Chapter 4 : Wave Equation

We live in a world of waves:

- "hearing": our ears detect waves of compression in the air
- "seeing": our eyes detect waves of electromagnetic radiation
- radio, television, mobile telephone networks use waves of electromagnetic radiation
- ocean waves
- earthquakes
- gravitational waves

(...)

Here we consider one of the simplest, yet extremely important wave equations, which for instance describes the vibrations of a musical stringed instrument (like a violin or guitar).

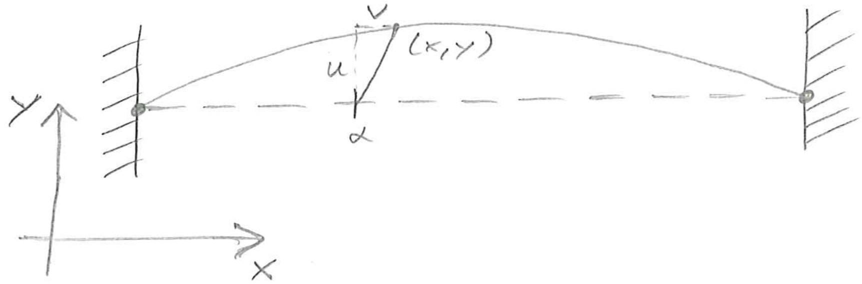
Vibrating strings: derivation of the governing equation

Consider a horizontally (tightly) stretched string whose ends are fixed and tied down (Think of a string in a violin).

If you slightly pull the string vertically upwards and then let it loose, we expect the string to start vibrating.

We now want to derive the governing equations describing such vibrations.

tightly stretched string in equilibrium



perturbed string
v: horizontal displacement
u: vertical displacement

Let α be the x-coordinate of a particle when the string is in equilibrium.

Assume that the slope of the string is small. Then the horizontal displacement v can be neglected and the motion is (approximately) entirely vertical $x = \alpha$. Correspondingly, the vertical displacement u depends only on x and t:

$$y = u(x,t)$$

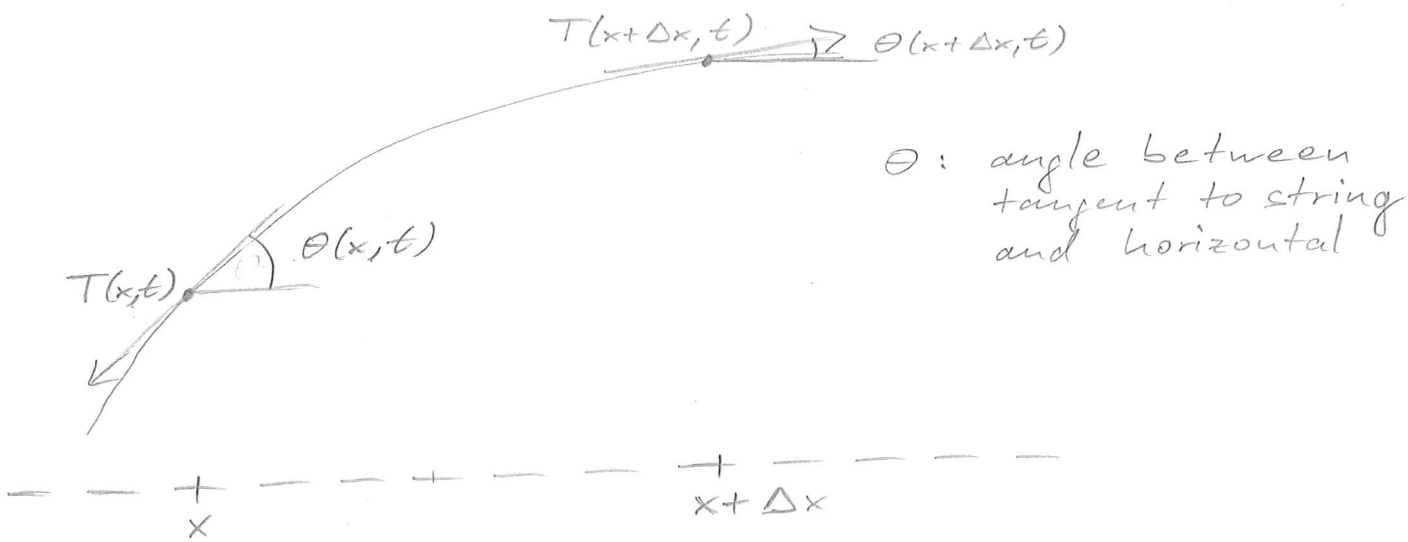
To describe the motion of the string we want to bring in Newton's law of motion

$$F = m \cdot a$$

force exerted on a body

acceleration of the body

Consider a very small segment of the string contained between x and $x+\Delta x$:
(think of this segment as a particle that is subject to several forces pulling it, which we now want to track)



Forces acting on the string segment:

- Assume that the body forces (such as gravitational force) act only vertically
- Assume that the string is perfectly flexible (it offers no resistance to bending). Then the force exerted by the rest of the string on the endpoints of the string segment is tangential to the string. This tangential force is called the tension in the string, denoted by $T(x,t)$, and is trying to stretch the string.

Let $\rho_0(x)$ be the mass density of the string. Then the total mass of the string segment is

$$\rho_0(x) \cdot \Delta x.$$

By Newton's law of motion, we obtain for the vertical motion of the string segment:

$$\underbrace{\varrho_0(x)\cdot\Delta x}_{\text{mass of string segment}}\cdot\underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{vertical acceleration of string segment}} = \left.\begin{array}{l} T(x+\Delta x,t)\cdot\sin(\theta(x+\Delta x,t)) \\ - T(x,t)\cdot\sin(\theta(x,t)) \end{array}\right\} \text{vertical components of tencile forces}$$

$$+ \varrho_0(x)\cdot\Delta x\cdot Q(x,t) \Big\} \text{vertical component of body force per unit mass}$$

Dividing by Δx and letting $\Delta x \to 0$ yields

$$\varrho_0(x)\cdot\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\Big(T(x,t)\cdot\sin(\theta(x,t))\Big) + \varrho_0(x)\cdot Q(x,t)$$

The slope of the string (at position x and time t) is given by

$$\frac{\partial u}{\partial x} = \frac{dy}{dx} = \tan(\theta(x,t))$$

For small angles θ (i.e. $|\theta| \ll 1$)

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \approx \sin(\theta)$$

$$\Rightarrow \frac{\partial u}{\partial x} \approx \sin(\theta)$$

Thus, we arrive at the equation

$$\left\| \quad \varrho_0(x)\cdot\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right) + \varrho_0(x)\cdot Q(x,t) \quad \right\|$$

For *perfectly elastic strings*, the tensile force $T(x,t)$ can be approximated by a constant T_0 (for small perturbations of the string), which leads to the PDE

$$\varrho_0(x) \cdot \frac{\partial^2 u}{\partial t^2} = T_0 \cdot \frac{\partial^2 u}{\partial x^2} + Q(x,t) \cdot \varrho_0(x)$$

Finally, if the only body force is gravity, then $Q(x,t) = -g$ can be neglected because the tensile forces are much larger:

$$\varrho_0(x) \cdot \frac{\partial^2 u}{\partial t^2} = T_0 \cdot \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}} \quad \text{where } c^2 = \frac{T_0}{\varrho_0(x)}$$

has dimension of velocity squared

For a uniform string, c is constant.

Boundary conditions

For a vibrating string of length L, the simplest (and most common) boundary condition is to have fixed ends (usually with fixed zero displacement):

$$u(0,t) = 0, \qquad u(L,t) = 0.$$

for all $t > 0$.

One can also give physical meaning to other boundary conditions such as $\frac{\partial u}{\partial x}(0,t)$ by considering one end of the string attached to a dynamical system, but these are much less common and we won't further discuss them here.

Solving the PDE for a vibrating string with fixed ends

We solve the one-dimensional wave equation for a uniform vibrating string of length L without external forces and fixed ends with zero displacement:

$$\text{(PDE)} \quad \frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L, \quad t > 0$$

$$\text{(BCs)} \quad \begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}$$

$$\text{(ICs)} \quad \begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x) \end{aligned}$$

Note that the wave equation contains second-order time derivatives; we therefore need two initial conditions, one for $u(x,0)$ and the other one for the first time derivative $\frac{\partial u}{\partial t}(x,0)$.
(Think of initial position and initial velocity; informally, one has to "integrate twice in time" to obtain the unknown $u(x,t)$ from the PDE involving $\frac{\partial^2 u}{\partial t^2}$; along the way <u>two</u> integration constants will come up that therefore have to be specified)

The (PDE) and the (BCs) are linear and homogeneous. Hence, it is reasonable to try to use the method of separation of variables to solve it:

Ansatz:

$$u(x,t) = \phi(x) \cdot h(t).$$

Plugging the ansatz into the (PDE):

$$\phi(x) \cdot \frac{d^2h}{dt^2} = c^2 \cdot h(t) \cdot \frac{d^2\phi}{dx^2}$$

$$\Rightarrow \frac{1}{c^2} \frac{1}{h(t)} \frac{d^2h}{dt^2} = \frac{1}{\phi(x)} \frac{d^2\phi}{dx^2} = -\lambda$$

minus sign for convenience

for some separation constant $\lambda \in \mathbb{R}$.

We obtain the following t-dependent ODE for $h(t)$

$$\frac{d^2h}{dt^2} = -\lambda c^2 h \quad , \quad t > 0$$

and the following familiar (x-dependent) boundary value problem for $\phi(x)$:

$$\begin{cases} \dfrac{d^2\phi}{dx^2} = -\lambda \cdot \phi \, , & 0 \le x \le L \\ \phi(0) = 0 \\ \phi(L) = 0 \end{cases}$$

We know that the boundary value problem has the eigenvalues

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 , \quad n = 1, 2, 3, \dots$$

with eigenfunctions

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right) , \quad n = 1, 2, 3, \dots$$

Then the t-dependent ODE

$$\frac{d^2h}{dt^2} = -c^2 \cdot \left(\frac{n\pi}{L}\right)^2 h(t) , \quad n = 1, 2, 3, \dots$$

has the general solution

$$h(t) = c_1 \cdot \cos\left(\frac{n\pi c}{L} \cdot t\right) + c_2 \cdot \sin\left(\frac{n\pi c}{L} \cdot t\right)$$

$\rightsquigarrow$ oscillating in time!

Using the superposition principle to put together the product solutions

$$\sin\left(\frac{n\pi x}{L}\right) \cdot \left(c_1 \cdot \cos\left(\frac{n\pi c}{L} \cdot t\right) + c_2 \cdot \sin\left(\frac{n\pi c}{L} \cdot t\right)\right)$$

as infinite linear combinations, we obtain solutions to the wave equation (PDE) of the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cdot \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi c}{L} \cdot t\right) + \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi c}{L} \cdot t\right)$$

The initial conditions are satisfied if

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \cdot \sin\left(\frac{n\pi x}{L}\right)$$

$$g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi c}{L} \cdot \sin\left(\frac{n\pi x}{L}\right).$$

Thus, writing the (ICs) $f(x)$ and $g(x)$ as Fourier sine series, we obtain for the coefficients

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$B_n \cdot \frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

<u>Interpretation of the results for a musical stringed instrument:</u>

- The vertical displacement is a linear combination of the simple product solutions

$$\sin\left(\frac{n\pi x}{L}\right)\left(A_n \cdot \cos\left(\frac{n\pi c}{L}\cdot t\right) + B_n \cdot \sin\left(\frac{n\pi c}{L}\cdot t\right)\right),$$

called the <u>normal modes of vibration</u>.

- The intensity of the produced sound depends on the amplitude $\sqrt{A_n^2 + B_n^2}$:

We can write

$$A_n \cos\left(\frac{n\pi c}{L} \cdot t\right) + B_n \cdot \sin\left(\frac{n\pi c}{L} \cdot t\right)$$

$$= \sqrt{A_n^2 + B_n^2} \cdot \sin\left(\frac{n\pi c}{L} \cdot t + \theta\right)$$

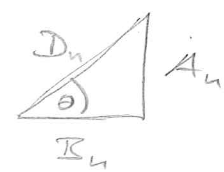
where $\theta = \arctan\left(\frac{A_n}{B_n}\right)$

<u>Derivation:</u>

$$A_n \cos\left(\frac{n\pi c}{L} \cdot t\right) + B_n \cdot \sin\left(\frac{n\pi c}{L} \cdot t\right)$$

$$= D_n \sin(\theta) \cos\left(\frac{n\pi c}{L} \cdot t\right) + D_n \cos(\theta) \sin\left(\frac{n\pi c}{L} \cdot t\right)$$

$$= D_n \sin\left(\frac{n\pi c}{L} \cdot t + \theta\right)$$



$\tan(\theta) = \frac{A_n}{B_n}$, $D_n = \sqrt{A_n^2 + B_n^2}$

$A_n = D_n \sin(\theta)$

$B_n = D_n \cos(\theta)$

using that

$$\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) = \sin(\alpha + \beta).$$

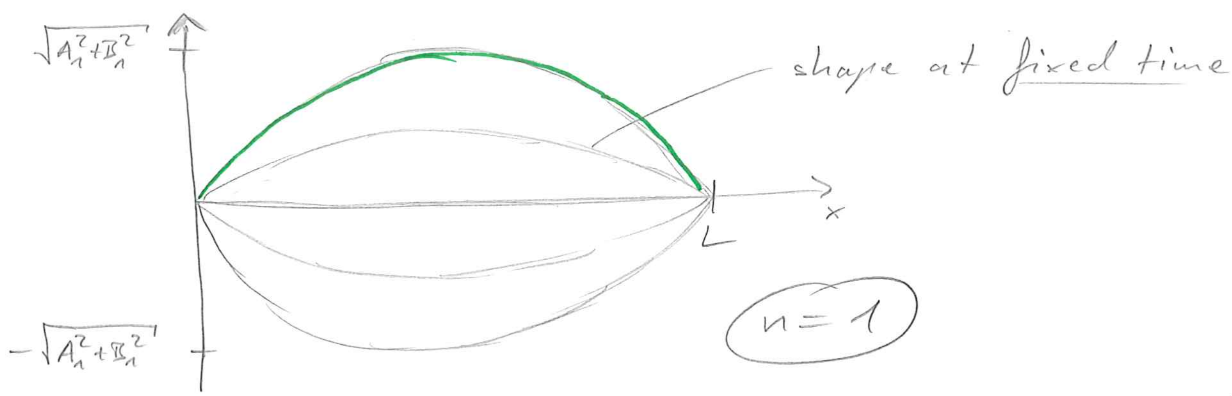
- The time dependence is just periodic with <u>circular frequency</u> given by $\frac{n\pi c}{L}$ (number of oscillations per 2π unit time).

Thus, the sound produced consists of these (possibly infinite) number of <u>natural frequencies</u>.

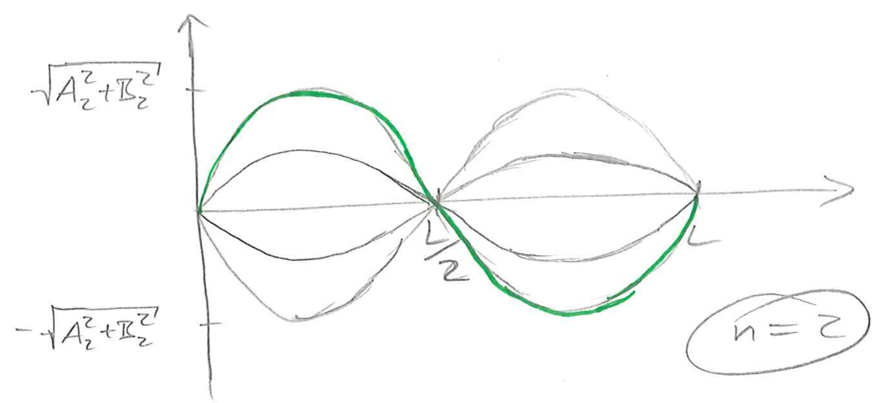
The normal mode $n=1$ is called the first harmonic or fundamental frequency.

⇝ Recall that $c = \sqrt{\frac{T_0}{\rho_0}}$: For a stringed instrument, the mass density ρ_0 cannot be changed, but a desired fundamental frequency can be produced by changing the length L or the tension T_0 of the string.

Motion of normal modes



standing waves



Standing Waves and Travelling Waves

Let's take another look at the expression for the normal modes of vibration

$$\sin\left(\frac{n\pi x}{L}\right)\cdot\left(A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n\cdot \sin\left(\frac{n\pi c}{L}\cdot t\right)\right)$$

$$= A_n\cdot \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi c}{L}\cdot t\right)$$

$$+ B_n\cdot \sin\left(\frac{n\pi}{L}\cdot x\right)\sin\left(\frac{n\pi c}{L}\cdot t\right)$$

Using the trigonometric identity

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$

we can write

$$A_n\cdot \sin\left(\frac{n\pi}{L}\cdot x\right)\cos\left(\frac{n\pi c}{L}\cdot t\right)$$

$$= \underbrace{\frac{A_n}{2}\cdot \sin\left(\frac{n\pi}{L}(x+ct)\right)}_{\substack{\text{wave travelling}\\ \text{to the left}\\ \text{(with velocity } -c)}} + \underbrace{\frac{A_n}{2}\cdot \sin\left(\frac{n\pi}{L}(x-ct)\right)}_{\substack{\text{wave travelling}\\ \text{to the right}\\ \text{(with velocity } c)}}$$

→ These two travelling waves must cancel out (by superposition principle) in such a way to produce a standing wave!

Similarly, using the trigonometric identity

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\cdot\Big(-\cos(\alpha+\beta) + \cos(\alpha-\beta)\Big),$$

we can write

$$B_n \sin\left(\tfrac{n\pi}{L}x\right)\sin\left(\tfrac{n\pi c}{L}t\right)$$

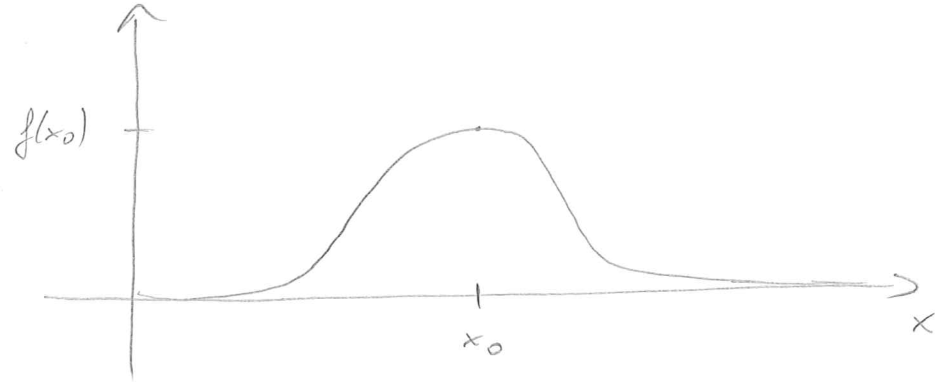
$$= \underbrace{-\frac{B_n}{2}\cos\left(\tfrac{n\pi}{L}(x+ct)\right)}_{\substack{\text{wave travelling}\\ \text{to the left}\\ \text{(with velocity } -c)}} + \underbrace{\frac{B_n}{2}\cdot\cos\left(\tfrac{n\pi}{L}(x-ct)\right)}_{\substack{\text{wave travelling}\\ \text{to the right}\\ \text{(with velocity } +c)}}$$

More generally, if you start from a "nice" (=twice differentiable) function $f(x)$ and define the time-dependent function

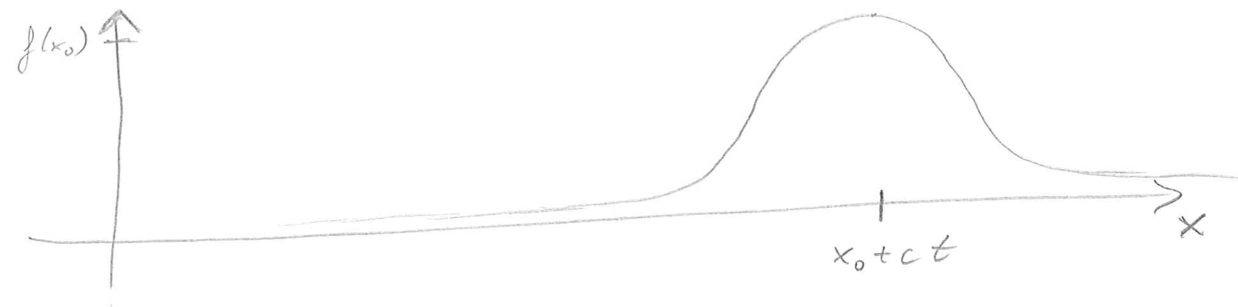
$$u(x,t) := f(x-ct),$$

then if you plot $x \mapsto u(x,t_n)$ at several consecutive times t_n, you will see how the shape of $f(x)$ travels to the right at speed c.

$u(x,0) = f(x)$



$u(x,t) = f(x-ct)$



Moreover, it turns out that this function $u(x,t)$ solves the wave equation

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - c^2 \cdot \frac{\partial^2 u}{\partial x^2} \\
&= \frac{\partial^2}{\partial t^2}\left(f(x-ct)\right) - c^2 \cdot \frac{\partial^2}{\partial x^2}\left(f(x-ct)\right) \\
&= (-c)^2 f''(x-ct) - c^2 \cdot f''(x-ct) \\
&= \underline{\underline{0}} \;!
\end{aligned}
$$

However, one has to be careful about satisfying the boundary conditions too.

Using the above trigonometric identities, you will show in Exercise 4.4.6 that the solution

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\cdot\left(A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right)\right)$$

to the IVP for the one-dim. wave equation

(PDE) $$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L$$

(BC) $$u(0,t) = 0$$
$$u(L,t) = 0$$

(IC) $$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

can always be written as

$$u(x,t) = \underbrace{S(x+ct)}_{\text{wave travelling to the left}} + \underbrace{R(x-ct)}_{\text{wave travelling to the right}}$$

for some functions S, R.

Comparison between 1D heat equation and 1D wave equation

We can now compute the solutions to the 1D heat equation

$$\frac{\partial v}{\partial t} = \underset{\substack{\uparrow \\ k=1}}{\cdot}\frac{\partial^2 v}{\partial x^2}, \qquad 0 \le x \le L, \quad t > 0$$

$$v(0,t) = 0$$
$$v(L,t) = 0$$
$$v(x,0) = f(x)$$

and the 1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = \underset{\substack{\uparrow \\ c=1}}{\cdot}\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L, \quad t > 0$$

$$u(0,t) = 0$$
$$u(L,t) = 0$$
$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = g(x).$$

Although the PDEs look relatively similar, their solutions behave <u>fundamentally different</u>:

The solution to the heat equation

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 t}$$

<u>decays exponentially</u> in time

(to the steady-state solution, for these ICs just zero) ...

while the solution to the wave equation

$$u(x,t) = \sum_{n=1}^{\infty} \left(\sin\left(\frac{n\pi x}{L}\right) \left(A_n \cos\left(\frac{n\pi c}{L}\cdot t\right) + B_n \cdot \sin\left(\frac{n\pi c}{L}\cdot t\right)\right)\right)$$

oscillates periodically in time (and in particular its amplitude does <u>not</u> decay).

This is also reflected by the fact that the wave equation admits a <u>conserved energy</u>, while such conserved quantities are not available for the heat equation.

For the solution $u(x,t)$ to the above wave equation with fixed ends note that we must also have that

$$\frac{\partial u}{\partial t}(0,t) = 0 = \frac{\partial u}{\partial t}(L,t).$$

Then we can compute that

$$\frac{\partial}{\partial t} \int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$= \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x} \underbrace{\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)}_{= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)} dx$$

integrate by parts

$$= \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial u}{\partial t}\, dx + \underbrace{\left[\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t}\right]_{x=0}^{x=L}}_{=0}$$

$$= \int_0^L \frac{\partial u}{\partial t} \underbrace{\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right)}_{=0} dx$$

$= 0$ ← u solves the wave equation

Thus, the <u>energy</u>

$$E(u) = \int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$$

is <u>conserved</u> (constant in time).

In contrast, for the solution $v(x,t)$ to the above heat equation we find that

$$\frac{\partial}{\partial t} \int_0^L v(x,t)^2 \, dx$$

$$= \int_0^L 2 v \frac{\partial v}{\partial t} dx$$

insert heat equation

$$= \int_0^L 2 v \cdot \frac{\partial^2 v}{\partial x^2} dx$$

integrate by parts

$$= \underbrace{\left[2 \cdot v \cdot \frac{\partial v}{\partial x}\right]_{x=0}^{x=L}}_{=0} - 2 \underbrace{\int_0^L \left(\frac{\partial v}{\partial x}\right)^2 dx}_{\geq 0}$$

$$= -2 \int_0^L \left(\frac{\partial v}{\partial x}\right)^2 dx \leq 0$$

Thus, $\int_0^L v(x,t)^2 dx$ is decreasing as time goes by! Integrating in time from $t=0$ to $t=T$ we find

$$\underbrace{\int_0^L v(x,T)^2 dx}_{\substack{\text{decreasing} \\ \text{(as T grows)}}} + 2 \underbrace{\int_0^T \int_0^L \left(\frac{\partial v}{\partial x}(x,t)\right)^2 dx\, dt}_{\substack{\text{increasing} \\ \text{(as T grows)}}} = \underbrace{\int_0^L v(x,0)^2 dx}_{\substack{\text{initial} \\ \text{condition}}}$$