Laplace's equation: Solutions and qualitative properties

Physical significance:

- arises in electrostatics, gravitation, fluid dynamics,...
- it is the steady-state heat equation:

The heat equation for a 2-dimensional domain is

$$\frac{\partial u}{\partial t} = k \cdot \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

where $u = u(x,y,t)$ is the temperature function.
Then steady-state solutions $u(x,y,t) = u(x,y)$ must satisfy

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(Laplace equation 2D)}$$

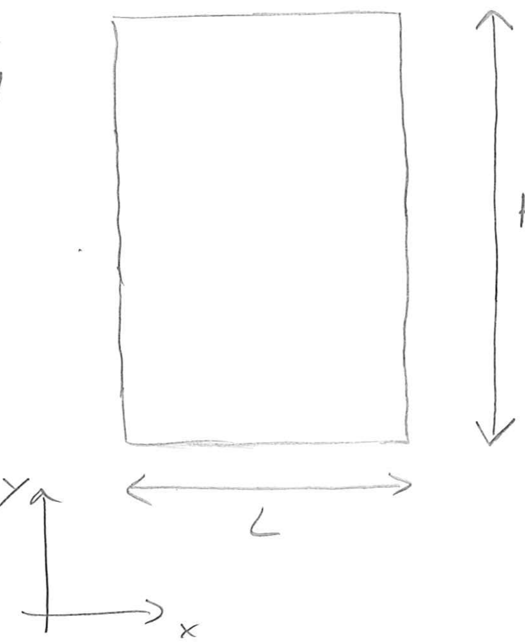
Notation:

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Laplace's equation inside a rectangle

We consider Laplace's equation in a rectangle $(0 \leq x \leq L,\ 0 \leq y \leq H)$ when the temperature is a prescribed function of position (independent of time) on the boundary:

$$\text{(PDE)} \quad \underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}_{\Delta u} = 0, \quad \begin{matrix} 0 \le x \le L \\ 0 \le y \le H \end{matrix}$$



$$\text{(BC1)} \quad u(0,y) = g_1(y)$$

$$\text{(BC2)} \quad u(L,y) = g_2(y)$$

$$\text{(BC3)} \quad u(x,0) = f_1(x)$$

$$\text{(BC4)} \quad u(x,H) = f_2(x)$$

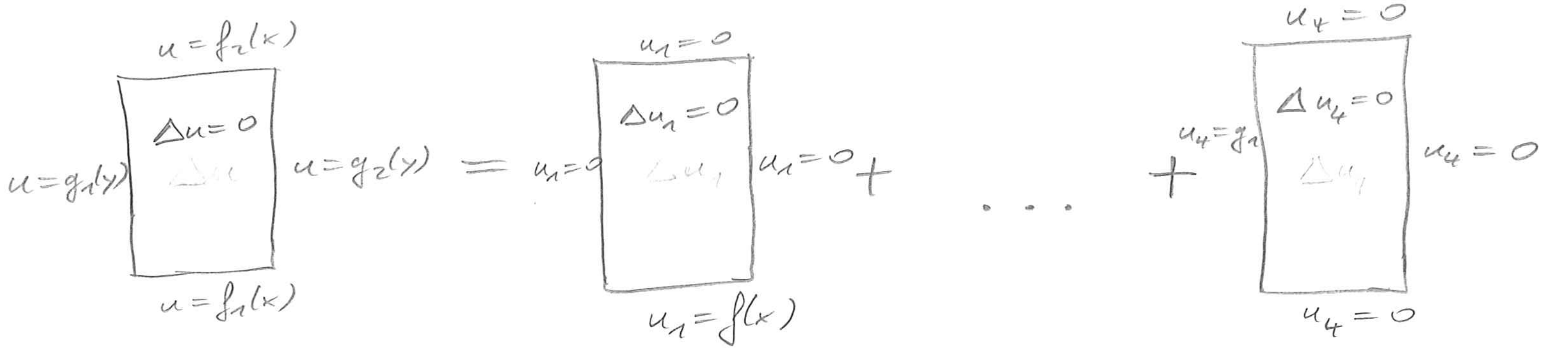
Note: The PDE is linear and homogeneous; the BCs are linear but not homogeneous!

Thus, we cannot apply the method of separation of variables directly (because when we separate variables, the boundary value problem determining the separation constant must have homogeneous boundary conditions)!

Trick: Use the superposition principle to break the problem into four problems, each having only one nonhomogeneous condition

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y),$$

where each $u_j(x,y)$, $j = 1,2,3,4$ satisfies:



Now we can try to solve for each u_j separately using the method of separation of variables, where we will have to match only one nonhomogeneous BC.

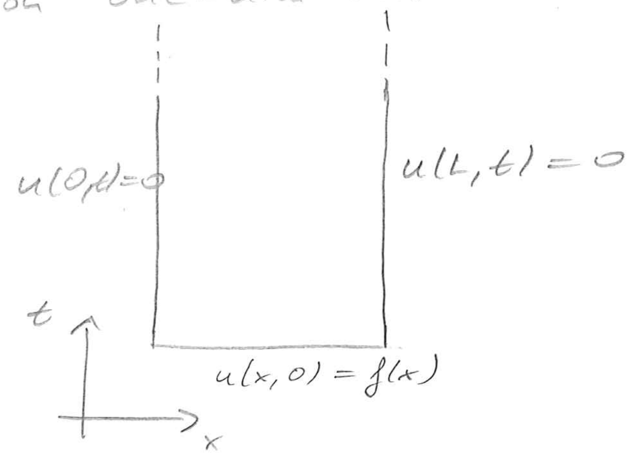
Analogy to heat equation on one-dim rod

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L$$

$$u(0,t) = 0$$

$$u(L,t) = 0$$

$$u(x,0) = f(x)$$



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Underlying principle:

Work with only one nonhomogeneous condition at a time, so that you can exploit the superposition principle correctly.

We treat the case of $u_4(x,y)$ in detail, all other cases work similarly:

$$\text{(PDE)} \qquad \frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0 \qquad \begin{matrix} 0 \le x \le L \\ 0 \le y \le H \end{matrix}$$

$$\text{(BCs)} \qquad \begin{aligned} u_4(0,y) &= g_1(y) \\ u_4(L,y) &= 0 \\ u_4(x,0) &= 0 \\ u_4(x,H) &= 0 \end{aligned}$$

First ignore the nonhomogeneous BC $u_4(0,y) = g_1(y)$, and make product ansatz

$$u_4(x,y) = h(x) \cdot \phi(y).$$

The three homogeneous BCs lead to

$$h(L) = 0$$

$$\phi(0) = 0$$

$$\phi(H) = 0.$$

Inserting the ansatz into our (PDE):

$$\frac{d^2 h}{dx^2} \cdot \phi(y) + h(x) \cdot \frac{d^2 \phi}{dy^2} = 0$$

$$\Rightarrow \quad \underbrace{\frac{1}{h(x)} \frac{d^2 h}{dx^2}}_{\substack{\text{depends} \\ \text{only on } x}} = \underbrace{- \frac{1}{\phi} \frac{d^2 \phi}{dy^2}}_{\substack{\text{depends} \\ \text{only on } y}} = \lambda$$

for some separation constant $\lambda \in \mathbb{R}$.

We obtain two ODEs:

$$\frac{d^2h}{dx^2} = \lambda \cdot h \quad , \quad 0 \le x \le L$$

$$\frac{d^2\phi}{dy^2} = -\lambda \cdot \phi \quad , \quad 0 \le y \le H$$

Note: We have two BCs for ϕ, namely $\phi(0) = 0$ and $\phi(H) = Y$. This gives a boundary value problem for ϕ, which will allow us to determine the eigenvalues λ!

Let's first solve the BVP for ϕ:

$$\frac{d^2\phi}{dy^2} = -\lambda \cdot \phi \quad , \quad 0 \le y \le H$$

$$\phi(0) = 0$$

$$\phi(H) = 0$$

We have already computed that the eigenvalues and eigenfunctions for this BVP are

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2 \quad , \quad n = 1, 2, \dots$$

$$\phi_n = \sin\left(\frac{n\pi y}{H}\right) \quad , \quad n = 1, 2, \dots$$

For these values of λ we now determine the solution(s) to the ODE for $h(x)$

$$\frac{d^2h}{dx^2} = \left(\frac{n\pi}{H}\right)^2 \cdot h \quad , \quad 0 \leq x \leq L,$$

satisfying

$$h(L) = 0$$

As fundamental systems of solutions for this ODE we can work with $\left\{ e^{+\left(\frac{n\pi}{H}\right)\cdot x}, e^{-\left(\frac{n\pi}{H}\right)\cdot x} \right\}$, $\left\{ \cosh\left(\frac{n\pi}{H}\cdot x\right), \sinh\left(\frac{n\pi}{H}\cdot x\right) \right\}$, but also (by translation invariance) $\left\{ \cosh\left(\frac{n\pi}{H}(x-L)\right), \sinh\left(\frac{n\pi}{H}(x-L)\right) \right\}$.

Any of these works, but the latter makes things particularly neat. As the general solution we then have

$$h(x) = a_1 \cdot \cosh\left(\frac{n\pi}{H}(x-L)\right) + a_2 \cdot \sinh\left(\frac{n\pi}{H}(x-L)\right).$$

To satisfy $h(L) = 0$, we must have $a_2 = 0$ (because $\cosh(0) = 1$ and $\sinh(0) = 0$).

Putting things together we have found the product solutions

$$u_4(x,y) = A \cdot \sinh\left(\frac{n\pi}{H}\cdot(x-L)\right) \cdot \sin\left(\frac{n\pi y}{H}\right), \quad n = 1, 2, \ldots$$

Now we try to combine these to also satisfy the nonhomogeneous BC!

By the superposition principle, any infinite linear combination

$$(*)\quad u_4(x,y) = \sum_{n=1}^{\infty} A_n \cdot \sinh\left(\frac{n\pi}{H}(x-L)\right) \cdot \sin\left(\frac{n\pi y}{H}\right), \quad A_n \in \mathbb{R},$$

solves $\Delta u_4 = 0$ and satisfies the three homogeneous BCs.

Correspondingly, let's require at $x=0$ that

$$g_1(y) = u_4(0,y) = \sum_{n=1}^{\infty} \underbrace{\left(A_n \cdot \sinh\left(\frac{n\pi}{H} \cdot (-L)\right)\right)}_{\text{constant coefficient}} \cdot \sin\left(\frac{n\pi y}{H}\right)$$

This is a (Fourier) sine series! By orthogonality of the sines $\left\{\sin\left(\frac{n\pi y}{H}\right)\right\}_{n=1}^{\infty}$ we have

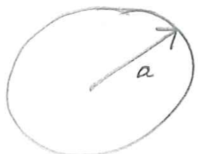
$$A_n \cdot \sinh\left(\frac{n\pi}{H}(-L)\right) = \frac{2}{H} \cdot \int_0^L g_1(y) \cdot \sin\left(\frac{n\pi y}{H}\right) dy$$

$$(**) \implies A_n = \frac{2}{H \cdot \sinh\left(\frac{n\pi}{H}(-L)\right)} \cdot \int_0^L g(y) \cdot \sin\left(\frac{n\pi y}{H}\right) dy$$

Thus, $(*)$ with $(**)$ is the desired solution to the problem for $u_4(x,y)$.

Laplace's equation for a circular disk

Consider Laplace's equation on a circular disk of radius a with prescribed temperature on the boundary of the disk:



Due to the geometry of the problem use polar coordinates (r, θ):

$$u = u(r, \theta), \quad \begin{array}{l} 0 \le r \le a \\ -\pi \le \theta \le \pi \end{array}$$

(PDE) $$\Delta u = \frac{1}{r} \frac{\partial}{\partial r}\left(r \cdot \frac{\partial u}{\partial r}\right) + \frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2} = 0$$

Laplace operator Δ in polar coordinates

(BC) $$u(a, \theta) = f(\theta)$$

The use of polar coordinates requires some additional compatibility conditions:

- polar coordinates are singular at $r = 0$; for physical reasons require

$$|u(0, \theta)| < \infty$$

- periodicity condition: $\theta = -\pi$ and $\theta = \pi$ correspond to the same point!

$$u(r, -\pi) = u(r, \pi)$$

$$\frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi)$$

Note: The compatibility conditions are all linear and homogeneous.

we try to use the method of separation of variables and make the product ansatz:

$$u(r,\theta) = \phi(\theta) \cdot G(r),$$

where we need to satisfy the three compatibility conditions, but ignore the nonhomogeneous BC $u(a,\theta) = f(\theta)$ for the moment:

$$\phi(-\pi) = \phi(\pi)$$
$$\frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi)$$

Inserting the product ansatz into the PDE

$$\frac{1}{r}\frac{d}{dr}\left(r \cdot \frac{dG}{dr}\right) \cdot \phi + \frac{1}{r^2} G(r) \cdot \frac{d^2\phi}{d\theta^2} = 0$$

Multiply by r^2 and divide by $G(r)\cdot\phi(\theta)$ to get

$$\frac{r}{G} \cdot \frac{d}{dr}\left(r \cdot \frac{dG}{dr}\right) + \frac{1}{\phi} \cdot \frac{d^2\phi}{d\theta^2} = 0$$

$$\Rightarrow \underbrace{\frac{r}{G} \cdot \frac{d}{dr}\left(r \cdot \frac{dG}{dr}\right)}_{\text{only } r\text{-dependent}} = \underbrace{-\frac{1}{\phi} \cdot \frac{d^2\phi}{d\theta^2}}_{\text{only } \theta\text{-dependent}} = \lambda$$

separation constant $\lambda \in \mathbb{R}$

We first solve the EVP for ϕ:

$$\frac{d^2\phi}{d\theta^2} = -\lambda \cdot \phi$$

$$\phi(-\pi) = \phi(\pi)$$

$$\frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi)$$

$\Rightarrow$ By analogy to the problem for the circular wire with $L = \pi$:

Eigenvalues : $\lambda = \left(\frac{n\pi}{L}\right)^2 = n^2$, $n = \underline{\underline{0}}, 1, 2, \ldots$

Eigenfunctions : $\sin(n\theta)$, $\cos(n\theta)$

Now we turn to the ODE for $G(r)$:

$$\frac{r}{G} \cdot \underbrace{\frac{d}{dr}\left(r \cdot \frac{dG}{dr}\right)}_{= r \cdot \frac{d^2G}{dr^2} + \frac{dG}{dr}} = \lambda = n^2$$

$$\Rightarrow r^2 \cdot \frac{d^2G}{dr^2} + r \cdot \frac{dG}{dr} - n^2 \cdot G = 0$$

From $|u(0,\theta)| < \infty$ we inherit the requirement

$$|G(0)| < \infty.$$

This is a 2^{nd} order linear ODE with variable coefficients, which are in general very difficult to solve explicitly.

Here we are lucky though, because the linear operator $r^2 \frac{d^2}{dr^2} + r \cdot \frac{d}{dr} - n^2$ has the property that any power $G(r) = r^n$ "reproduces" itself!

Substitute $G(r) = r^\mu$ to get

$$\mu \cdot (\mu-1) \cdot r^\mu + \mu \cdot r^\mu - n^2 \cdot r^\mu = 0$$

$$\Rightarrow \quad (\mu(\mu-1) + \mu - n^2) \cdot r^\mu = 0$$

$$\Rightarrow \quad (\mu^2 - n^2) \cdot r^\mu = 0$$

$$\Rightarrow \quad \mu = \pm n$$

Thus, for $n \neq 0$ the general solution is

$$G(r) = c_1 \cdot r^n + c_2 \cdot r^{-n}, \qquad c_1, c_2 \in \mathbb{R}.$$

For $n = 0$, one solution is $r^0 = 1$ (constant sol.).
To find another independent solution, note that for $n = 0$

$$\frac{d}{dr}\left(r \cdot \frac{dG}{dr}\right) = 0$$

$$\Rightarrow \quad \frac{dG}{dr} = \frac{\text{const.}}{r}$$

$$\Rightarrow \quad G(r) = c \cdot \ln(r)$$

Hence, for $n = 0$ the general solution is

$$G(r) = \bar{c}_1 + \bar{c}_2 \cdot \ln(r), \qquad c_1, c_2 \in \mathbb{R}.$$

The condition $|G(0)| < \infty$ rules out the $\ln(r)$ solution as well as the r^{-n} solution for $n \geq 1$. Thus,

$$G(r) = c \cdot r^n, \qquad n \geq 0.$$

Using the superposition principle we may put together all product solutions to arrive at the solution(s)

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n \cdot r^n \cdot \cos(n\theta) + \sum_{n=1}^{\infty} B_n \cdot r^n \cdot \sin(n\theta),$$

$$0 \leq r \leq a$$
$$-\pi \leq \theta \leq \pi$$

To satisfy the nonhomogeneous BC $u(a,\theta) = f(\theta)$ we must have

$$f(\theta) = \sum_{n=0}^{\infty} \underbrace{A_n \cdot a^n}_{\text{modified coefficient}} \cdot \cos(n\theta) + \sum_{n=1}^{\infty} \underbrace{B_n \cdot a^n}_{\text{modified coefficient}} \cdot \sin(n\theta).$$

Using the orthogonality formulas for the family of $\cos(n\theta)$, $\sin(n\theta)$ (as in the treatment of the circular wire with $\underline{\underline{L=\pi}}$) we conclude that

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta,$$

$$A_n \cdot a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta, \quad n \geq 1,$$

$$B_n \cdot a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta, \quad n \geq 1.$$