- We usually describe how hot an object is by its temperature, not by its thermal energy density.

  $u(x,t) :=$ temperature

- The precise relation between the thermal energy density and the temperature depends on the material:

  mass density (mass per unit vol)

  $$e(x,t) = c(x) \cdot \rho(x) \cdot u(x,t)$$

  specific heat
  (thermal energy that must be supplied to a unit mass of a substance to raise its temperature by one unit)

- Thus, (*2) becomes

  $$(*3) \quad c(x) \cdot \rho(x) \cdot \frac{\partial u}{\partial t} = - \frac{\partial \phi}{\partial x} + Q$$

- How does the heat flux $\phi(x,t)$ depend on the temperature $u(x,t)$?

  Fourier's law of heat conduction

  $$\phi = - K_0 \cdot \frac{\partial u}{\partial x}$$

  thermal conductivity
  (again a material dependent quantity)

Observe that Fourier's law captures several qualitative properties of heat flow with which are familiar from experiments and daily life:

(i) If the temperature is constant in a region, no heat energy flows.

(ii) If there are temperature differences, the heat energy flows from the hotter region to the colder region.

(iii) The greater the temperature differences (for the same material), the greater is the flow of heat energy.

(iv) The flow of heat energy will vary for different materials, even with the same temperature differences.

- Substituting Fourier's law into (*3) yields a PDE for the temperature $u(x,t)$:

$$(*4) \quad c \cdot \rho \cdot \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(-K_0 \cdot \frac{\partial u}{\partial x}\right) + Q$$

- In a <u>uniform rod</u>, the specific heat c, the mass density ρ, and the thermal conductivity K_0 are just constants (and are <u>not</u> functions of the location x)

Then (*4) becomes

$$c \cdot \rho \, \frac{\partial u}{\partial t} = K_0 \cdot \frac{\partial^2 u}{\partial x^2} + Q.$$

- If also $Q = 0$ (i.e. no heat sources), we arrive at

$$\boxed{\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}}$$

<u>heat equation</u>

where

$$k = \frac{K_0}{c \cdot \rho}$$ <u>thermal diffusivity</u>

- In order to solve this PDE we need to be given an <u>initial condition</u> (IC):

$$u(x, t=0) = f(x)$$

initial temperature distribution

<u>Q:</u> Is this enough information to predict the future temperature?

No, we need to know what happens at the two boundaries, $x = 0$ and $x = L$, i.e. we need two <u>boundary conditions</u>.

The appropriate boundary condition depends on the physical mechanism in effect at each end!

Boundary Conditions

Prescribed temperature:

The temperature of one end of the rod, for example at $x = 0$, may be approximated by a prescribed temperature

$$u(0,t) = u_{B}(t),$$

where $u_{B}(t)$ is the temperature of a fluid bath (or reservoir) with which the rod is in contact.

Insulated boundary

Sometimes it is possible to describe the heat flow rather than the temperature

$$-K_0(0) \cdot \frac{\partial u}{\partial x}(0,t) = \phi(t), \quad \leftarrow \left(\text{remember Fourier's law of heat conduction}\right)$$

where $\phi(t)$ is given.

When an end is perfectly insulated, there is no heat flow at the boundary and thus

$$\frac{\partial u}{\partial x}(0,t) = 0.$$

Newton's law of cooling

This is a mixture of the previous two cases taking into account that (when the end is not insulated), the heat flow leaving the rod is proportional to the temperature difference between the end of the rod and the prescribed external temperature

$$- K_0 \cdot \frac{\partial u}{\partial x}(0,t) = - H \cdot \left(u(0,t) - u_{\mathrm{I}}(t) \right)$$

heat transfer coefficient $H > 0$

⇝ Check that the signs make sense!

Summary:

Three different kinds of boundary conditions. For example, at $x = 0$,

$u(0,t) = u_{\mathrm{I}}(t)$ — prescribed temperature ("Dirichlet BC")

$-K_0(0) \cdot \frac{\partial u}{\partial x}(0,t) = \phi(t)$ — prescribed heat flux ("Neumann BC")

$-K_0(0) \cdot \frac{\partial u}{\partial x}(0,t) = -H \cdot \left(u(0,t) - u_{\mathrm{I}}(t) \right)$ — Newton's law of cooling ("Robin BC")

Note:

In the limit $H \to 0$, Newton's law of cooling approaches the insulated boundary condition.

In the limit $H \to \infty$, Newton's law of cooling approaches the prescribed temperature condition.