Chapter 3: Fourier Series

As we developed the method of separation of variables in the previous chapter, we relied on the fact that "any reasonable" initial condition can be written as an infinite series of sines or cosines or both, known as a <u>Fourier series</u>:

Given a function $f(x)$ on an interval $-L \leq x \leq +L$, can we write

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right), \quad -L \leq x \leq +L$$

for suitable coefficients a_n, b_n?

In which sense does such an infinite series converge to $f(x)$? What conditions on $f(x)$ are needed?

We try to answer these questions in this chapter. To this end we first need to introduce some definitions.

Definition:

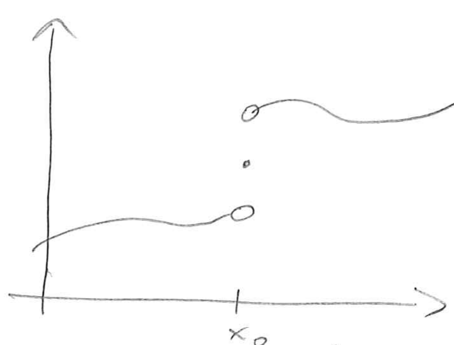
Let $[c,d]$ be an interval. A function $f:[c,d]\to\mathbb{R}$ is piecewise smooth on $[c,d]$ if the interval $[c,d]$ can be partitioned into finitely many consecutive subintervals such that inside each subinterval $f(x)$ and its derivative $f'(x)$ are continuous.

Moreover, $f(x)$ is only allowed to have jump discontinuities at the intersections of the subintervals.

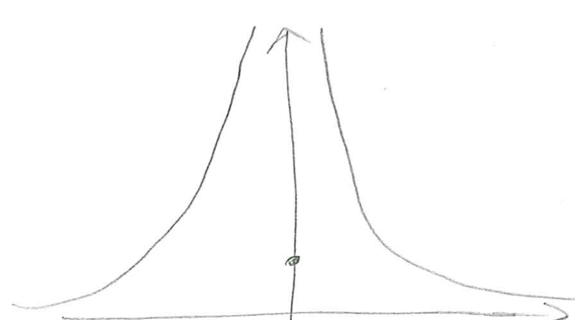
→ For us a "reasonable" function is piecewise smooth.

Recall:

A function $f(x)$ has a jump discontinuity at a point $x=x_0$ if the one-sided limits $f(x_0-)=\lim\limits_{x\uparrow x_0} f(x)$ and $f(x_0+)=\lim\limits_{x\downarrow x_0} f(x)$ exist and are unequal.

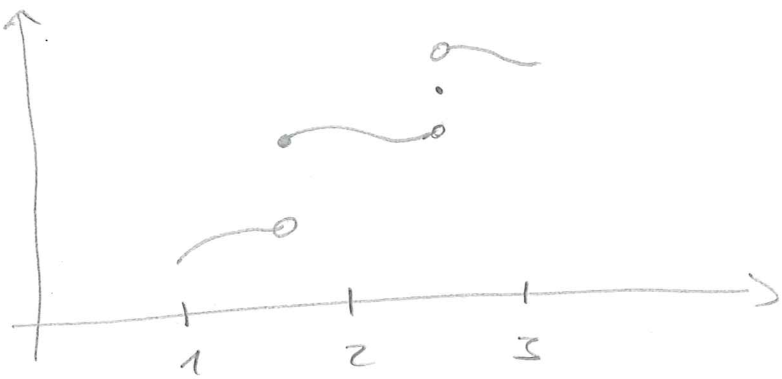


Example of a function with a jump discontinuity at $x=x_0$



The function $f(x)=\begin{cases}\frac{1}{x^2}, & x\neq 0\\ 1, & x=0\end{cases}$ does not have a jump discontinuity at $x_0=0$

Example of a piecewise smooth function on $[1,3]$:



Definition:

A function $f: \mathbb{R} \to \mathbb{R}$ (defined on the whole line) is periodic with period P if

$$f(x+P) = f(x) \qquad \text{for all } x \in \mathbb{R}.$$

Example:

The functions $\cos\left(\frac{n\pi x}{L}\right)$, $n = 0, 1, 2, \ldots$, and $\sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, 3, \ldots$, are periodic with period $2L$.

[Hence, the series

$$a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right)$$

is periodic with period $2L$]

Given a function $f(x)$ on an interval $-L \leq x \leq +L$, we introduce the <u>periodic extension</u> $f_{per}(x)$ of $f(x)$ to the whole line by

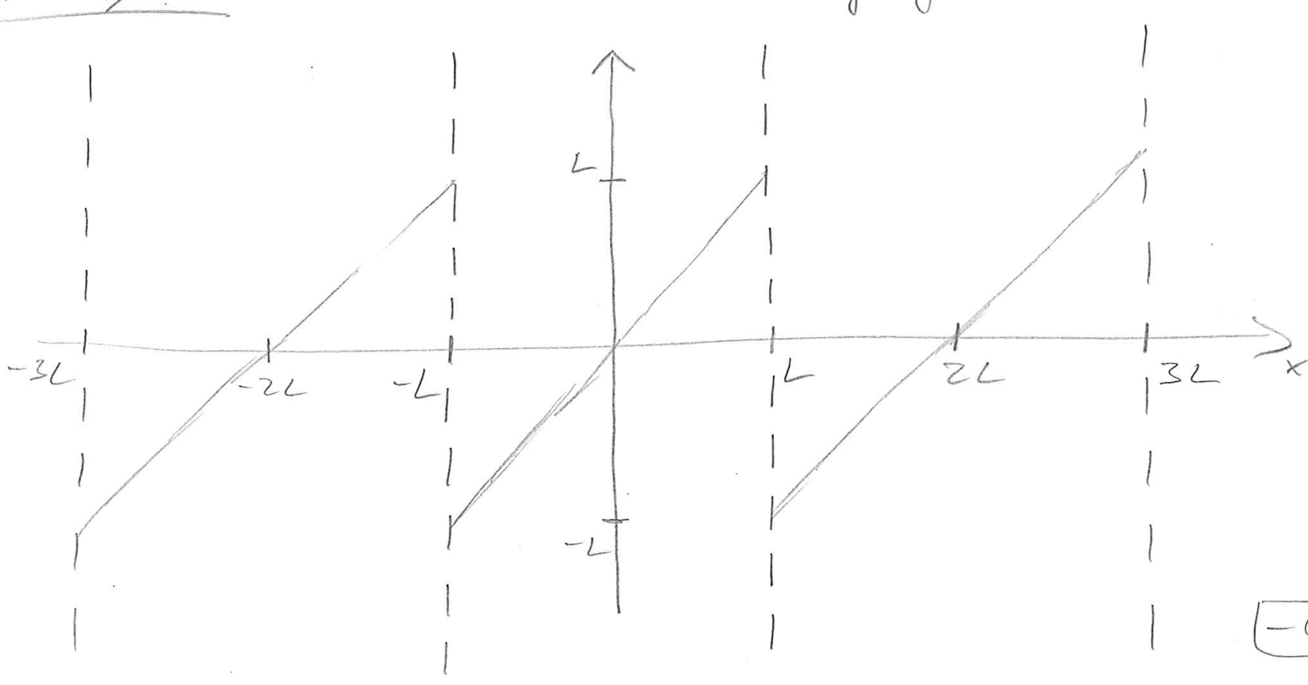
$$f_{per}(x \pm m \cdot 2L) := f(x) \qquad \text{for } -L < x \leq +L$$
$$\text{and any integer } m \in \mathbb{Z}.$$

In other words, in order to sketch the periodic extension of $f(x)$, simply sketch $f(x)$ for $-L < x \leq +L$ and then keep repeating the pattern by translating the original sketch.

<u>Note:</u>

Unless $f(-L) = f(+L)$, the periodic extension of $f(x)$ from the interval $-L \leq x \leq +L$ to the whole line has jump discontinuities at $x = L + m \cdot 2L$, $m \in \mathbb{Z}$.

<u>Example:</u> Periodic extension of $f(x) = x$, $-L \leq x \leq +L$



Definition: (Fourier series)

Given a function $f(x)$ over the interval $-L \leq x \leq +L$, we define its Fourier series $(Sf)(x)$ to be the infinite series

$$(Sf)(x) := a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right),$$

["S" for "sum"]

where the Fourier coefficients a_n, b_n are defined by

$$a_0 := \frac{1}{2L} \cdot \int_{-L}^{+L} f(x)\, dx,$$

$$a_n := \frac{1}{L} \int_{-L}^{+L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx,$$

$$b_n := \frac{1}{L} \int_{-L}^{+L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Note:

- For any piecewise smooth function on $[-L, L]$, the Fourier coefficients are well-defined. (In contrast, for the function $f(x) = \frac{1}{x^2}$, the integral $a_0 := \frac{1}{2L} \int_{-L}^{L} \frac{1}{x^2} dx$ does not exist due to the singularity at $x=0$; but $f(x) = \frac{1}{x^2}$ is not piecewise smooth on any interval $[-L, L]$).
- The Fourier series $(Sf)(x)$ is defined on the whole line, i.e. for any $x \in \mathbb{R}$.

The following fundamental theorem describes in which sense the Fourier series $(Sf)(x)$ of a piecewise smooth function converges and what its relation to the original function $f(x)$ is:

Fourier's theorem:

Let $f: [-L, +L] \to \mathbb{R}$ be piecewise smooth. Then for any $x \in \mathbb{R}$, the Fourier series $(Sf)(x)$ converges (pointwise)

(i) to the periodic extension $f_{per}(x)$ of f, where the periodic extension is continuous;

(ii) to the average of the two one-sided limits

$$\frac{1}{2}\left(f_{per}(x+) + f_{per}(x-)\right)$$

where the periodic extension has a jump discontinuity.

Recall:

$$f_{per}(x_0+) := \lim_{x \downarrow x_0} f(x), \quad f_{per}(x_0-) := \lim_{x \uparrow x_0} f(x).$$

$\rightsquigarrow$ We will not discuss the proof of this theorem, but I'll give you some intuition later.

We conclude that for a piecewise smooth function $f: [-L, L] \to \mathbb{R}$, it holds that

- for $-L < x < +L$ (inside the interval)
$$(Sf)(x) = \frac{1}{2}\left(f(x+) + f(x-)\right)$$

- for $-L < x < +L$ where $f(x)$ is continuous [then $f(x-) = f(x+)$]
$$(Sf)(x) = f(x)$$

- at the endpoints $x = -L$ and $x = +L$
$$(Sf)(+L) = (Sf)(-L) = \frac{1}{2} \cdot \left(f((-L)+) + f((+L)-)\right)$$

- outside the interval $-L \leq x \leq +L$, the values of $(Sf)(x)$ can be determined from the above using that the Fourier series $(Sf)(x)$ is periodic with period $2L$.

Example:

Let
$$f(x) = \begin{cases} 0, & -L \leq x < \frac{L}{2} \\ 1, & \frac{L}{2} \leq x \leq L \end{cases}$$

Determine the Fourier series $(Sf)(x)$ of the function $f(x)$ over the interval $-L \leq x \leq +L$.

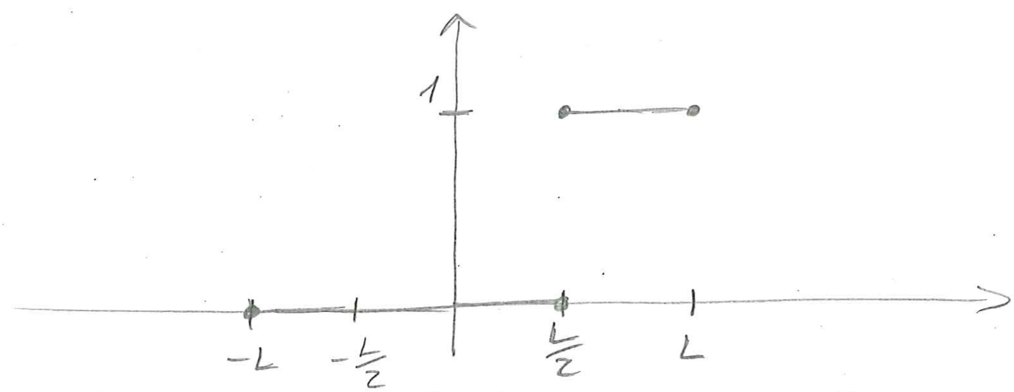
$\leadsto$ Note that $f(x)$ is piecewise smooth, so we can apply Fourier's theorem.

Strategy:

(1) Sketch $f(x)$ over $-L \le x \le +L$

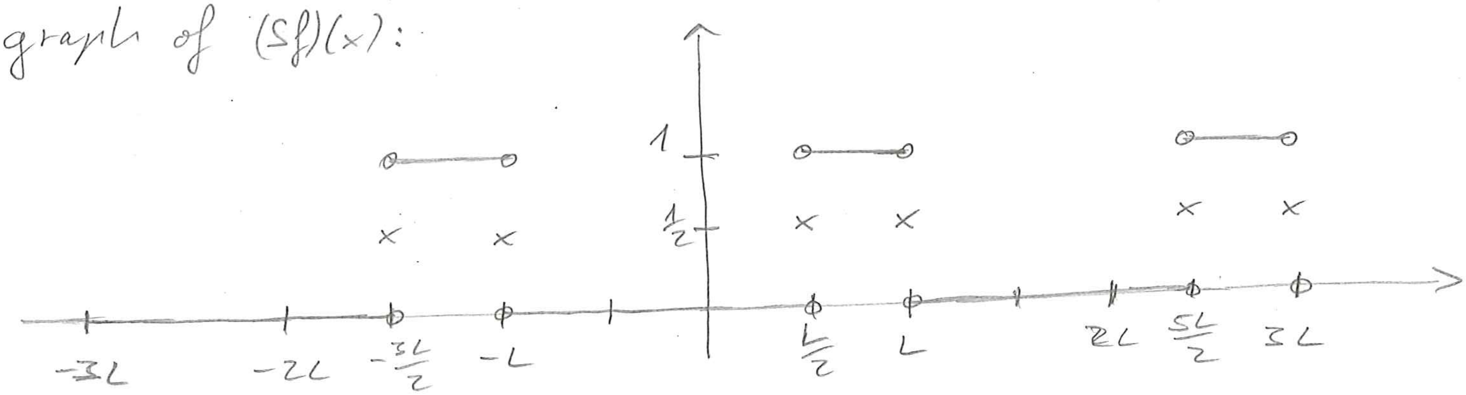
(2) Sketch the periodic extension $f_{per}(x)$ of $f(x)$

(3) Determine the jump discontinuities of the periodic extension $f_{per}(x)$ and determine the average of the one-sided limits of $f_{per}(x)$ there.
[Mark those average values with an x on the graph]



graph of $f(x)$ on $[-L,+L]$

graph of $(Sf)(x)$:



Values of $(Sf)(x)$ on $[-L,+L]$:

$$(Sf)(x) = \begin{cases} \frac{1}{2} & , x = L \\ 1 & , \frac{L}{2} < x < L \\ \frac{1}{2} & , x = \frac{L}{2} \\ 0 & , -L < x < \frac{L}{2} \\ \frac{1}{2} & , x = -L \end{cases}$$

Fourier coefficients

Thanks to Fourier's theorem and the above strategy, we actually do not have to compute the Fourier coefficients a_n, b_n in order to sketch the graph of the Fourier series of a piecewise smooth function!

However, it is still important to know how to calculate Fourier coefficients.

Sometimes this can be a tricky exercise in evaluating integrals (where integration by parts is often helpful).

Example: (continued)

Fourier coefficients of

$$f(x) = \begin{cases} 0, & -L \leq x < \frac{L}{2} \\ 1, & \frac{L}{2} \leq x \leq L \end{cases}$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\,dx = \frac{1}{2L} \int_{\frac{L}{2}}^{L} 1\,dx = \frac{1}{2L} \cdot \frac{L}{2} = \underline{\underline{\frac{1}{4}}}$$

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L} \int_{\frac{L}{2}}^{L} \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L} \left[\frac{L}{n\pi} \cdot \sin\left(\frac{n\pi x}{L}\right) \right]_{x=\frac{L}{2}}^{x=L} \\ &= \frac{1}{n\pi} \Big(\underbrace{\sin(n\pi)}_{=0} - \sin\left(\frac{n\pi}{2}\right) \Big) = \underline{\underline{-\frac{1}{n\pi} \sin\left(\frac{n\pi}{2}\right)}} \end{aligned}$$

By carefully evaluating $\sin\left(\frac{n\pi}{2}\right)$, one could further simplify this to

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & , \quad n = 4m \\ 1 & , \quad n = 4m+1 \\ 0 & , \quad n = 4m+2 \\ -1 & , \quad n = 4m+3 \end{cases} , \quad m \in \mathbb{Z}.$$

Similarly,

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L} \int_{\frac{L}{2}}^{L} \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L} \left[-\frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right) \right]_{x=\frac{L}{2}}^{x=L} \\ &= \frac{1}{n\pi} \left(-\cos(n\pi) + \cos\left(\frac{n\pi}{2}\right) \right). \end{aligned}$$

Interlude: (Honors Material)

The convergence of the Fourier series of a function $f(x)$ can be a delicate matter. Different types of convergence are possible and this all crucially depends on the smoothness of $f(x)$. [pointwise convergence, uniform convergence, L^2 convergence]

Generally speaking, one can say that the smoother $f(x)$, the ~~more~~ decay of its Fourier coefficients and the stronger the convergence of the Fourier series.

To illustrate this, assume that $f(x)$ is periodic and continuously differentiable. Then we find for its Fourier coefficients a_n by integration by parts:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{= \frac{L}{n\pi}\frac{d}{dx}\left(\sin\left(\frac{n\pi x}{L}\right)\right)} dx$$

integration by parts

$$= \frac{1}{L}\cdot\frac{L}{n\pi}\cdot \underbrace{\left[f(x) \underbrace{\sin\left(\frac{n\pi x}{L}\right)}\right]_{x=-L}^{x=L}}_{=0}$$

$$- \frac{1}{L}\cdot\frac{L}{n\pi}\int_{-L}^{L} f'(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$= -\frac{1}{n\pi}\int_{-L}^{L} f'(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

Thus, for all $n \geq 1$ (since $|\sin(\frac{n\pi x}{L})| \leq 1$)

$$|a_n| \leq \frac{1}{n\cdot\pi}\int_{-L}^{L} |f'(x)|\, dx \Rightarrow |a_n| \leq \frac{C(f)}{n}$$

If f was twice continuously differentiable, we could integrate by parts once more and infer that

$$|a_n| \leq \frac{C(f)}{n^2}$$

stronger decay!

Hence, more smoothness for $f(x)$ implies more decay of the Fourier coefficients a_n, which in turn implies stronger convergence of the Fourier series.

Fourier Sine and Cosine Series

We will see in the following that the series of sines only and the series of cosines only are just special cases of a Fourier series.

Fourier Sine Series

Definition:

An odd function $f(x)$ satisfies the equation

$$f(-x) = -f(x).$$

Examples

$\sin(x)$, $\sin(5x)$, x, x^3, ...

Note:

- The sketch of an odd function for $x<0$ is minus the mirror image for $x>0$
- The integral of an odd function over a symmetric interval is zero

$$\int_{-L}^{+L} f(x)\,dx = 0.$$

Fourier series of odd functions

Let $f(x)$, $-L \leq x \leq +L$, be odd. Then

$$a_0 = \frac{1}{2L} \cdot \int_{-L}^{+L} \underbrace{f(x)}_{\text{odd}}\, dx = 0$$

$$a_n = \frac{1}{L} \cdot \int_{-L}^{+L} \underbrace{f(x) \cdot \cos\left(\frac{n\pi x}{L}\right)}_{\text{odd function!}}\, dx = 0$$

(the product of an odd ~~and~~ an even function is odd)

Thus, the Fourier series $(Sf)(x)$ of an odd function $f(x)$ is an infinite series of sines only.

$$(Sf)(x) = \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right),$$

where the coefficients b_n can be simplified a bit

$$(*1) \quad b_n := \frac{1}{L} \int_{-L}^{+L} f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

use change of variables $x \mapsto -x$ on $[-L, 0]$ and use ~~that~~ $f(x)$ and $\sin\left(\frac{n\pi x}{L}\right)$ are odd

Fourier sine series

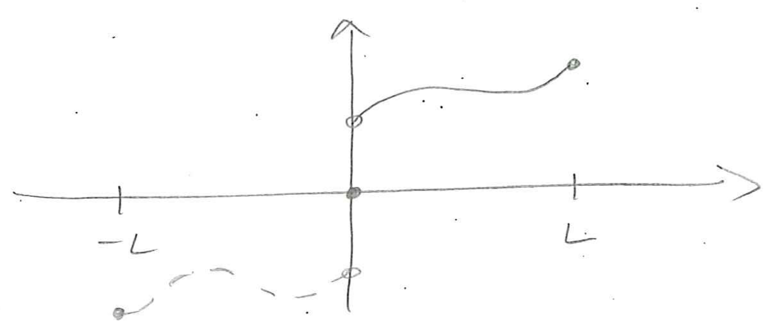
When we used the method of separation of variables to solve the heat equation on a rod of length L with zero boundary conditions, we needed to express a given initial condition $f(x)$ on $0 \leq x \leq L$ as an infinite sine series

$$f(x) \overset{?}{=} \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right).$$

This looks like the Fourier series of an odd function, but $f(x)$ is only defined for $0 \leq x \leq L$ (so it does not make sense to think of $f(x)$ as odd)!

Idea: Define the odd extension $f_{odd}(x)$ of a given function $f(x)$ on $0 \leq x \leq L$ by

$$f_{odd}(x) := \begin{cases} f(x), & 0 < x \leq L \\ -f(-x), & -L \leq x < 0 \\ 0, & x = 0 \end{cases}$$



Since the odd extension $f_{odd}(x)$ is by construction an odd function, its Fourier series just consists of sines

$$S(f_{odd})(x) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right),$$

where from (*1) we have that

$$B_n = \frac{2}{L} \int_0^L \underbrace{f_{odd}(x)}_{= f(x) \text{ on } 0 \leq x \leq L\,!} \sin\left(\frac{n\pi x}{L}\right) dx$$

Since $f_{odd}(x)$ is identical to $f(x)$ on $0 \leq x \leq L$, we can restrict the Fourier series of $f_{odd}(x)$ to the interval $0 \leq x \leq L$ to obtain the <u>Fourier sine series of $f(x)$</u>

$$f(x) \sim \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right), \quad 0 \leq x \leq L$$

with

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

$\rightsquigarrow$ We use the convention from our textbook:
"$\sim$" means that $f(x)$ is on the left-hand-side and the Fourier (sine) series is on the right-hand side, but that the two functions may be quite different!

Example:

Sketch the Fourier sine series of

$$f(x) = 1 - x \quad , \quad 0 \le x \le 1$$

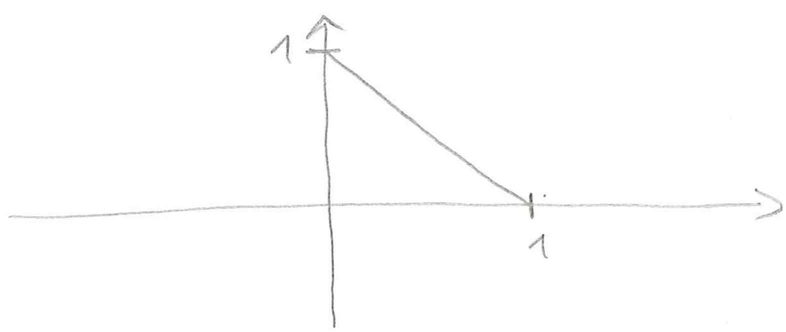
⇝ Strategy:

(1) Sketch $f(x)$ (for $0 \le x \le 1$)

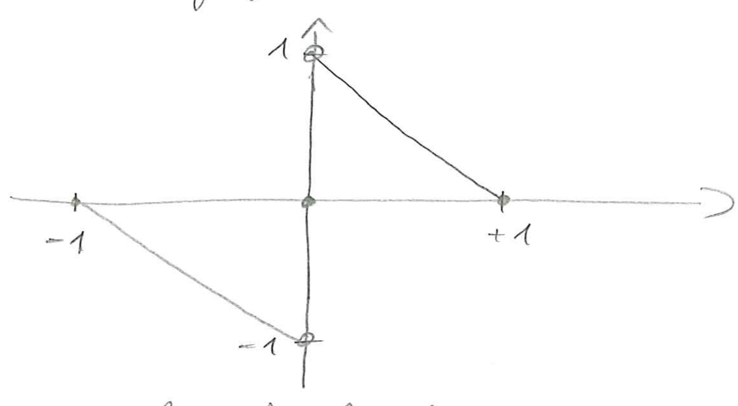
(2) Sketch the odd extension $f_{odd}(x)$ of $f(x)$

(3) Extend f_{odd} as a periodic function to the whole line

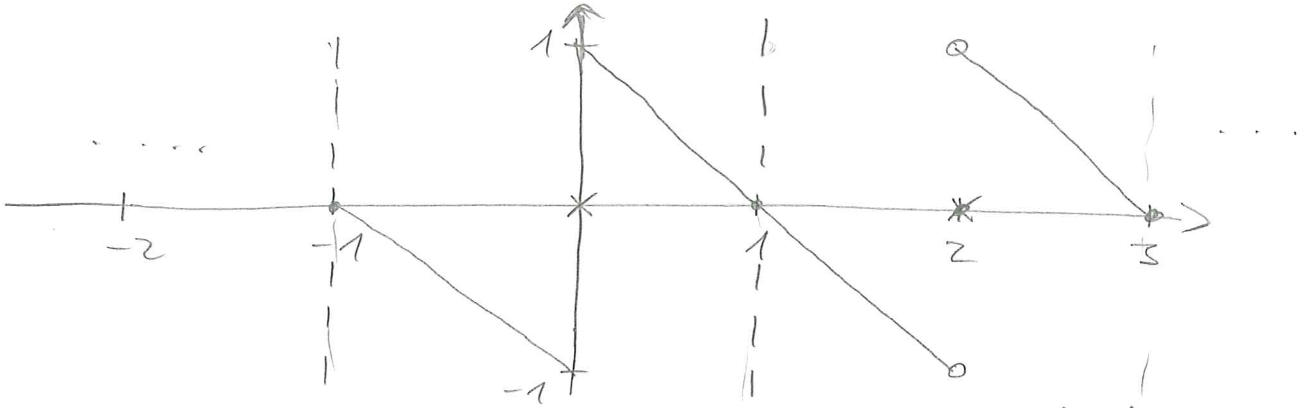
(4) Mark an x at the average of the one-sided limits where the periodic extension of f_{odd} has a jump discontinuity



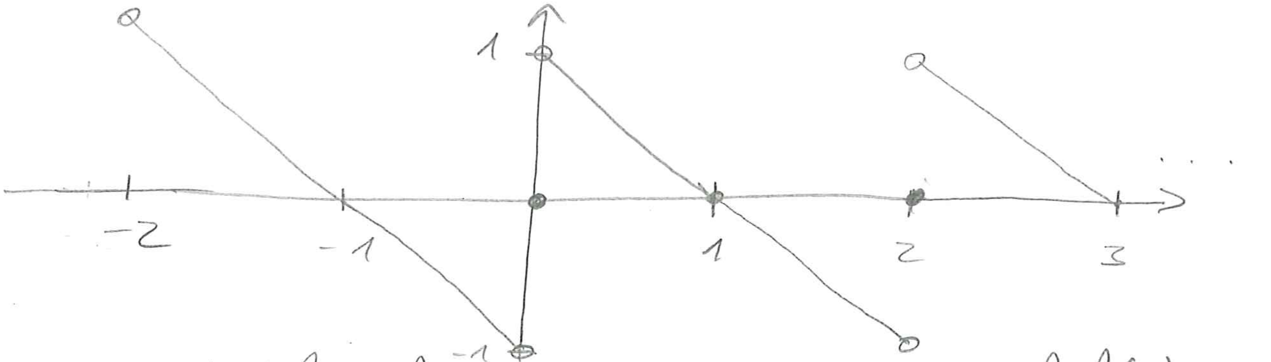
sketch of $f(x) = 1 - x$, $0 \le x \le 1$



sketch of $f_{odd}(x)$, $-1 \le x \le 1$



sketch of periodic extension of f_{odd}

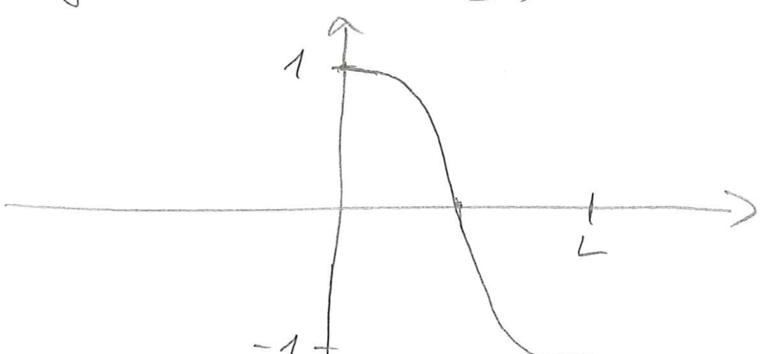


sketch of Fourier sine series of $f(x)$

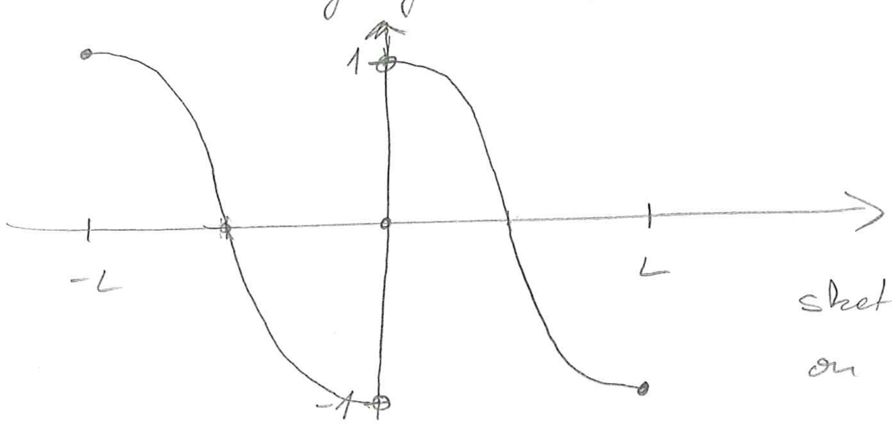
Example:

Sketch the Fourier sine series of

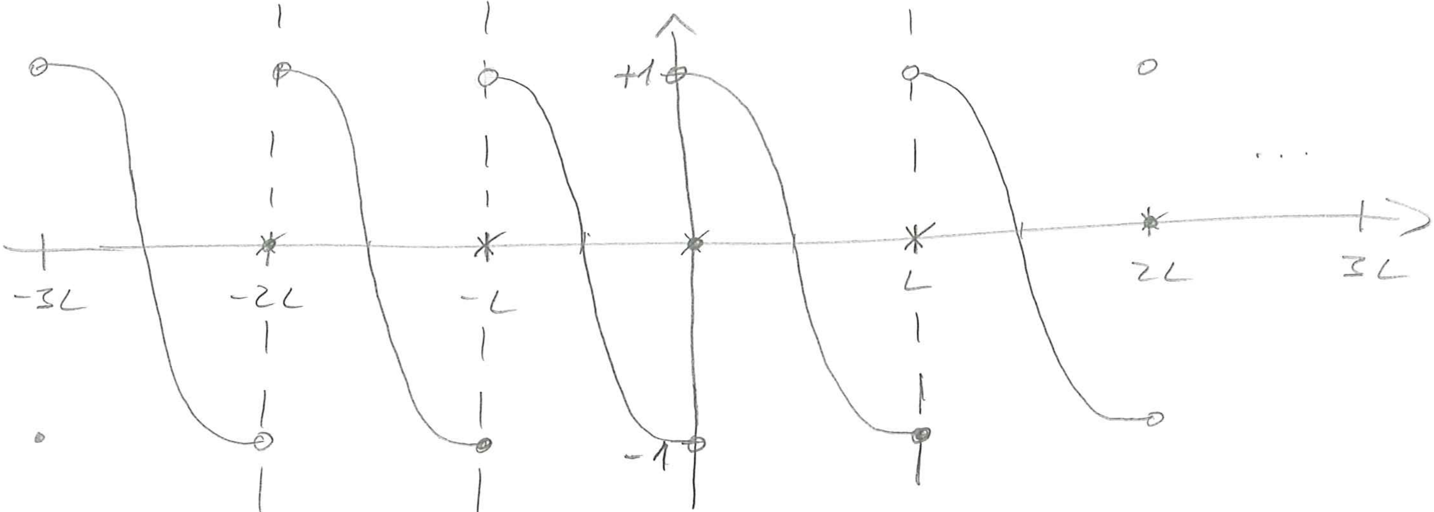
$$f(x) = \cos\left(\frac{\pi x}{L}\right), \quad 0 \le x \le L$$



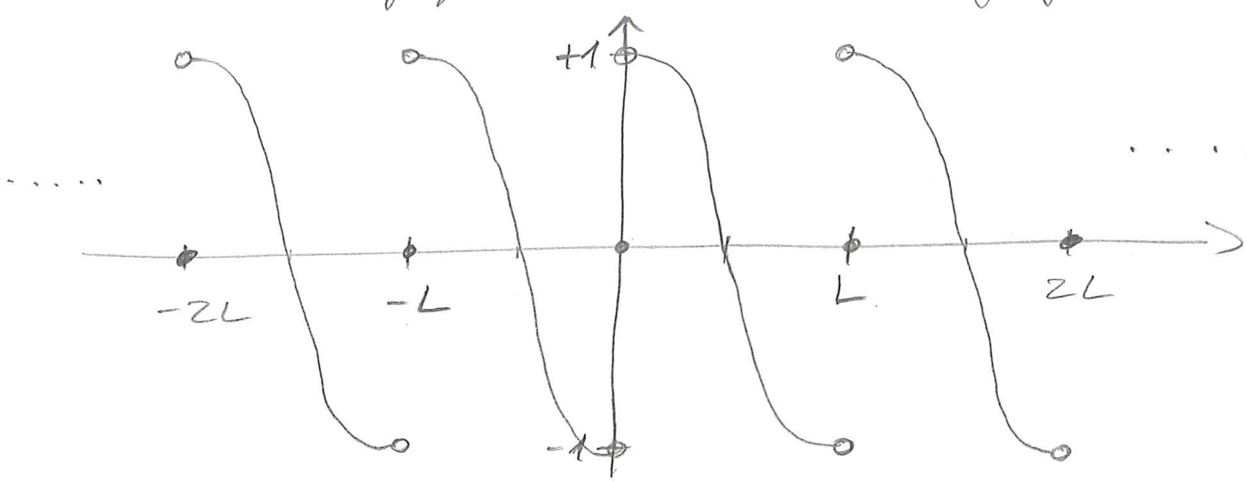
sketch of $f(x)$ on $0 \le x \le L$



sketch of $f_{odd}(x)$ on $-L \le x \le +L$



Sketch of periodic extension of $f_{odd}(x)$



Sketch of Fourier sine series of $\cos\left(\frac{\pi x}{L}\right)$

Fourier cosine series

Definition:

An even function $f(x)$ satisfies the equation

$$f(-x) = f(x)$$

Examples

$x^2, \quad x^4, \quad \cos(x), \quad \cos(3x), \ldots$

Note:

The sketch of an even function for $x < 0$ is the mirror image of that for $x > 0$

Fourier series of even functions

Let $f(x)$ on $-L \leq x \leq +L$ be even. Then

$$b_n = \frac{1}{L} \int_{-L}^{+L} \underbrace{f(x) \cdot \sin\left(\frac{n\pi x}{L}\right)}_{\text{odd function}} dx = 0$$

(even times odd gives odd function)

Thus, the Fourier series $(Sf)(x)$ of an even function $f(x)$ is an infinite series of cosines only

$$(Sf)(x) = \sum_{n=0}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right),$$

where the formulas for the coefficients a_n can be simplified a bit

$$a_0 = \underbrace{\frac{1}{2L} \cdot \int_{-L}^{L} f(x)\,dx}_{= 2 \cdot \int_0^L f(x)\,dx \quad \text{since } f(x) \text{ even}} = \underline{\underline{\frac{1}{L} \int_0^L f(x)\,dx}},$$

(*2)

$$a_n = \frac{1}{L} \underbrace{\int_{-L}^{L} f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx}_{\substack{\text{even function} \\ (\text{even times even} \\ \text{gives even function})}} = \underline{\underline{\frac{2}{L} \cdot \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx}}$$

In our applications of the method of separation of variables it is sometimes necessary to write a given function $f(x)$ on an interval $0 \leq x \leq L$ as an infinite cosine series

$$f(x) \overset{?}{=} \sum_{n=0}^{\infty} \cos\left(\frac{n\pi x}{L}\right).$$

Analogously to the case of an infinite sine series, this looks like the Fourier series of an even function, but $f(x)$ is only defined for $0 \leq x \leq L$.

Correspondingly, we introduce the <u>even extension</u> of a given function $f(x)$ on $0 \le x \le L$ by

$$f_{even}(x) := \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L \le x \le 0. \end{cases}$$

Then

$$(Sf_{even})(x) = \sum_{n=0}^{\infty} A_n \cdot \cos\left(\frac{n\pi x}{L}\right), \quad -L \le x \le +L,$$

where from (*2) we have

$$A_0 = \frac{1}{L} \cdot \int_0^L f(x)\,dx,$$

$$A_n = \frac{2}{L} \cdot \int_0^L f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx$$

In this manner we obtain the <u>Fourier cosine series</u> of a function $f(x)$ on the interval $0 \le x \le L$:

$$f(x) \sim \sum_{n=0}^{\infty} A_n \cdot \cos\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L,$$

with

$$A_0 = \frac{1}{L} \int_0^L f(x)\,dx,$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx.$$

Example: Sketch the Fourier cosine series of

$$f(x) = x \ , \quad 0 \le x \le L$$

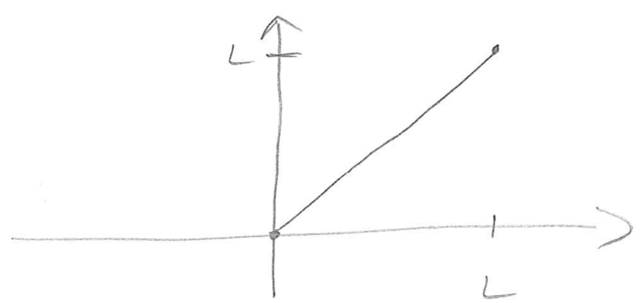
$\leadsto$ Strategy:

(1) Sketch $f(x)$ over the interval $0 \le x \le L$

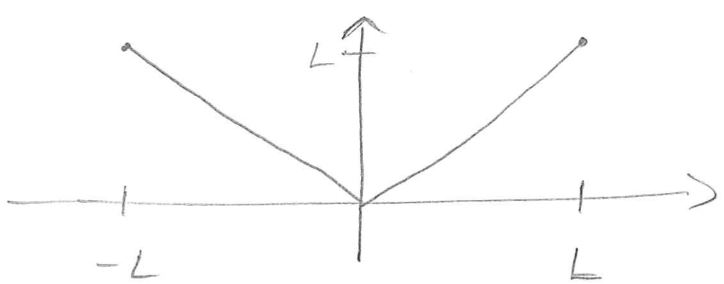
(2) Sketch the even extension of $f(x)$ over the interval $-L \le x \le +L$

(3) Extend as a periodic function on the whole line (with period $2L$)

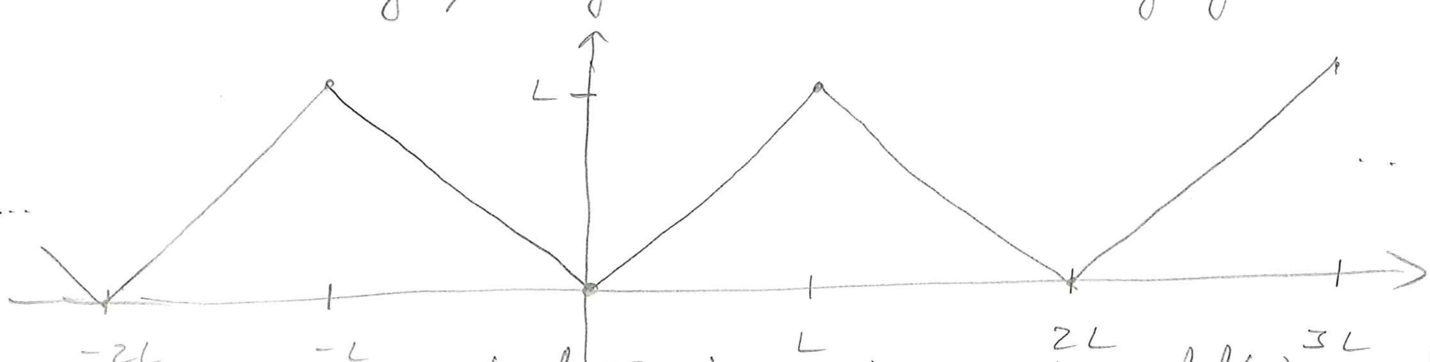
(4) Mark $\times$ at points of discontinuity



graph of $f(x)$ over $0 \le x \le L$



graph of even extension of $f(x)$



graph of Fourier cosine series of $f(x)$

Even and odd parts

An arbitrary function $f(x)$ on $-L \leq x \leq +L$ (which is neither even nor odd) can be decomposed into its even and odd parts

$$f(x) = \underbrace{\frac{1}{2}\left(f(x) + f(-x)\right)}_{\substack{\text{even part } f_e(x) \\ \text{of } f(x)}} + \underbrace{\frac{1}{2}\left(f(x) - f(-x)\right)}_{\substack{\text{odd part } f_o(x) \\ \text{of } f(x)}}$$

Then the Fourier series of $f(x)$

$$(Sf)(x) = \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right)}_{\substack{\text{Fourier cosine} \\ \text{series of } f_e(x)}} + \underbrace{\sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right)}_{\substack{\text{Fourier sine} \\ \text{series of } f_o(x)}}$$

equals the Fourier cosine series of $f_e(x)$ plus the Fourier sine series of $f_o(x)$.

Continuous Fourier series

Q: Given a piecewise smooth function $f(x)$ on an interval $-L \leq x \leq +L$:
Under which conditions on $f(x)$ is its Fourier series $(Sf)(x)$ a continuous function?

By Fourier's theorem, the Fourier series $(Sf)(x)$ assumes the values of the periodic extension $f_{per}(x)$ of $f(x)$, where $f_{per}(x)$ is continuous, and otherwise the average of the one-sided limits $\frac{1}{2}(f_{per}(x+) + f_{per}(x-))$.
For $f_{per}(x)$ to be continuous, and hence $(Sf)(x)$ to be continuous, the function $f(x)$ has to be continuous on $-L \leq x \leq +L$ and must satisfy $f(-L) = f(+L)$.

We summarize:

Given a piecewise smooth function $f(x)$ on an interval $-L \leq x \leq +L$, then its Fourier series $(Sf)(x)$ is continuous and equals $f(x)$ for $-L \leq x \leq +L$ if and only if $f(x)$ is continuous and $f(-L) = f(+L)$.

From this observation we can also quickly deduce the following results about the continuity of the Fourier cosine and Fourier sine series of a function $f(x)$ given on an interval $0 \leq x \leq L$:

- Given a piecewise smooth function $f(x)$ on an interval $0 \leq x \leq L$, the Fourier cosine series of $f(x)$ is <u>continuous</u> and <u>equals</u> $f(x)$ for $0 \leq x \leq L$ if and only if $f(x)$ is continuous.

- Given a piecewise smooth function $f(x)$ on an interval $0 \leq x \leq L$, the Fourier sine series of $f(x)$ is <u>continuous</u> and <u>equals</u> $f(x)$ for $0 \leq x \leq L$ if and only if $f(x)$ is <u>continuous</u> and both $f(0) = 0$ and $f(L) = 0$.

Complex form of Fourier series

The Fourier series of a function $f(x)$ on an interval $-L \leq x \leq +L$

$$(Sf)(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

with

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\, dx,$$

$$a_n = \frac{1}{L} \int_{-L}^{+L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{+L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx,$$

can also be written in terms of complex exponentials instead of sines and cosines.

To this end recall Euler's formula

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

or

$$\cos(\theta) = \frac{1}{2}\left(e^{+i\theta} + e^{-i\theta}\right),$$

$$\sin(\theta) = \frac{1}{2i}\left(e^{+i\theta} - e^{-i\theta}\right).$$

Thus,

$$(Sf)(x) = a_0 + \sum_{n=1}^{\infty} \frac{1}{2}(a_n - ib_n)\, e^{+i\cdot\frac{n\pi x}{L}}$$

$$+ \sum_{n=1}^{\infty} \frac{1}{2}(a_n + ib_n)\, e^{-i\cdot\frac{n\pi x}{L}}.$$

To have only $e^{-i\frac{n\pi x}{L}}$ terms, we change the dummy index in the first summation, replacing n by $-n$, to get

$$(Sf)(x) = a_0 + \sum_{n=-1}^{-\infty} \frac{1}{2}\left(a_{(-n)} - ib_{(-n)}\right) e^{+i\frac{n\pi x}{L}}$$

$$+ \sum_{n=1}^{\infty} \frac{1}{2}(a_n + ib_n)\, e^{-i\cdot\frac{n\pi x}{L}}.$$

Note that from the definition

$$a_n = \frac{1}{L}\int_{-L}^{+L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

we have $a_{(-n)} = a_n$ for any $n\in\mathbb{Z}$.

and from

$$b_n = \frac{1}{L}\int_{-L}^{+L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

we have

$$b_{(-n)} = -b_n \quad \text{for any } n\in\mathbb{Z}$$

$$\left(\begin{array}{lll} \text{since} & \cos\left(\frac{(-n)\pi x}{L}\right) = \cos\left(\frac{n\pi x}{L}\right) \\ \text{and} & \sin\left(\frac{(-n)\pi x}{L}\right) = -\sin\left(\frac{n\pi x}{L}\right) \end{array}\right).$$

Thus, if we define

$$c_0 = a_0,$$

$$c_n = \frac{a_n + i b_n}{2}, \quad n \in \mathbb{Z} \setminus \{0\},$$

then $(Sf)(x)$ simply becomes

$$\boxed{(Sf)(x) = \sum_{n=-\infty}^{+\infty} c_n \cdot e^{-i \cdot \frac{n\pi x}{L}}.}$$

complex form of the Fourier series $(Sf)(x)$

In this form the complex Fourier coefficients are

$$c_n = \frac{1}{2}(a_n + i b_n) \qquad \text{for } n \in \mathbb{Z} \setminus \{0\}$$

$$= \frac{1}{2L} \int_{-L}^{+L} f(x) \underbrace{\left(\cos\left(\frac{n\pi x}{L}\right) + i \sin\left(\frac{n\pi x}{L}\right) \right)}_{= e^{+i \frac{n\pi x}{L}} \text{ Euler's formula}} dx$$

$$= \frac{1}{2L} \int_{-L}^{+L} f(x)\, e^{+i \cdot \frac{n\pi x}{L}} dx.$$

Since $c_0 = a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x) \cdot \underbrace{1}_{= e^{+i \cdot \frac{0 \cdot \pi x}{L}}} dx$,

we have the simple unified formula

$$\boxed{c_n = \frac{1}{2L} \int_{-L}^{+L} f(x)\, e^{+i \cdot \frac{n\pi x}{L}}, \quad n \in \mathbb{Z}}$$

→ The complex form of the Fourier series of a function will be a useful analogy when we introduce the Fourier transform.

Complex orthogonality

Definition:

A complex-valued function $\phi(x)$ is orthogonal to a complex-valued function $\psi(x)$ over the interval $a \leq x \leq b$ if

$$\int_a^b \overline{\phi(x)} \cdot \psi(x) \, dx = 0.$$

Observe that the functions $\left\{ e^{+i \cdot \frac{n\pi x}{L}} \right\}_{n \in \mathbb{Z}}$ are orthogonal over $-L \leq x \leq +L$:

$$\int_{+L}^{+L} \overline{e^{+i \cdot \frac{n\pi x}{L}}} \cdot e^{+i \frac{m\pi x}{L}} \, dx$$

$$= \int_{-L}^{+L} e^{+i \cdot \frac{(-n+m) \cdot \pi x}{L}} \, dx$$

$$= \begin{cases} 0 & , \quad n \neq m, \\ 2L & , \quad n = m. \end{cases}$$

Similarly to using the orthogonality of sines and cosines, we can use the orthogonality of $\left\{ e^{+i \frac{n\pi x}{L}} \right\}_{n \in \mathbb{Z}}$ to directly determine the formula for the Fourier coefficients

$$c_n = \frac{1}{2L} \int_{-L}^{+L} f(x)\, e^{+i \cdot \frac{n\pi x}{L}}\, dx$$

by multiplying the complex form of the Fourier series

$$(Sf)(x) = \sum_{m=-\infty}^{+\infty} c_m \cdot e^{-i \cdot \frac{m\pi x}{L}}$$

by $e^{+i \cdot \frac{n\pi x}{L}}$ and then integrate over $-L \leq x \leq L$.