# Chapter 2 : Method of Separation of Variables

We develop a technique called the method of separation of variables to solve certain PDEs. We will do so at the example of the heat equation for a one-dimensional rod with zero boundary conditions

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2} , \qquad \begin{matrix} 0 < x < L \\ t > 0 \end{matrix}$$

$$(\mathrm{BC}) \quad \begin{matrix} u(0,t) = 0 \\ u(L,t) = 0 \end{matrix}$$

$$(\mathrm{IC}) \quad u(x,0) = f(x)$$

Our strategy will be to first look for "special solutions" of the form

$$u(x,t) = \phi(x) \cdot G(t) \quad \rightarrow \text{ "separation of variables"}$$

and to then "add these up to match the initial condition $f(x)$".

More generally, the method of separation of variables is used when the PDE and the boundary conditions are linear and homogeneous.

Linearity and homogeneity are fundamental concepts for the study of PDEs. We should therefore first carefully define and discuss these more abstract notions.

## Linearity and Homogeneity

Before defining what a linear PDE and a linear operator are, let's look at some examples:

Algebraic equations:

| linear | nonlinear |
| --- | --- |
| $x + 2y = 0$ | $x^3 = 2x$ |
| $3x - y = 1$ | |

ODEs:

| linear | nonlinear |
| --- | --- |
| $\frac{dy}{dt} + t^2 \cdot y = \cos(3t)$ | $\frac{dy}{dt} = t^2 + e^y$ |

PDEs:

| linear | nonlinear |
| --- | --- |
| $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ | $\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2$ |

Heuristically, what distinguishes linear equations from nonlinear ones? How does one recognize by eye a linear equation:

- the unknown quantity only appears to the first power and does not
- appear inside (transcendental) functions like sin or log
- there may be terms that do not involve the unknown at all and it is OK for the independent variable to appear in nonlinear functions.

Let's now give a formal definition of linear:

Definition:

An operation $L$ on functions (usually referred to as an operator) is linear if it satisfies

$$L(c_1 \cdot u_1 + c_2 \cdot u_2) = c_1 \cdot L(u_1) + c_2 \cdot L(u_2)$$

for any two functions $u_1, u_2$ and any constants $c_1, c_2$.

Examples of linear operators:

- differentiation of $u$: $L(u) := \frac{du}{dx}$
- multiplication of $u$ by a given function : $L(u) = x^2 \cdot u(x)$
- integration : $L(u) = \int_0^1 u(x)\,dx$

Claim: The heat operator

$$L(u) = \frac{\partial u}{\partial t} - k \cdot \frac{\partial^2 u}{\partial x^2}$$

is linear

Proof:

We have

$$L(c_1 \cdot u_1 + c_2 \cdot u_2) = \frac{\partial}{\partial t}(c_1 \cdot u_1 + c_2 \cdot u_2) - k \cdot \frac{\partial^2}{\partial x^2}(c_1 \cdot u_1 + c_2 \cdot u_2)$$

$$= c_1 \cdot \frac{\partial u_1}{\partial t} + c_2 \cdot \frac{\partial u_2}{\partial t} - k \cdot c_1 \frac{\partial^2 u_1}{\partial x^2} - k \cdot c_2 \cdot \frac{\partial^2 u_2}{\partial x^2}$$

$$= c_1 \cdot \left( \frac{\partial u_1}{\partial t} - k \cdot \frac{\partial^2 u_1}{\partial x^2} \right) + c_2 \cdot \left( \frac{\partial u_2}{\partial t} - k \cdot \frac{\partial^2 u_2}{\partial x^2} \right)$$

$$= c_1 \cdot L(u_1) + c_2 \cdot L(u_2)$$

$\boxtimes$

Definition:

A linear equation is an equation of the form

$$L(u) = f,$$

where $L$ is a linear operator, $f$ is a "given" or "known" function and $u$ is the unknown.

Thus, the heat equation is a linear PDE

$$L(u) = \frac{\partial u}{\partial t} - k \cdot \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

or in the more common form

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2} + f(x,t)$$

The $u$-independent terms enter the picture in the role of $f$.
This leads to an absolutely crucial distinction:

**Definition:**

A linear equation $L(u) = f$ is homogeneous if $f = 0$; it is nonhomogeneous if $f \neq 0$.

A fundamental property of solutions to linear and homogeneous equations is that they can be added together in the following sense:

**Principle of Superposition**

If $u_1, u_2$ satisfy a linear homogeneous equation, then any linear combination $c_1 \cdot u_1 + c_2 \cdot u_2$ for $c_1, c_2 \in \mathbb{R}$ also satisfies the same linear homogeneous equation.

Proof:

Let $L$ be the linear operator and suppose $L(u_1) = L(u_2) = 0$.
Then by linearity of $L$,

$$L(c_1 \cdot u_1 + c_2 \cdot u_2) = c_1 \cdot \underbrace{L(u_1)}_{=0} + c_2 \cdot \underbrace{L(u_2)}_{=0} = 0.$$

$\boxtimes$

The concepts of linearity and homogeneity also apply to boundary conditions.

Examples of linear boundary conditions

$$u(0,t) = f(t)$$
$$\frac{\partial u}{\partial x}(L,t) = g(t)$$

homogeneous if and only if $f(t) = 0$, $g(t) = 0$

$$\frac{\partial u}{\partial x}(0,t) = 0$$

A nonlinear boundary condition would for example be:

$$\frac{\partial u}{\partial x}(L,t) = u(L,t)^2.$$

⇝ The method of separation of variables can apply to determine the solution(s) of a PDE if the PDE and the boundary conditions are linear and homogeneous.

# Heat equation with zero temperatures at finite ends

We now introduce the method of separation of variables in the context of solving the heat equation for a one-dimensional rod $(0 < x < L)$ with no sources and both ends immersed in a $0°$ temperature bath:

(PDE) $\quad \dfrac{\partial u}{\partial t} = k \cdot \dfrac{\partial^2 u}{\partial x^2}, \qquad \begin{matrix} 0 < x < L \\ t > 0 \end{matrix}, \qquad k > 0$

(BC) $\quad \begin{matrix} u(0,t) = 0 \\ u(L,t) = 0 \end{matrix}$

(IC) $\quad u(x,0) = f(x)$

We first seek to determine special solutions of the form

$$(*) \quad u(x,t) = \underbrace{\phi(x)}_{\text{only a function of } x} \cdot \underbrace{G(t)}_{\text{only a function of } t}$$

Substituting $(*)$ into the heat equation $\dfrac{\partial u}{\partial t} = k \cdot \dfrac{\partial^2 u}{\partial x^2}$, we find

$$\phi(x) \cdot \frac{dG}{dt} = k \cdot \frac{d^2\phi}{dx^2} \cdot G$$

Dividing by $k \cdot \phi(x) \cdot G(t)$ ("separating variables"), we find

$$\underbrace{\frac{1}{k \cdot G(t)} \cdot \frac{dG}{dt}}_{\substack{\text{function of} \\ t \text{ only!}}} = \underbrace{\frac{1}{\phi(x)} \cdot \frac{d^2\phi}{dx^2}}_{\substack{\text{function of} \\ x \text{ only!}}}$$

Key observation:

The LHS depends only on $t$, while the RHS depends only on $x$. Thus, the identity can only hold for all $t$ and $x$, if in fact both sides are equal to the same constant!

Hence, we must have

$$\frac{1}{k \cdot G} \cdot \frac{dG}{dt} = \frac{1}{\phi} \cdot \frac{d^2\phi}{dx^2} = -\lambda$$

the minus sign is just for convenience in later computations!

for some arbitrary constant $\lambda \in \mathbb{R}$, known as the <u>separation constant</u>.

This gives rise to two <u>simpler</u> ODEs for $G(t)$ and $\phi(x)$:

$$\frac{dG}{dt} = -\lambda \cdot k \cdot G,$$

$$\frac{d^2\phi}{dx^2} = -\lambda \cdot \phi.$$

Now our product solution $u(x,t) = \phi(x) \cdot G(t)$ shall also satisfy the BCs

$$u(0,t) = 0 \quad \Rightarrow \quad \phi(0) \cdot G(t) = 0$$

So either $\phi(0) = 0$ or $G(t) = 0$ (for all $t$).

Note that $G(t) = 0$ would imply $u(x,t) = 0$, which is a trivial solution that we already know. To get non-trivial solutions, we require

$$\phi(0) = 0$$

and analogously at the other end

$$\phi(L) = 0.$$

## The time-dependent equation

We now have to solve the two ODEs separately. ~~Start~~ with the time-dependent one which does not come with additional conditions

$$\frac{dG}{dt} = -\lambda k \cdot G.$$

Making an exponential ansatz $G(t) = e^{r \cdot t}$ and substituting, we must have $r = -\lambda \cdot k$, hence the general solution is

$$G(t) = c \cdot e^{-\lambda k t}, \quad c \in \mathbb{R}.$$

Now observe:

$\lambda > 0$ : $G(t)$ decays exponentially as $t$ increases (recall that $k > 0$)

$\lambda = 0$ : $G(t)$ constant in time

$\lambda < 0$ : $G(t)$ grows exponentially as $t$ increases
$\rightsquigarrow$ we do not expect this for a heat equation without external sources!

$\Rightarrow$ Expect: $\lambda \geq 0$
(we introduced the artificial minus sign further above to have this expectation for non-negative $\lambda$)