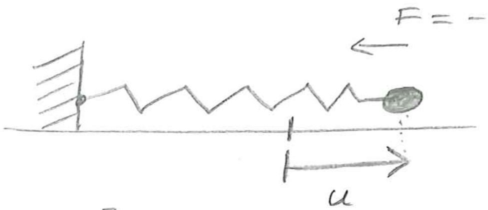
Chapter 1

What is a partial differential equation (PDE)?

Recall: Ordinary differential equations (ODEs)

Example: Motion of a mass on a spring



spring constant k
mass m

$$\frac{d^2u}{dt^2} = -\frac{k}{m}\cdot u$$

⤳ Newton's 2nd law of motion
⤳ Hooke's law

General solution:

$$u(t) = c_1 \cdot \cos\left(\sqrt{\tfrac{k}{m}}\, t\right) + c_2 \cdot \sin\left(\sqrt{\tfrac{k}{m}}\, t\right)$$

Definition: PDE

Let $t, x, y, \ldots$ be independent variables and let u be an unknown function (dependent variable) of at least two of these.

A PDE is an equation that relates the independent variables, the dependent variable u, and the partial derivatives of u.

The order of a PDE is the highest derivative that appears.

PDEs are ubiquitous in the natural sciences and engineering. It is somehow a miracle that phenomena in nature and physical processes can be described so well in the language of mathematics, in particular in terms of ODEs and PDEs!

Examples of important 2nd order PDEs:

- Laplace's equation: $u = u(x,y)$ independent variables x, y

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

⇝ electrostatics, steady-states in heat conduction

- Heat equation: $u = u(t,x)$ independent variables: t, x (t: time, x: spatial coordinate)

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

⇝ conduction of heat, diffusion processes

- Wave equation: $u = u(t,x)$ independent variables: t, x

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

⇝ models wave propagation:
vibration of strings or membranes,
propagation of light (Maxwell's equations of electromagnetism in vacuum)

- Schrödinger's equation $\qquad u = u(t,x)$ complex-valued

$$i \cdot \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0$$

imaginary unit $i = \sqrt{-1}$

$\rightsquigarrow$ fundamental evolution equation of quantum mechanics

In fact, these are the four fundamental types of 2^{nd} order PDEs.

Goals of this course:

- learn how PDEs arise in physical problems
- develop solution techniques (separation of variables, Fourier series, Fourier transform, method of characteristics,...)
- classification of PDEs and differences between properties of solutions of PDEs in various classes

Derivation of the heat equation from physical principles

(more specifically here: derivation of the conduction of heat in a one-dimensional rod)

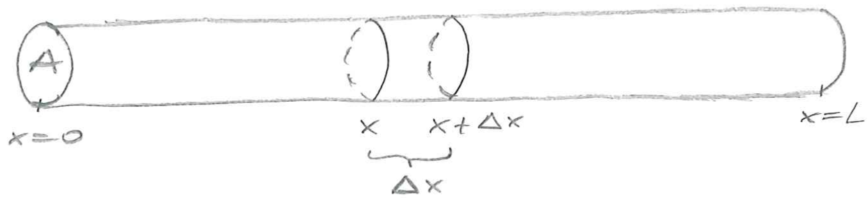
Without being too precise let us first try to define the physical quantities involved:

Temperature is a physical quantity expressing hot and cold. The temperature of a body is related to its internal energy or thermal energy (think of it as how strongly the molecules in say an iron rod swing or how fast the molecules in a gas move about).
Heat is energy in transfer.

Two fundamental principles of physics:

- conservation of energy
- "heat flows from hotter to colder regions" (Fourier's law of heat conduction)

Consider a rod of constant cross-sectional area A oriented in the x-direction (from $x=0$ to $x=L$):



- Thermal energy density

$$e(x,t) := \text{amount of thermal energy per unit volume}$$

- Then the thermal energy (or heat energy) in the above thin slice of the rod is given by

$$\text{thermal energy} = e(x,t) \cdot A \cdot \Delta x$$

(assuming that the thermal density $e(x,t)$ is approximately constant in the slice)

- By the conservation of energy principle, the thermal energy in the thin slice can only change in time due to heat flowing across the edges and being generated inside
(the lateral surfaces are assumed to be insulated)

$$(*1) \quad \begin{matrix}\text{rate of change}\\ \text{of thermal energy}\\ \text{in time}\end{matrix} = \begin{matrix}\text{heat energy}\\ \text{flowing across}\\ \text{boundaries per}\\ \text{unit time}\end{matrix} + \begin{matrix}\text{thermal energy}\\ \text{generated inside}\\ \text{per unit time}\end{matrix}$$

For the thin slice we have

$$\begin{array}{l}\text{rate of change} \\ \text{of thermal energy} \\ \text{in time}\end{array} = \frac{\partial}{\partial t}\Big(e(x,t)\, A\, \Delta x \Big)$$

- Heat flux:

$$\phi(x,t) := \begin{array}{l}\text{amount of thermal energy per unit time} \\ \underline{\text{flowing to the right}} \text{ per unit surface area}\end{array}$$

⇝ Note: If $\phi(x,t) < 0$, then thermal energy is flowing to the left!

- Heat sources:

$$Q(x,t) := \begin{array}{l}\text{thermal energy per unit volume} \\ \text{generated per unit time}\end{array}$$

- Thus, from (*1) we obtain that

$$\frac{\partial}{\partial t}\Big(e(x,t)\cdot A \cdot \Delta x \Big) \approx \phi(x,t)\cdot A - \phi(x+\Delta x, t)\cdot A + Q(x,t)\cdot A \cdot \Delta x$$

Dividing by $A\cdot\Delta x$ yields

$$\frac{\partial e}{\partial t}(x,t) \approx \underbrace{\frac{\phi(x,t) - \phi(x+\Delta x, t)}{\Delta x}}_{\xrightarrow{\Delta x \to 0}\; -\frac{\partial \phi}{\partial x}(x,t)} + Q(x,t)$$

In the limit $\Delta x \to 0$ we find

$$(*2) \qquad \frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q$$