Chapter 7 : Higher dimensionals PDEs

Over the last weeks we have solved several types of linear PDEs using the method of separation of variables, but all involved only two independent variables

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq L \quad \text{(1D heat equation)}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq L \quad \text{(1D wave equation)}$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{(2D Laplace equation)}$$

We now discuss how one can push the method of separation of variables to also solve analogous PDEs with more than two independent variables:

* Linear heat equation in 2 or 3 space dimensions posed on arbitrary (connected) bounded domains Ω:

$$\frac{\partial u}{\partial t} = k \cdot \Delta u$$

where

$$\Delta u = \begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}, & (x,y) \in \Omega \subset \mathbb{R}^2 \quad \text{(2D)} \\[2ex] \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}, & (x,y,z) \in \Omega \subset \mathbb{R}^3 \quad \text{(3D)} \end{cases}$$

* Linear wave equation in 2 or 3 space dimensions posed on arbitrary (connected) bounded domains Ω:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \Delta u ,$$

where (again)

$$\Delta u = \begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} , & (x,y) \in \Omega \subset \mathbb{R}^2 \quad \text{2D} \\ \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} , & (x,y,z) \in \Omega \subset \mathbb{R}^3 \quad \text{3D} \end{cases}$$

Separation of the time variable

* 2D wave equation (vibrating membrane of any shape)

Let $\Omega \subset \mathbb{R}^2$ be an arbitrary (connected) bounded region. We consider

$$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) , \quad (x,y) \in \Omega$$

with homogeneous boundary conditions, usually $u = 0$ along the whole boundary of Ω, and initial conditions

$$u(x,y,0) = \alpha(x,y),$$

$$\frac{\partial u}{\partial t}(x,y,0) = \beta(x,y).$$

We start by seeking solutions of the form

$$u(x,y,t) = h(t) \cdot \phi(x,y),$$

i.e. we separate spatial and temporal parts of u. Note that $\phi(x,y)$ is still an unknown function of <u>two</u> independent variables x, y.

Plugging this ansatz into the wave equation, we find that

$$\phi(x,y) \cdot \frac{d^2 h}{dt^2} = c^2 \cdot h(t) \cdot \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)$$

$$\Rightarrow \underbrace{\frac{1}{c^2 \cdot h(t)} \cdot \frac{d^2 h}{dt^2}}_{\text{only } t\text{-dependent}} = \underbrace{\frac{1}{\phi(x,y)} \cdot \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)}_{\text{only } (x,y)\text{-dependent}} = -\lambda$$

for some separation constant $\lambda \in \mathbb{R}$.

Thus, we obtain the equations

$$(*1) \quad \frac{d^2 h}{dt^2} = -\lambda c^2 \cdot h$$

$$(*2) \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda \cdot \phi$$

Note that $(*1)$ is an <u>ODE</u> for $h(t)$ and here $\lambda > 0$ corresponds to oscillating solutions (for $\lambda > 0$ the general solution is $h(t) = c_1 \cdot \cos(\sqrt{\lambda}\, c t) + c_2 \cdot \sin(\sqrt{\lambda}\, c t)$)

In contrast, (*2) is still a PDE (involving the two independent variables x and y) for ϕ, subject to the same homogeneous boundary conditions as the 2D wave equation. (usually $\phi = 0$ on the whole boundary of Ω).
$\rightsquigarrow$ we have to determine for which values of λ (eigenvalues) this PDE has non-trivial solutions.

* 2D heat equation (heat conduction in any region)

If we make the analogous solution ansatz

$$u(x,y,t) = h(t) \cdot \phi(x,y)$$

for the 2D heat equation

$$\frac{\partial u}{\partial t} = k \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad , \quad (x,y) \in \Omega \subset \mathbb{R}^2,$$

we are led to solve the equations

$$\frac{dh}{dt} = -\lambda \cdot k \cdot h,$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda \cdot \phi$$

for some separation constant $\lambda \in \mathbb{R}$.

* Similarly in higher space dimensions, say for the 3D heat equation, the corresponding product ansatz

$$u(x,y,z,t) = \phi(x,y,z) \cdot h(t)$$

leads to the analogous equations

$$\frac{dh}{dt} = -\lambda k \cdot h,$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -\lambda \cdot \phi,$$

for some separation constant $\lambda \in \mathbb{R}$.

Summary

In higher space dimensions ($\geq 2D$), we are led to solve the following eigenvalue problem for the spatial part $\phi(x,y)$ (in 2D) or $\phi(x,y,z)$ (in 3D)

(*1) $\boxed{\Delta \phi = -\lambda \cdot \phi}$ in $\Omega \subset \mathbb{R}^2$ (or $\Omega \subset \mathbb{R}^3$)

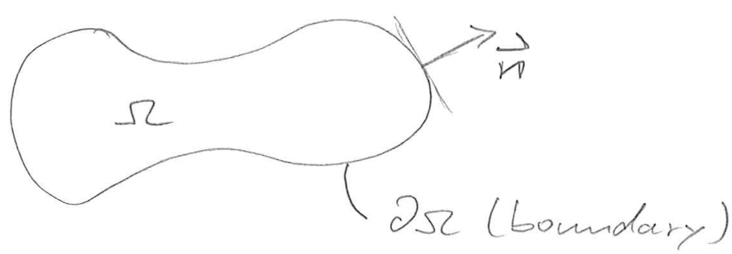
Helmholtz equation

with ϕ satisfying a homogeneous boundary condition of the general type

$$\alpha \cdot \phi + \beta \nabla \phi \cdot \vec{n} = 0 \quad \text{on } \partial\Omega$$

boundary of Ω

where α, β may depend on x, y (and z) and $\vec{n}$ is the outward normal vector to the boundary $\partial\Omega$ of Ω:



Remarks:

* In one space dimension, (*1) is just the usual (and by now very familiar) boundary value problem

$$\frac{d^2\phi}{dx^2} = -\lambda \cdot \phi \qquad 0 \leq x \leq L$$

* $\lambda = 0$ is just Laplace's equation on Ω

* $\beta = 0$: "zero / fixed" boundary condition

  $\alpha = 0$: "insulated" boundary condition

* The Helmholtz equation (*1) can only be solved explicitly for simple geometries of the domain Ω (like rectangles or circles), but certain general properties of the eigenvalue problem (*1) can be proved for arbitrary domains, see the next lectures.

Solutions to the Helmholtz equation

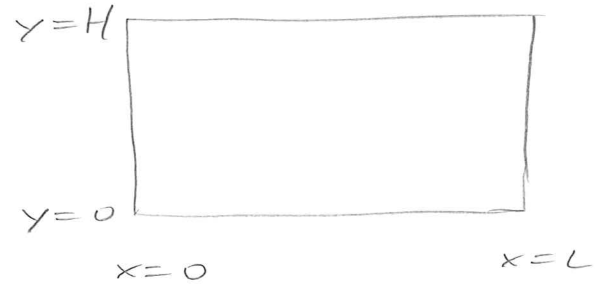
$\Delta \phi = -\lambda \phi$ on a rectangle

We now determine the solutions to the eigenvalue problem

$$\Delta \phi = -\lambda \phi$$

on a rectangle $0 \le x \le L$, $0 \le y \le H$ with zero boundary conditions:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda \cdot \phi$$



$$u(0,y) = 0$$
$$u(L,y) = 0$$
$$u(x,0) = 0$$
$$u(x,H) = 0$$

We try to look for product solutions

$$\phi(x,y) = f(x) \cdot g(y).$$

Plugging this ansatz into the Helmholtz equation, we get that

$$\frac{d^2 f}{dx^2} \cdot g(y) + f(x) \cdot \frac{d^2 g}{dy^2} = -\lambda \cdot f \cdot g$$

Dividing by $f(x) \cdot g(y)$ and rearranging,

$$\frac{1}{f(x)} \cdot \frac{d^2 f}{dx^2} + \frac{1}{g(y)} \frac{d^2 g}{dy^2} = -\lambda$$

$$\Rightarrow \underbrace{\frac{1}{f(x)} \cdot \frac{d^2 f}{dx^2}}_{\text{only } x\text{-dependent}} = \underbrace{-\lambda - \frac{1}{g(y)} \frac{d^2 g}{dy^2}}_{\text{only } y\text{-dependent}} = -\mu$$

<u>another</u> separation constant $\mu \in \mathbb{R}$

Thus, we obtain two ODEs

$$(*4) \quad \frac{d^2 f}{dx^2} = -\mu \cdot f \qquad , \quad 0 \leq x \leq L$$

$$(*5) \quad \frac{d^2 g}{dy^2} = -(\lambda - \mu) \cdot g \qquad , \quad 0 \leq y \leq H.$$

To determine the values of λ and μ (which lead to non-trivial solutions) we have to use the boundary conditions

$$f(0) = f(L) = 0$$

$$g(0) = g(H) = 0$$

The familiar boundary value problem (*4) has the eigenvalues and eigenfunctions

$$\mu_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1,2,3,\ldots$$

$$f_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1,2,3,\ldots$$

Infact, the boundary value problem (*5) is also familiar to us, only that we view $(\lambda-\mu)$ as an eigenvalue.

For each value μ_n, we solve (*5) and obtain that

$$\lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2, \quad m = 1,2,3,\ldots$$

λ now depends on n and m

with eigenfunction

$$g_{nm}(y) = \sin\left(\frac{m\pi y}{H}\right), \quad m = 1,2,3,\ldots$$

Thus,

$$\lambda_{nm} = \mu_n + \left(\frac{m\pi}{H}\right)^2 = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2,$$

for $n = 1,2,3,\ldots$
$m = 1,2,3,\ldots$

Hence, we have found the product solutions

$$\phi_{nm}(x,y) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right), \quad \begin{array}{l} n=1,2,3,\dots \\ m=1,2,3,\dots \end{array}$$

to $\Delta\phi = -\lambda\phi$ with associated eigenvalues

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 > 0.$$

Vibrating rectangular membrane

We now apply the above results to determine the solutions to the 2D wave equation on a rectangular membrane with zero boundary conditions

$$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad \begin{array}{l} 0 \le x \le L \\ 0 \le y \le H \end{array}$$

(BCs) $\quad u(0,y,t) = 0 \qquad u(x,0,t) = 0$

$\qquad\quad u(L,y,t) = 0 \qquad u(x,H,t) = 0$

(ICs) $\quad u(x,y,0) = \alpha(x,y)$

$\qquad\quad \frac{\partial u}{\partial t}(x,y,0) = \beta(x,y)$

Previously, we computed that here the product ansatz

$$u(x,y,t) = h(t)\cdot\phi(x,y)$$

leads to the equations

$$(*6) \quad \frac{d^2h}{dt^2} = -\lambda c^2 h$$

$$(*7) \quad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\lambda\phi \qquad \begin{array}{l} 0 \le x \le L \\ 0 \le y \le H \end{array}$$

For each eigenvalue $\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{n\pi}{H}\right)^2$, $n,m = 1,2,\ldots$ of $(*7)$, the general solution to the ODE $(*6)$ is given by

$$h(t) = c_1 \cdot \cos(\sqrt{\lambda_{nm}}\, ct) + c_2 \cdot \sin(\sqrt{\lambda_{nm}}\, ct).$$

By the superposition principle, we conclude that

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_n \cdot \cos(\sqrt{\lambda_{nm}}\, ct)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_n \cdot \sin(\sqrt{\lambda_{nm}}\, ct)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

is a solution to the 2D wave equation satisfying the zero boundary conditions.

Finally, we determine the coefficients A_{nm}, B_{nm} from the initial conditions

$$u(x,y,0) = \alpha(x,y)$$

$$\frac{\partial u}{\partial t}(x,y,0) = \beta(x,y)$$

For example, $u(x,y,0)$ leads to the double Fourier series

$$\alpha(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{nm} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$$

Note that for every fixed $0 \le x \le L$,

$$\left(\sum_{n=1}^{\infty} A_{nm} \sin\left(\frac{n\pi x}{L}\right)\right)$$

depends only on m and must be the Fourier sine series coefficients of $\alpha(x,y)$ in y (for the fixed x) over the interval $0 \le y \le H$.

Thus,

$$\left(\sum_{n=1}^{\infty} A_{nm} \sin\left(\frac{n\pi x}{L}\right)\right) = \frac{2}{H}\int_0^H \alpha(x,y)\sin\left(\frac{m\pi y}{H}\right)dy$$

for every $m = 1, 2, \dots$. Now for every m, the left-hand side must be the Fourier sine series (in x) of the right-hand side (as a function of x). Hence, for m and n

$$A_{nm} = \frac{2}{L}\int_0^L\left(\frac{2}{H}\int_0^H \alpha(x,y)\sin\left(\frac{m\pi y}{H}\right)dy\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\cdot\frac{2}{H}\int_0^L\int_0^H \alpha(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dy\,dx$$

Similarly, we can determine the coefficients B_{nm} from

$$\beta(x,y) = \frac{\partial u}{\partial t}(x,y,0)$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}}\, B_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)$$

One finds that

$$c \cdot \sqrt{\lambda_{nm}}\, B_{nm} = \frac{2}{L}\frac{2}{H} \int_0^L \int_0^H \beta(x,y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) dy\, dx.$$

<u>Remarks:</u>

* The special product solutions

$$\left(A_{nm} \cos\left(\sqrt{\lambda_{nm}}\, ct\right) + B_{nm} \sin\left(\sqrt{\lambda_{nm}}\, ct\right)\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)$$

are called <u>modes of vibration</u> (as with the vibrating string)

* Our approach to determine the eigenvalues of $\Delta \phi = -\lambda \phi$ for a rectangle of course also works for higher-dimensional boxes.

Interlude: General results for the eigenvalue problem $\Delta\phi = -\lambda\phi$

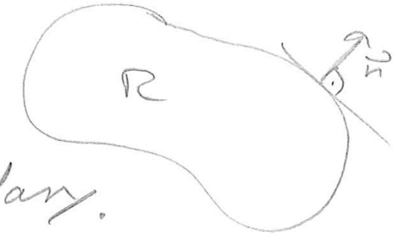
(following Sec. 7.4)

In solving the heat and the wave equation on higher-dimensional spatial domains ($\geq$ 2D), we were led to consider the following multidimensional eigenvalue problem (Helmholtz equation)

$$(*) \qquad \Delta\phi = -\lambda\phi \quad \text{on some region } R \subset \mathbb{R}^n$$

with homogeneous boundary conditions

$$a\cdot\phi + b\,\nabla\phi\cdot\vec{n} = 0,$$



where a and b may depend on the position on the boundary.

The equation $(*)$ can be generalized to

$$(**) \qquad \nabla\cdot(p\nabla\phi) + q\cdot\phi = -\lambda\sigma\phi,$$

where p, q, σ are functions of the position.

In one space dimension $(*)$ is called a regular Sturm-Liouville eigenvalue problem.

The Helmholtz equation (*) can only be solved explicitly for very simple geometries like the circle or the rectangle (as we have worked out).

However, one can still prove several general properties of solutions to the Helmholtz equation (*) on any (reasonable) domain. The proofs are beyond the scope of our course, but we state and illustrate the general theorem (for a two-dimensional spatial domain):

Theorem:

1. All eigenvalues are real.

2. There exists an infinite number of eigenvalues. There is a smallest eigenvalue, but no largest one.

3. Corresponding to an eigenvalue, there may be many eigenfunctions.

4. The eigenfunctions $\phi(x,y)$ form a "complete" set, meaning that

any piecewise smooth function $f(x,y)$ can be represented by a generalized Fourier series of the eigenfunctions:

$$(***) \quad f(x,y) \sim \sum_{\lambda} a_\lambda \phi_\lambda (x,y)$$

5. Eigenfunctions belonging to different eigenvalues ($\lambda_1 \neq \lambda_2$) are orthogonal over the entire region R

$$\iint_R \phi_{\lambda_1} \cdot \phi_{\lambda_2} \, dx\,dy = 0 \quad \text{if } \lambda_1 \neq \lambda_2 .$$

6. An eigenvalue λ can be related to the eigenfunction by the so-called <u>Rayleigh quotient</u>

$$\lambda = \frac{-\oint \phi \nabla\phi \cdot \vec{n} \, ds + \iint_R |\nabla \phi|^2 dx\,dy}{\iint_R \phi^2 \, dx\,dy} ,$$

where $\oint \ldots ds$ is a closed line integral over the entire boundary.

Illustration:

For the Helmholtz equation on a rectangle

$$\Delta \phi = -\lambda \phi \qquad \begin{array}{l} 0 \le x \le L \\ 0 \le y \le H \end{array}$$

with zero boundary conditions

$$\phi(0,y) = 0 \qquad \phi(x,0) = 0$$

$$\phi(L,y) = 0 \qquad \phi(x,H) = 0$$

we determined the eigenvalues

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2, \qquad \begin{array}{l} n = 1,2,\ldots \\ m = 1,2,\ldots \end{array}$$

with associated eigenfunctions

$$\phi_{nm}(x,y) = \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right).$$

On 3.:

Consider for instance the case $L = H$, then

$$\lambda_{nm} = n^2 + m^2,$$

thus $\lambda_{nm} = \lambda_{mn}$ and so λ_{nm} for $n \neq m$ is a multiple eigenvalue.

On 4.:

We derived that any reasonable function $f(x,y)$ can be written as

$$f(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)$$

with

$$A_{nm} = \frac{4}{L \cdot H} \int_0^L \int_0^H f(x,y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) dy\, dx$$

On 5.:

In the homeworks you will verify that

$$\int_0^L \int_0^H \phi_{nm}(x,y)\, \phi_{kl}(x,y)\, dx\, dy = 0$$

if $(n,m) \neq (k,l)$.

More generally, the orthogonality of the eigenfunctions ϕ_{λ_i} can be used to determine the generalized Fourier coefficients a_λ:

Multiply $(***)$ by ϕ_{λ_i} and integrate over R

$$\iint_R f(x,y)\, \phi_{\lambda_i}(x,y)\, dx\, dy = \sum_\lambda a_\lambda \underbrace{\iint_R \phi_\lambda(x,y)\, \phi_{\lambda_i}(x,y)\, dx\, dy}_{= \begin{cases} 0 & , \lambda \neq \lambda_i \\ \iint \phi_{\lambda_i}^2 & , \lambda = \lambda_i \end{cases}}$$

$$\Rightarrow \quad a_{\lambda_i} = \frac{\iint_R f(x,y)\,\phi_{\lambda_i}(x,y)\,dx\,dy}{\iint_R \phi_{\lambda_i}(x,y)^2\,dx\,dy}$$

Note that

$$\int_0^L \int_0^H \left(\sin\left(\frac{n\pi x}{L}\right)\cdot\sin\left(\frac{n\pi y}{L}\right)\right)^2 dx\,dy = \frac{L}{2}\cdot\frac{H}{2}.$$

On 6.:

One can derive this formula by multiplying

$$\Delta\phi = -\lambda\phi$$

by ϕ and then integrating over the region R

$$\iint_R \phi\cdot\underbrace{\Delta\phi}_{=\nabla\cdot\nabla\phi}\,dx\,dy = -\lambda\iint_R \phi^2\,dx\,dy$$

Recall the divergence identity

$$\iint_R \nabla\cdot\vec{A}\,dx\,dy = \oint \vec{A}\cdot\vec{n}\,ds$$

Thus

$$\iint_R \underbrace{\phi\,\nabla\cdot\nabla\phi}_{=\nabla\cdot(\phi\nabla\phi)-|\nabla\phi|^2}\,dx\,dy = \iint_R \nabla\cdot(\phi\nabla\phi)\,dx\,dy - \iint_R |\nabla\phi|^2\,dx\,dy$$

$$= \oint \phi\nabla\phi\cdot\vec{n}\,ds - \iint_R |\nabla\phi|^2\,dx\,dy$$

$$\Rightarrow \oint \phi \nabla\phi \cdot \vec{n}\, ds - \iint_R |\nabla\phi|^2 dx\,dy = -\lambda \iint_R \phi^2 dx\,dy$$

$$\Rightarrow \lambda = \frac{-\oint \phi\nabla\phi\cdot\vec{n} + \iint_R |\nabla\phi|^2 dx\,dy}{\iint_R \phi^2 dx\,dy}$$

Now, if we have zero boundary conditions $\phi = 0$ along the boundary of R, then

$$\oint \underbrace{\phi}_{=0} \nabla\phi\cdot\vec{n} = 0$$

$$\Rightarrow \lambda = \frac{\iint_R |\nabla\phi|^2 dx\,dy}{\iint_R \phi^2 dx\,dy} \geq 0,$$

so we know that all eigenvalues must be non-negative!

<u>Outlook:</u> Helmholtz equation on the disk

$$\Delta \phi = -\lambda \cdot \phi \quad \text{on}$$

with zero boundary conditions.

Use polar coordinates (r, θ) with $0 \leq r \leq a$ and $-\pi \leq \theta \leq \pi$. Recall that in polar coordinates

$$\Delta \phi = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}.$$

Try method of separation of variables and make the product ansatz

$$u(r, \theta) = f(r) \cdot g(\theta)$$

with boundary and compatibility conditions

$$f(a) = 0 \quad , \quad |f(0)| < \infty$$

$$g(-\pi) = g(+\pi), \quad \frac{dg}{d\theta}(-\pi) = \frac{dg}{d\theta}(+\pi).$$

Then

$$\frac{1}{r} \frac{d}{dr}\left(r \frac{df}{dr} \right) \cdot g + \frac{1}{r^2} \frac{d^2 g}{d\theta^2} \cdot f = -\lambda f \cdot g$$

$$\Rightarrow \quad \frac{1}{f} \cdot \frac{1}{r} \frac{d}{dr}\left(r \frac{df}{dr} \right) + \frac{1}{g} \cdot \frac{1}{r^2} \frac{d^2 g}{d\theta^2} = -\lambda$$

Multiply by r^2 and rearrange

$$-\frac{1}{g} \frac{d^2 g}{d\theta^2} = \frac{r}{f} \frac{d}{dr}\left(r \frac{df}{dr} \right) + \lambda r^2 = \mu$$

for some separation constant $\mu \in \mathbb{R}$.

The familiar boundary value problem

$$\frac{d^2 g}{d\theta^2} = -\mu \cdot g \quad , \quad -\pi \leq \theta \leq \pi$$

$$g(-\pi) = g(+\pi)$$

$$\frac{dg}{d\theta}(-\pi) = \frac{dg}{d\theta}(+\pi)$$

has the eigenvalues

$$\mu_m = m^2 \quad , \quad m = 0, 1, 2, \dots$$

with eigenfunctions $\sin(m\theta)$, $\cos(m\theta)$.

Thus, for each $m = 0, 1, 2, \dots$ we have to solve

$$\frac{r}{f} \frac{d}{dr}\left(r \frac{df}{dr}\right) + \lambda r^2 = m^2$$

$$\Rightarrow \quad r^2 \frac{d^2 f}{dr^2} + r \frac{df}{dr} + (\lambda r^2 - m^2) f = 0$$

By the change of variables $z = \sqrt{\lambda} \cdot r$ this becomes

$$\boxed{z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - m^2) f = 0}$$

<u>Bessel's differential equation of order m</u>

For each non-negative integer $m \geq 0$ it has the general solution

$$f(z) = c_1 \cdot J_m(z) + c_2 \cdot Y_m(z),$$

where $J_m(z)$, $Y_m(z)$ are the so-called Bessel functions of the first and second kind.
Since $|f(0)|<\infty$ but $Y_m(z)\to\infty$ as $z\to 0$, we require $c_2=0$.
Then from the boundary condition $f(a)=0$ and from the (infinitely many) zeros of J_m one can determine all possible admissible eigenvalues λ_{nm}.

The Bessel functions $\{J_m\}_{m\geq 0}$ also form a complete set, which then leads to a full solution of the Helmholtz equation, see Section 7.7 for the details.