A set of functions each member of which is orthogonal to every other member is called an orthogonal set of functions,

for instance the family of sine functions $\left\{ \sin\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty}$.

Brief summary

The method of separation of variables for linear homogeneous PDEs proceeds in two main steps:

1. Hunt for product solutions
$$u(x,t) = \phi(x) \cdot G(t).$$
During this step we only use the homogeneous conditions (PDE) as well as (BC), and ignore the initial condition.

2. Superpose the product solutions (form a linear combination or an infinite series of them) and solve for the coefficients to match the initial condition (IC).

Worked example: Heat conduction in a rod with insulated ends

Let's determine the solutions to the following heat equation:

(PDE) $$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,\ t > 0$$

(BC) $$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(L,t) = 0$$ ⇝ Neumann boundary conditions

(IC) $$u(x,0) = f(x)$$

⇝ The main difference to the previous problem is that here we have Neumann boundary conditions (ends of the rod are thermally insulated), while before we had Dirichlet boundary conditions (ends of the rod at 0°).

The (PDE) and (BC) are linear and homogeneous, so we can use the method of separation of variables:

We make a product ansatz

$$u(x,t) = \phi(x) \cdot G(t).$$

As before we must then have

$$\frac{dG}{dt} = -\lambda k \cdot G$$

$$\frac{d^2\phi}{dx^2} = -\lambda \cdot \phi$$

for some separation constant $\lambda \in \mathbb{R}$.

Then

$$G(t) = e^{-\lambda k t}$$

and we are left to find all non-trivial solutions to the following Boundary value problem

$$\frac{d^2\phi}{dx^2} = -\lambda \cdot \phi$$

$$\left.\begin{aligned} \frac{d\phi}{dx}(0) &= 0 \\ \frac{d\phi}{dx}(L) &= 0 \end{aligned}\right\} \rightsquigarrow \text{now with Neumann BCs!}$$

Again have to distinguish three cases for the sign of λ:

<u>1^{st} case:</u> $\lambda > 0$

General solution:

$$\phi(x) = c_1 \cdot \cos(\sqrt{\lambda}\, x) + c_2 \cdot \sin(\sqrt{\lambda}\, x), \quad c_1, c_2 \in \mathbb{R}$$

$$\Rightarrow \frac{d\phi}{dx} = \sqrt{\lambda} \cdot \left(-c_1 \cdot \sin(\sqrt{\lambda}\, x) + c_2 \cdot \cos(\sqrt{\lambda}\, x)\right)$$

From $\frac{d\phi}{dx}(0) = 0$ we infer $c_2 = 0$

and then $\frac{d\phi}{dx}(L) = 0$ leads to

$$0 = -\sqrt{\lambda} \cdot c_1 \cdot \sin(\sqrt{\lambda} \cdot L)$$

As before we conclude that

$$\sqrt{\lambda} \cdot L = n \cdot \pi, \quad n = 1, 2, 3, \dots$$

Hence, the eigenvalues are

$$\lambda_n = \left(\frac{n \cdot \pi}{L}\right)^2 , \quad n = 1, 2, 3, \ldots$$

with associated eigenfunctions

$$\phi_n(x) = c_1 \cdot \cos\left(\frac{n\pi x}{L}\right) , \quad n = 1, 2, 3, \ldots$$

which are now cosines!

The resulting product solutions are

$$u(x,t) = A \cdot \cos\left(\frac{n\pi x}{L}\right) \cdot e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 t} , \quad n = 1, 2, 3, \ldots$$

where $A \in \mathbb{R}$ is arbitrary.

2^nd case: $\lambda = 0$

If $\lambda = 0$, then $\frac{d^2\phi}{dx^2} = 0$ has the general solution

$$\phi(x) = c_1 + c_2 \cdot x . \quad \Rightarrow \quad \frac{d\phi}{dx}(x) = c_2 .$$

Then $\frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(L) = 0$ implies $c_2 = 0$.

Thus, for any constant $c_1 \in \mathbb{R}$, $\phi(x) = c_1$ is a solution to the boundary value problem.

Correspondingly, since $e^{-\lambda k t} = 1$ for $\lambda = 0$, in this case we obtain from the product solution ansatz that

$$u(x,t) = A_0$$

for any constant $A_0 \in \mathbb{R}$ is also a solution to (PDE) satisfying the (BC).

3rd case: $\lambda < 0$

⤳ You will show on HW3 (Problem 2.4.4) that for $\lambda < 0$ the boundary value problem has no non-trivial solutions.

By the superposition principle we find that

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cdot \cos\left(\frac{n\pi x}{L}\right) e^{-k\cdot\left(\frac{n\pi}{L}\right)^2 t}$$

(*)

$$= \sum_{n=0}^{\infty} A_n \cdot \cos\left(\frac{n\pi x}{L}\right) e^{-k\cdot\left(\frac{n\pi}{L}\right)^2 t}$$

note that $\cos\left(\frac{n\pi x}{L}\right) = 1$ and $e^{-k\left(\frac{n\pi}{L}\right)^2 t} = 1$ for $n=0$

is a solution to (PDE) satisfying (BC).

Correspondingly, by evaluating at $t=0$, it follows that we can solve the IVP for any initial condition $f(x)$ that can be written as

$$(**)\quad f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cdot \cos\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L.$$

⤳ We will see that "any reasonable" initial condition $f(x)$ can be written as such a cosine series.

To determine the coefficients A_n for a given initial condition $f(x)$, we use the following <u>orthogonality relation for cosines</u>

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cdot \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & , \quad n \neq m \\ \frac{L}{2} & , \quad n = m \neq 0 \\ L & , \quad n = m = 0 \end{cases}$$

$\rightsquigarrow$ Check this! See also Problem 2.3.6.

As before, now multiply (**) by $\cos\left(\frac{m\pi x}{L}\right)$ and integrate $\int_0^L \dots dx$ to get

$$\int_0^L f(x) \cos\left(\frac{m\pi x}{L}\right) dx = \sum_{n=0}^{\infty} A_n \cdot \underbrace{\int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx}_{= \begin{cases} 0 , & n \neq m \\ \frac{L}{2} , & n=m\neq 0 \\ L , & n=m=0 \end{cases}}$$

By the orthogonality relations for cosines it follows that

$$A_0 = \frac{1}{L} \int_0^L f(x)\, dx,$$

$$A_m = \frac{2}{L} \int_0^L f(x) \cdot \cos\left(\frac{m\pi x}{L}\right) dx.$$

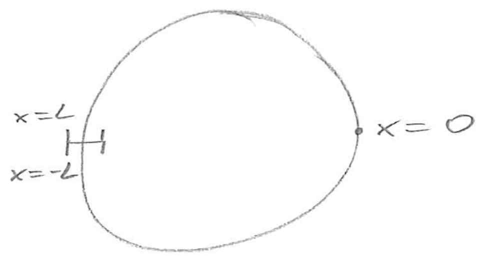
<u>Observe</u>: In (*) all terms with $n \geq 1$ are exponentially decaying as $t \to \infty$ due to the $e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 t}$ factor. Thus,

$$\lim_{t \to \infty} u(x,t) = A_0 = \frac{1}{L} \int_0^L f(x)\, dx,$$

which is the average of the initial temperature distribution! Recall that we expected this to happen from physical considerations!

<u>Worked example:</u> Heat conduction in a thin circular ring

We consider the heat conduction in a thin wire (with lateral sides insulated) that is bent into the shape of a circle and we assume perfect thermal contact at its connected ends



⇝ For "technical convenience" it is helpful here to say that the wire has length $2L$.

(PDE) $$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}, \qquad -L \le x \le L, \quad t > 0$$

(BC) $$\left.\begin{aligned} u(-L,t) &= u(L,t) \\ \frac{\partial u}{\partial x}(-L,t) &= \frac{\partial u}{\partial x}(L,t) \end{aligned}\right\}$$ We assume perfect thermal contact at the connected ends, thus, temperature and heat flux should be continuous there.

(IC) $$u(x,0) = f(x)$$

Here we speak of mixed boundary conditions or periodic boundary conditions. Note that they are <u>linear</u> and <u>homogeneous</u>!

We use the method of separation of variables and make the product ansatz

$$u(x,t) = \phi(x) \cdot G(t).$$

As before we obtain the solutions $G(t) = c \cdot e^{-\lambda k t}$, $\lambda \in \mathbb{R}$, for the time-dependent ODE, and the following boundary value problem

$$\begin{cases} \dfrac{d^2\phi}{dx^2} = -\lambda \cdot \phi \quad , \quad -L \leq x \leq L \\ \phi(-L) = \phi(L) \\ \dfrac{d\phi}{dx}(-L) = \dfrac{d\phi}{dx}(L) \end{cases}$$

<u>1^{st} case:</u> $\lambda > 0$

General solution:

$$\phi(x) = c_1 \cdot \cos(\sqrt{\lambda}\, x) + c_2 \cdot \sin(\sqrt{\lambda} \cdot x) \quad , \quad c_1, c_2 \in \mathbb{R}.$$

Thus, the BC $\phi(-L) = \phi(L)$ implies

$$c_1 \cdot \cos(\sqrt{\lambda} \cdot (-L)) + c_2 \cdot \sin(\sqrt{\lambda} \cdot (-L))$$

$$= c_1 \cdot \cos(\sqrt{\lambda} \cdot L) + c_2 \cdot \sin(\sqrt{\lambda} \cdot L).$$

Now cosine is even, i.e. $\cos(\sqrt{\lambda} \cdot (-L)) = \cos(\sqrt{\lambda} \cdot L)$, while sine is odd, i.e. $\sin(\sqrt{\lambda} \cdot (-L)) = -\sin(\sqrt{\lambda} \cdot L)$

$$\Rightarrow \quad 2\, \underline{c_2} \cdot \sin(\sqrt{\lambda} \cdot L) = 0$$

From

$$\frac{d\phi}{dx} = \sqrt{\lambda} \cdot \left(-c_1 \cdot \sin(\sqrt{\lambda}\, x) + c_2 \cdot \cos(\sqrt{\lambda}\, x)\right)$$

and the other BC $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$, again using that cosine is even, while sine is odd, we obtain

$$2\, \underline{\underline{c_1}} \cdot \sin(\sqrt{\lambda} \cdot L) = 0$$

Thus, if $\sin(\sqrt{\lambda}\cdot L) \neq 0$, then we must have $c_1 = c_2 = 0$, which is just the trivial solution.

For non-trivial solutions we therefore need

$$\sin(\sqrt{\lambda}\cdot L) = 0,$$

which again leads to the eigenvalues

$$\lambda = \left(\frac{n\cdot\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$

but now with corresponding eigenfunctions

$$\cos\left(\frac{n\pi x}{L}\right) \quad \underline{\text{and}} \quad \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots$$

$\rightsquigarrow$ we did not conclude that we must have $c_1 = 0$ or $c_2 = 0$

$\rightsquigarrow$ It was convenient to choose the length $2L$ for the wire to get the same eigenvalues here as in the previous examples

<u>2^{nd} case:</u> $\lambda = 0$

General solution to $\frac{d^2\phi}{dx^2} = 0$:

$$\phi(x) = c_1 + c_2\cdot x, \quad c_1, c_2 \in \mathbb{R}$$

Then the BC $\phi(-L) = \phi(L)$ implies

$$c_1 - c_2\cdot L = c_1 + c_2\cdot L$$

$$\Rightarrow \quad 2c_2\cdot L = 0 \quad \Rightarrow \quad c_2 = 0$$

$$\Rightarrow \quad \phi(x) = c_1 \quad \text{and} \quad \frac{d\phi}{dx} = 0$$ which certainly satisfies the other BC $\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(L)$

Hence, any constant function

$$\phi(x) = c_1, \quad c_1 \in \mathbb{R}$$

is an eigenfunction for the eigenvalue $\lambda = 0$.

3rd case: $\lambda < 0$

$\rightsquigarrow$ Exercise: Show that only the trivial solution $\phi(x) = 0$ is possible here.

Superposing all of the above product solutions, we obtain the following solution to the IVP

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) \cdot e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 t} + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

We will see that any "reasonable" initial condition $f(x)$, $-L \leq x \leq L$, satisfying periodic boundary conditions, can be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right), \quad -L \leq x \leq L.$$

To determine the coefficients a_0, a_n, b_n from $f(x)$ we proceed as in the previous examples using the following orthogonality properties for any non-negative integers n, m

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cdot \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \\ 2L, & n = m = 0 \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cdot \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0, & n \neq m \\ L, & n = m = 0 \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0$$

We obtain that (see page 64 in textbook for details)

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

$$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \cos\left(\frac{m\pi x}{L}\right) dx$$

$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

Summary:

In the previous three worked examples we have seen that the eigenvalues and eigenfunctions for the x-dependent ODE

$$\frac{d^2\phi}{dx^2} = -\lambda \cdot \phi$$

crucially depend on the types of boundary conditions.
(See also table 2.4.1 on page 65).