Heat equation with zero temperatures at finite ends

We now introduce the method of separation of variables in the context of solving the heat equation for a one-dimensional rod $(0 < x < L)$ with no sources and both ends immersed in a $0°$ temperature bath:

(PDE) $\quad \dfrac{\partial u}{\partial t} = k \cdot \dfrac{\partial^2 u}{\partial x^2}, \quad \begin{array}{c} 0 < x < L \\ t > 0 \end{array}, \quad k > 0$

(BC) $\quad \begin{array}{l} u(0,t) = 0 \\ u(L,t) = 0 \end{array}$

(IC) $\quad u(x,0) = f(x)$

We first seek to determine special solutions of the form

(*) $\quad u(x,t) = \phi(x) \cdot G(t)$ ← only a function of t

↑ only a function of x

Substituting (*) into the heat equation $\dfrac{\partial u}{\partial t} = k \cdot \dfrac{\partial^2 u}{\partial x^2}$, we find

$$\phi(x) \cdot \frac{dG}{dt} = k \cdot \frac{d^2\phi}{dx^2} \cdot G$$

Dividing by $k \cdot \phi(x) \cdot G(t)$ ("separating variables"), we find

$$\underbrace{\frac{1}{k \cdot G(t)} \cdot \frac{dG}{dt}}_{\substack{\text{function of} \\ t \text{ only!}}} = \underbrace{\frac{1}{\phi(x)} \cdot \frac{d^2\phi}{dx^2}}_{\substack{\text{function of} \\ x \text{ only!}}}.$$

Key observation:

The LHS depends only on t, while the RHS depends only on x. Thus, the identity can only hold for all t and x, if in fact both sides are equal to the same constant!

Hence, we must have

$$\frac{1}{k \cdot G} \cdot \frac{dG}{dt} = \frac{1}{\phi} \cdot \frac{d^2\phi}{dx^2} = -\lambda$$

the minus sign is just for convenience in later computations!

for some arbitrary constant $\lambda \in \mathbb{R}$, known as the <u>separation constant</u>.

This gives rise to two <u>simpler</u> ODEs for $G(t)$ and $\phi(x)$:

$$\frac{dG}{dt} = -\lambda \cdot k \cdot G,$$

$$\frac{d^2\phi}{dx^2} = -\lambda \cdot \phi.$$

Now our product solution $u(x,t) = \phi(x) \cdot G(t)$ shall also satisfy the BCs

$$u(0,t) = 0 \quad \Rightarrow \quad \phi(0) \cdot G(t) = 0$$

So either $\phi(0) = 0$ or $G(t) = 0$ (for all t).

Note that $G(t) = 0$ would imply $u(x,t) = 0$, which is a trivial solution that we already know.

To get non-trivial solutions, we require

$$\phi(0) = 0$$

and analogously at the other end

$$\phi(L) = 0.$$

The time-dependent equation

We now have to solve the two ODEs separately. ~~Start~~ With the time-dependent one which does not come with additional conditions

$$\frac{dG}{dt} = -\lambda k \cdot G.$$

Making an exponential ansatz $G(t) = e^{r \cdot t}$ and substituting, we must have $r = -\lambda \cdot k$, hence the general solution is

$$G(t) = c \cdot e^{-\lambda k t}, \quad c \in \mathbb{R}.$$

Now observe:

$\lambda > 0$: $G(t)$ decays exponentially as t increases (recall that $k > 0$)

$\lambda = 0$: $G(t)$ constant in time

$\lambda < 0$: $G(t)$ grows exponentially as t increases

$\rightsquigarrow$ We do not expect this for a heat equation without external sources!

$\Rightarrow$ Expect: $\lambda \geq 0$

(We introduced the artificial minus sign further above to have this expectation for non-negative λ)

$\rightsquigarrow$ We will soon see that only certain values of λ (≥ 0) are allowable.

Boundary Value Problem

We turn to the x-dependent ODE with two homogeneous boundary conditions

$$\begin{cases} \dfrac{d^2\phi}{dx^2} = -\lambda \cdot \phi \\ \phi(0) = 0 \\ \phi(L) = 0 \end{cases}$$

<u>boundary value problem</u>

Note that $\phi(x) = 0$ is a (so-called) trivial solution. We will now see that for certain special values of λ, called <u>eigenvalues</u>, this boundary value problem has <u>non-trivial</u> solutions, called <u>eigenfunctions</u> (associated with the eigenvalues λ).

Let's try to determine the non-trivial solutions, make the ansatz

$$\phi(x) = e^{rx}$$

Then we must have

$$r^2 = -\lambda.$$

Distinguish the following cases:

- $\lambda > 0 \leadsto r = \pm i\sqrt{\lambda}$
- $\lambda = 0 \leadsto r = 0$
- $\lambda < 0 \leadsto r = \pm\sqrt{-\lambda}$

(• λ has non-zero imaginary part) $\leadsto$ ignore this case

→ From considering the time-dependent ODE we expect that only $\lambda \geq 0$ is possible for physical reasons; we will see here that $\lambda < 0$ would lead only to the trivial solution anyway.

1^{st} case: $\lambda > 0$

Then two independent solutions are $e^{\pm i\sqrt{\lambda}x}$.
To have real-valued independent solutions we rather choose $\cos(\sqrt{\lambda}x)$ and $\sin(\sqrt{\lambda}x)$.

Thus, the general solution is

$$\phi(x) = c_1 \cdot \cos(\sqrt{\lambda}x) + c_2 \cdot \sin(\sqrt{\lambda}x)$$

for some constants c_1, c_2.

Now we have to match the BCs:

$$\phi(0) = 0 \implies c_1 = 0 \implies \phi(x) = c_2 \cdot \sin(\sqrt{\lambda} \cdot x)$$

$$\phi(L) = 0 \implies \sin(\sqrt{\lambda} \cdot L) = 0$$

Hence, $\sqrt{\lambda} \cdot L$ must be a zero of the sine function:

$$\sqrt{\lambda} \cdot L = n \cdot \pi \quad , \quad n = 1, 2, 3, \ldots$$

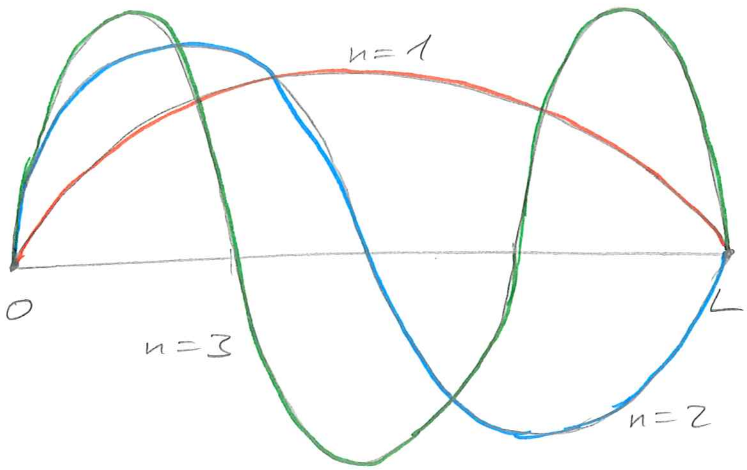
$\sqrt{\lambda} > 0$, so only positive integers

The special eigenvalues λ are thus positive integers

$$\lambda = \left(\frac{n \cdot \pi}{L}\right)^2 \quad , \quad n = 1, 2, 3, \ldots$$

with corresponding eigenfunctions

$$\phi(x) = c \cdot \sin\left(\frac{n\pi x}{L}\right) \quad , \quad c \in \mathbb{R}.$$



2^{nd} case: $\lambda = 0$

If $\lambda = 0$, we consider $\frac{d^2\phi}{dx^2} = 0$ with general solution

$$\phi(x) = c_1 \cdot x + c_2 .$$

Then the boundary condition $\phi(0) = 0$ implies $c_2 = 0$ and $\phi(L) = 0$ yields

$$0 = c_1 \cdot L \quad \Rightarrow \quad c_1 = 0 \quad (\text{since } L > 0).$$

Hence, for $\lambda = 0$ we only get the trivial solution $\phi(x) = 0$.

3^{rd} case: $\lambda < 0$

If $\lambda < 0$, then $\frac{d^2\phi}{dx^2} = \underbrace{-\lambda}_{>0} \cdot \phi$ has the two independent solutions $e^{+\sqrt{-\lambda}\cdot x}$ and $e^{-\sqrt{-\lambda}\, x}$, thus the general solution is

$$\phi(x) = c_1 \cdot e^{+\sqrt{-\lambda}\cdot x} + c_2 \cdot e^{-\sqrt{-\lambda}\cdot x}, \quad c_1, c_2 \in \mathbb{R}.$$

To match the BCs we must have

$$0 = c_1 + c_2 \quad \Rightarrow \quad c_2 = -c_1$$

$$0 = c_1 \cdot e^{+\sqrt{-\lambda}\cdot L} + \underbrace{c_2}_{=-c_1} \cdot e^{-\sqrt{-\lambda}\cdot L}$$

$$= c_1 \cdot \underbrace{\left(e^{+\sqrt{-\lambda}\cdot L} - e^{-\sqrt{-\lambda}\cdot L} \right)}_{\neq 0 \text{ for any } L>0!}$$

$$\Rightarrow \quad c_1 = 0 \quad \text{and} \quad c_2 = 0$$

Thus, for $\lambda < 0$ we only find the trivial solution $\phi(x) \equiv 0$.

<u>Summary</u>:

We have found that the boundary value problem

$$\frac{d^2\phi}{dx^2} + \lambda \cdot \phi = 0,$$

$$\phi(0) = 0,$$

$$\phi(L) = 0,$$

only has non-trivial solutions when $\lambda > 0$; more precisely for the eigenvalues

$$\lambda = \left(\frac{n\cdot\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$

with corresponding eigenfunctions

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots$$

Product solutions

By putting together the solutions to the time-dependent problem and to the boundary value problem that we have found, we obtain the following product solutions to the heat equation

$$\boxed{u(x,t) = B \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 \cdot t}, \quad n = 1, 2, 3, \ldots}$$

→ Note that these special solutions are all exponentially decaying as $t \to \infty$.

Initial value problems (IVPs)

We now hope to be able to use these product solutions to solve IVPs for the heat equation

$$\frac{\partial u}{\partial t} - k \cdot \frac{\partial^2 u}{\partial x^2} = 0$$

$$\text{(BC)} \quad \begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}$$

$$\text{(IC)} \quad u(x,0) = f(x)$$

for arbitrary (!?) initial conditions $u(x,0) = f(x)$.