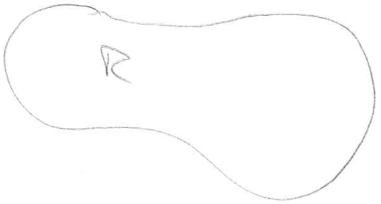
Definition:

We say that a PDE is well-posed if there exists a unique solution that depends continuously on the (nonhomogeneous) data, i.e. the solution only varies a small amount if the data are slightly changed.

⇝ This is an important property for a PDE to be relevant for problems in physics because in nature things usually only change by small degrees.

We can use the maximum and minimum principles to show that Laplace's equation is well-posed:

Let $R \subset \mathbb{R}^2$ be a region and let $f(x)$ and $g(x)$ be boundary data (for the boundary of R).



Let $\Delta u = 0$ in R with
$u(x) = f(x)$ on the boundary of R
and let $\Delta v = 0$ in R with $v(x) = g(x)$ on the boundary of R.

$\Longrightarrow$ $\Delta(u-v) = 0$ in R with
$(u-v)(x) = f(x) - g(x)$ on the boundary of R

$\Longrightarrow$ By maximum / minimum principle:

$$\underset{\underline{\underline{\text{bdry}}}}{\min}(u-v) = \min(f-g) \leq u - v \leq \max(f-g) = \underset{\underline{\underline{\text{bdry}}}}{\max}(u-v)$$

Hence, if $f(x)$ and $g(x)$ differ only a little bit, then the corresponding solutions u and v to Laplace's equation also only differ a little bit!

Moreover, we can prove that the solution to Laplace's equation is unique:

Proof: (by contradiction)

Suppose there are two solutions u and v ($\Delta u = 0$ and $\Delta v = 0$) with the same boundary data ($u = f(x)$ and $v(x) = f(x)$ on the boundary).

By the maximum/minimum principle

$$0 = \underbrace{\min(f-g)}_{\substack{\min(u-v)\\ \text{boundary}}} \leq u - v \leq \underbrace{\max(f-f)}_{\substack{\max(u-v)\\ \text{boundary}}} = 0$$

$$\Rightarrow \quad u - v = 0 \quad \Rightarrow \quad \underline{\underline{u = v}}$$

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