Product solutions

By putting together the solutions to the time-dependent problem and to the boundary value problem that we have found, we obtain the following product solutions to the heat equation

$$\boxed{u(x,t) = B \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 \cdot t}, \quad n = 1, 2, 3, \dots}$$

→ Note that these special solutions are all exponentially decaying as $t \to \infty$.

Initial value problems (IVPs)

We now hope to be able to use these product solutions to solve IVPs for the heat equation

$$\frac{\partial u}{\partial t} - k \cdot \frac{\partial^2 u}{\partial x^2} = 0$$

$$(BC) \quad \begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}$$

$$(IC) \quad u(x,0) = f(x)$$

for arbitrary (!?) initial conditions $u(x,0) = f(x)$.

Observe that at time $t=0$, the special product solutions are of the form

$$u(x,0) = B \cdot \sin\left(\frac{n\pi x}{L}\right).$$

Thus, if the (IC) $f(x)$ happens to be a multiple of $\sin\left(\frac{n\pi x}{L}\right)$ for some $n = 1,2,3,\dots$, say $4 \cdot \sin\left(\frac{3\pi x}{L}\right)$, then we already know that

$$u(x,t) = 4 \cdot \sin\left(\frac{3\pi x}{L}\right) \cdot e^{-k \cdot \left(\frac{3\pi}{L}\right)^2 t}$$

is the (unique) solution to the IVP for such special initial data.

Recall:

By the superposition principle, given any solutions $u_1, u_2, \dots, u_M$ of a linear homogeneous PDE, then any finite linear combination

$$c_1 \cdot u_1 + c_2 \cdot u_2 + \dots + c_M \cdot u_M = \sum_{n=1}^{M} c_n \cdot u_n$$

is also a solution.

Thus, for any $M \in \mathbb{N}$ and any $B_1, \dots, B_M \in \mathbb{R}$,

$$u(x,t) = \sum_{n=1}^{M} B_n \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 \cdot t}$$

is a solution to the heat equation (on $0 \le x \le L$) with zero boundary conditions.

Correspondingly, if the initial condition $f(x)$ is of the form

$$f(x) = \sum_{n=1}^{M} B_n \cdot \sin\left(\frac{n\pi x}{L}\right)$$

for some $M \in \mathbb{N}$ and constants $B_1, \dots, B_n$, then we also already have the (unique) solution to the corresponding IVP.

<u>Q</u>: What to do when the initial condition $f(x)$ is <u>not</u> a finite linear combination of such $\sin\left(\frac{n\pi x}{L}\right)$ functions?

We will soon discuss so-called Fourier series and we will see that "any reasonable" initial condition $f(x)$ can be written as an infinite linear combination of $\sin\left(\frac{n\pi x}{L}\right)$, known as a type of Fourier series

$$f(x) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right)$$

the coefficients $B_n = B_n(f)$ depend on $f(x)$ of course.

Then we expect that the corresponding (unique) solution to the heat equation with $f(x)$ as (IC) is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right) e^{-k\cdot\left(\frac{n\pi}{L}\right)^2 t}$$

with B_n from above

→ We will have to discuss the convergence of such infinite series and how such an infinite series can be a solution to our heat equation.

But first we find out how to determine the coefficients B_n for a given initial condition $f(x)$:

Orthogonality of Sines

We assume that it is possible to write an initial condition $f(x)$ as

$$(\#1) \quad f(x) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } 0 \leq x \leq L.$$

- To determine the coefficients B_n from $f(x)$, we will use the following important identity:

$$(\#2) \quad \int_0^L \sin\left(\frac{n\pi x}{L}\right) \cdot \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & \underline{\underline{m \neq n}} \\ \frac{L}{2}, & \underline{\underline{m = n}} \end{cases}$$

for any integers m, n

[→ Deriving (#2) will be part of HW2!]

- Now multiply (#1) by $\sin\left(\frac{m\pi x}{L}\right)$ for any $m \in \mathbb{N}$

$$f(x) \cdot \sin\left(\frac{m\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot \sin\left(\frac{m\pi x}{L}\right)$$

and integrate from $x = 0$ to $x = L$

$$\int_0^L f(x)\cdot \sin\left(\frac{m\pi x}{L}\right) dx = \sum_{n=1}^{\infty} B_n \cdot \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right)\cdot \sin\left(\frac{m\pi x}{L}\right) dx}_{\text{by } (\#2) \begin{cases} 0, & n \neq m \\ \frac{L}{2}, & n = m \end{cases}}$$

Note: One has to justify, in principle, that integration and summation can be interchanged

the only non-zero term occurs when $n = m$!

$$\Rightarrow \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = B_m \cdot \frac{L}{2}$$

$$\Rightarrow \boxed{B_m = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx}$$

Orthogonality

Just like two vectors $\vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \vec{k}$ and $\vec{w} = w_1 \hat{i} + w_2 \hat{j} + w_3 \vec{k}$ are orthogonal (perpendicular) if

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 = 0,$$

we can think of two functions $f(x), g(x)$ on an interval $0 \leq x \leq L$ to be orthogonal; we say that $f(x), g(x)$ are orthogonal if

$$\int_0^L f(x) g(x)\, dx = 0.$$

A set of functions each member of which is orthogonal to <u>every other</u> member is called an <u>orthogonal set of functions</u>, for instance the family of sine functions $\left\{ \sin\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty}$.