Equilibrium temperature distribution

We now begin to determine solutions to the heat equation

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2} \quad , \quad 0 \le x \le L$$

(IC) $u(x,0) = f(x)$

for two types of boundary conditions by seeking special equilibrium or steady-state solutions

$$u(x,t) = u(x),$$

which are independent of time.
Thus, $\frac{\partial u}{\partial t} = 0$ and the heat equation just becomes an ODE

$$\frac{d^2 u}{dx^2} = 0.$$

By integrating twice, we obtain its general solution

$$u(x) = C_1 \cdot x + C_2$$ → graph: straight line

for some constants C_1, C_2.

→ In doing steady state calculations, the initial conditions are usually ignored (to a certain extent).

Steady prescribed temperature BCs

Suppose that the boundary conditions at $x=0$ and $x=L$ are steady

$$u(0,t) = T_1$$
$$u(L,t) = T_2$$

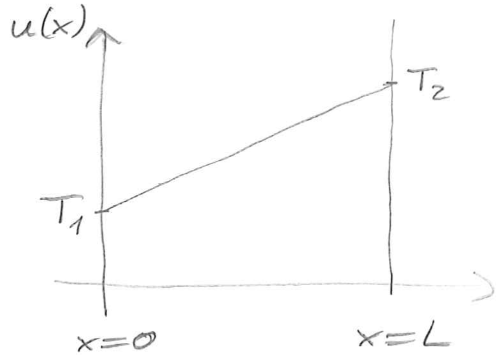
for fixed (temperatures) T_1, T_2.

Then we have for an equilibrium solution

$$u(0) = T_1 \implies C_2 = T_1$$
$$u(L) = T_2 \implies C_1 \cdot L + \underbrace{C_2}_{=T_1} = T_2$$

Thus, $C_1 = \dfrac{T_2 - T_1}{L}$ and

$$\boxed{u(x) = T_1 + \frac{T_2 - T_1}{L} \cdot x}$$



Expectation for the time-dependent case: If we wait for a very long time, we would imagine that the influence of the steady temperatures at the two ends dominates.

It is therefore reasonable to expect that in the long run the initial temperature distribution is forgotten and the solution approaches the equilibrium solution

$$\lim_{t \to \infty} u(x,t) = u(x) = T_1 + \frac{T_2 - T_1}{L} \cdot x.$$

$\rightsquigarrow$ We will derive in the next weeks that this happens indeed.

Insulated boundaries

Let's do another steady-state calculation in the case of insulated boundaries at $x = 0$ and at $x = L$:

$$\frac{\partial u}{\partial x}(0,t) = 0,$$

$$\frac{\partial u}{\partial x}(L,t) = 0.$$

The general solution for the equilibrium temperature distribution is (from above)

$$u(x) = C_1 \cdot x + C_2$$

for some constants C_1, C_2

Here the insulated boundary conditions are satisfied if $C_1 = 0$.

Thus,

$$u(x) = C_2$$

is an equilibrium solution for any constant C_2 (no uniqueness!).

For the time-dependent problem, we would again expect that in the long run the solution approaches an equilibrium state

$$\lim_{t\to\infty} u(x,t) = u(x) = C_2 .$$

But physically it does not make sense that it should be an arbitrary equilibrium temperature C_2.
It should have something to do with the initial condition $u(x,0) = f(x)$.

We should have conservation of the total thermal energy in the rod (since no heat can flow in or out at the ends)!

$$\int_0^L \underbrace{e(x,t)}_{= c \cdot \rho \cdot u(x,t)} dx = \text{constant for all } t \geq 0$$

thermal energy density

Hence, for any $t > 0$

$$\int_0^L c \cdot \rho \cdot u(x,t)\,dx = \int_0^L c \cdot \rho \cdot \underbrace{u(x,0)}_{= f(x)}\,dx$$

$$\Rightarrow \int_0^L \underbrace{u(x,t)}_{\xrightarrow{t \to \infty} C_2}\,dx = \int_0^L f(x)\,dx$$

$$\Rightarrow \underbrace{\int_0^L C_2\,dx}_{= C_2 \cdot L} = \int_0^L f(x)\,dx$$

$$\Rightarrow \boxed{C_2 = \frac{1}{L}\int_0^L f(x)\,dx}$$ average of the initial temperature distribution

$\rightsquigarrow$ We will soon derive that this is indeed the limiting equilibrium temperature in the case of insulated boundary conditions.

Heat equation in higher space dimensions

$$\frac{\partial u}{\partial t} = k \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \text{(two space dimensions)}$$

$$\frac{\partial u}{\partial t} = k \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad \text{(three space dimensions)}$$