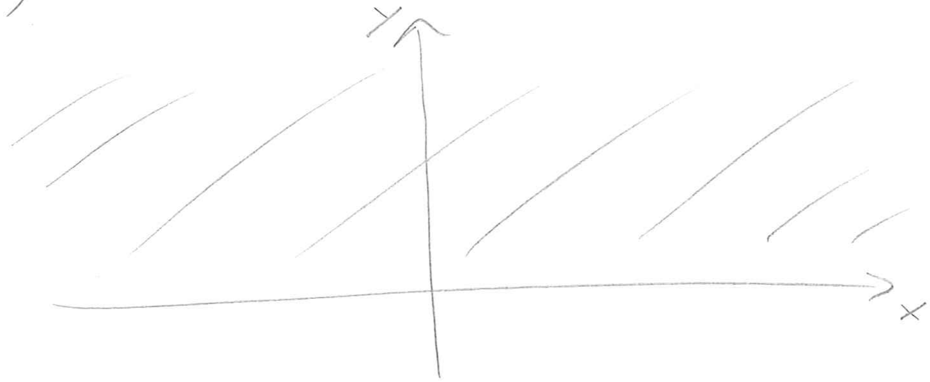
Example 2: Laplace's equation in a half-plane

We consider Laplace's equation in the half-plane $\{(x,y) \in \mathbb{R}^2 \mid y > 0\}$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \,, \quad \begin{matrix} -\infty < x < +\infty \\ y > 0 \end{matrix}.$$

subject to the boundary condition

$$u(x,0) = f(x)$$



If $f(x) \to 0$ as $x \to \pm\infty$, it is reasonable to assume the following three other boundary conditions for the (unknown) solution

$$\lim_{x \to +\infty} u(x,y) = 0 \quad \text{for all fixed } y \geq 0,$$

$$\lim_{x \to -\infty} u(x,y) = 0 \quad \text{for all fixed } y \geq 0,$$

$$\lim_{y \to +\infty} u(x,y) = 0 \quad \text{for all fixed } x \in \mathbb{R}.$$

Since for every $y>0$, $u(x,y)$ is defined for all $-\infty<x<\infty$, and decays as $x\to\pm\infty$, for every $y>0$ we can take the Fourier transform in x of $u(x,y)$:

$$\bar{u}(\omega,y) = \frac{1}{2\pi}\cdot\int_{-\infty}^{+\infty} u(x,y)\, e^{i\omega x}\, dx$$

$$u(x,y) = \int_{-\infty}^{+\infty} \bar{u}(\omega,y)\, e^{-i\omega x}\, d\omega$$

Taking the Fourier transform in x of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

we obtain

$$\mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}\right) + \mathcal{F}\left(\frac{\partial^2 u}{\partial y^2}\right) = 0$$

$$\Rightarrow \underbrace{(-i\omega)^2}_{=-\omega^2}\,\bar{u} + \frac{\partial^2 \bar{u}}{\partial y^2} = 0$$

$$\Rightarrow \left\| \frac{\partial^2 \bar{u}}{\partial y^2} = \omega^2 \bar{u} \right\|$$

- Since $u(x,y)\to 0$ as $y\to+\infty$, the Fourier transform in x, $\bar{u}(\omega,y)$, also vanishes as $y\to+\infty$,

$$\bar{u}(\omega,y) \to 0 \quad \text{as } y\to+\infty.$$

- Moreover, at $y=0$, $\bar{u}(\omega,0)$ is the Fourier transform in x of the boundary condition $f(x)$,

$$\bar{u}(\omega,0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\, e^{i\omega x}\, dx$$

- The general solution of the ODE (for any $\omega \in \mathbb{R}$)

$$\frac{\partial^2 \overline{u}}{\partial y^2} = \omega^2 \overline{u}$$

is

$$\overline{u}(\omega, y) = a(\omega) \cdot e^{+\omega y} + b(\omega) \cdot e^{-\omega y}.$$

We now determine the coefficients $a(\omega)$ and $b(\omega)$ from the boundary conditions

(1) $\overline{u}(\omega, y) \to 0$ as $y \to +\infty$,

(2) $\overline{u}(\omega, 0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)\, e^{i\omega x}\, dx$

We have to be careful with (1); note that when $\omega > 0$, $e^{+\omega y} \to \infty$ as $y \to \infty$ while if $\omega < 0$, $e^{\omega y} \to 0$ as $y \to \infty$.

Thus, we need to have

$$\overline{u}(\omega, y) = \begin{cases} b(\omega)\, e^{-\omega y} & , \omega > 0 \\ a(\omega)\, e^{+\omega y} & , \omega < 0. \end{cases}$$

It is therefore more convenient to write

$$\overline{u}(\omega, y) = c(\omega) \cdot e^{-|\omega| y}.$$

From the boundary condition (2) we then get that $c(\omega) = \overline{u}(\omega, 0)$ is just the Fourier transform of $f(x)$.

$$\Rightarrow \quad \overline{u}(\omega, y) = \mathcal{F}(f)(\omega) \cdot e^{-|\omega| y}$$

Hence, by the convolution theorem,

$$u(x,y) = \mathcal{F}^{-1}(\bar{u}(\omega,y)) = \frac{1}{2\pi} f * \mathcal{F}^{-1}(e^{-|\omega|y})$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\bar{x})\, g(x-\bar{x},y)\,dy,$$

where $g(x,y)$ is the inverse Fourier transform of $e^{-|\omega|y}$:

$$\mathcal{F}^{-1}(e^{-|\omega|y})(x) = \int_{-\infty}^{+\infty} e^{-|\omega|y}\, e^{-i\omega x}\,d\omega$$

$$= \int_{-\infty}^{0} \underbrace{e^{\omega y}\, e^{-i\omega x}}_{=e^{(y-ix)\omega}}\,d\omega + \int_{0}^{\infty} \underbrace{e^{-\omega y}\cdot e^{-i\omega x}}_{=e^{(-y-ix)\omega}}\,d\omega$$

$$= \left[\frac{1}{y-ix}\, e^{(y-ix)\omega}\right]_{-\infty}^{0} + \left[\frac{1}{-y-ix}\, e^{(-y-ix)\omega}\right]_{0}^{\infty}$$

note that $y>0$

$$= +\frac{1}{y-ix} - \frac{1}{-y-ix}$$

$$= \frac{1}{y-ix} + \frac{1}{y+ix}$$

$$= \frac{2y}{x^2+y^2}$$

Thus, the solution $u(x,y)$ is

$$\Bigg\| \; u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\bar{x})\,\frac{2y}{(x-\bar{x})^2+y^2}\,d\bar{x} \; \Bigg\|.$$

<u>Note:</u> We derived this solution formula under the assumption that $f(x) \to 0$ as $x \to \pm\infty$; but in fact, this solution formula is valid under much weaker assumptions,

the integral just has to be convergent (for which it suffices that $f(x)$ is bounded).

Example of a boundary condition:

Consider the simple boundary condition

$$f(x) = \begin{cases} 0 & , \quad x < 0 \\ 1 & , \quad x > 0 \end{cases}$$

Then the solution to the Laplace equation on the half-plane with BC $f(x)$ is

$$u(x,y) = \frac{1}{2\pi} \int_0^\infty \frac{2y}{\underbrace{(x-\bar{x})^2}_{=(\bar{x}-x)^2}+y^2}\, d\bar{x}$$

$$= \frac{1}{\pi} \int_0^\infty \frac{1}{y} \cdot \frac{1}{1+\left(\frac{\bar{x}-x}{y}\right)^2}\, d\bar{x}$$

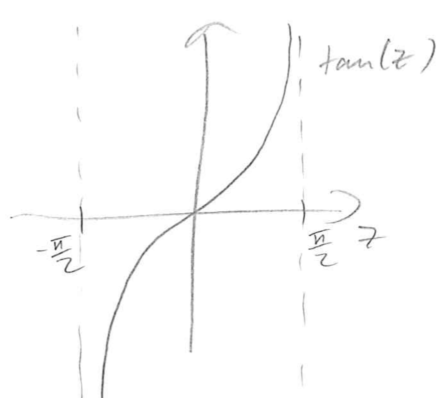
$$= \frac{1}{\pi} \left[\arctan\left(\frac{\bar{x}-x}{y}\right) \right]_{\bar{x}=0}^{\bar{x}=\infty}$$

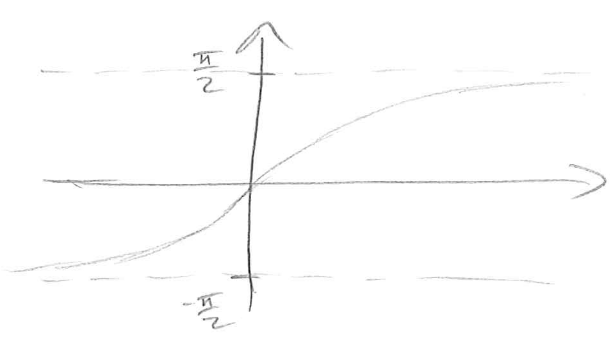
$$= \frac{1}{\pi} \left(\frac{\pi}{2} - \left(\arctan\left(-\frac{x}{y}\right)\right) \right)$$

$$= \frac{1}{\pi} \left(\frac{\pi}{2} + \arctan\left(\frac{x}{y}\right) \right)$$

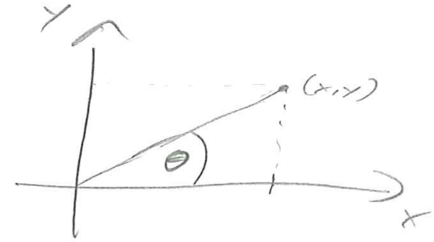
$\frac{d}{d\bar{x}} \arctan(\bar{x}) = \frac{1}{1+\bar{x}^2}$

$\Rightarrow \frac{d}{d\bar{x}}\left(\arctan\left(\frac{\bar{x}-x}{y}\right)\right) = \frac{1}{y} \frac{1}{1+\left(\frac{\bar{x}-x}{y}\right)^2}$





If we let θ denote the angle that the point (x,y) makes with the x-axis, then



$$\tan(\theta) = \frac{y}{x}$$

and

$$\tan\left(\frac{\pi}{2}-\theta\right) = \frac{x}{y}$$

$$\Rightarrow \frac{\pi}{2} - \theta = \arctan\left(\frac{x}{y}\right)$$

Thus, we may write the solution $u(x,y)$ more succinctly as

$$u(x,y) = \frac{1}{\pi}\left(\frac{\pi}{2} + \frac{\pi}{2} - \theta\right) = 1 - \frac{\theta}{\pi}$$

Note: For such a simple BC we can check the answer using polar coordinates (r,θ):

$\Delta u = 0$ becomes $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

with BCs $u(r,0) = 1$ and $u(r,\pi) = 0$

Since the BCs only depend on the angle, the solution will only depend on the angle $u = u(\theta)$.

$\Rightarrow \frac{d^2 u}{d\theta^2} = 0$ has the solution $u(\theta) = 1 - \frac{\theta}{\pi}$ satisfying $u(0) = 1$, $u(\pi) = 0$.

We summarize some important properties of the Fourier transform that we have discussed: (see also table 10.4.1 in our textbook)

$f(x) = \int_{-\infty}^{+\infty} F(\omega) e^{-i\omega x} d\omega$	$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\bar{x}) e^{+i\omega \bar{x}} d\bar{x}$
$\frac{\partial f}{\partial x}$	$(-i\omega) F(\omega)$
$\frac{\partial^2 f}{\partial x^2}$	$(-i\omega)^2 F(\omega)$
$\frac{\partial f}{\partial t}$	$\frac{\partial F}{\partial t}$
$\frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} f(\bar{x})\, g(x-\bar{x})\, d\bar{x}$	$F(\omega) \cdot G(\omega)$
$f(x-\beta)$	$e^{+i\omega\beta} F(\omega)$
$x \cdot f(x)$	$-i \frac{dF}{d\omega}$