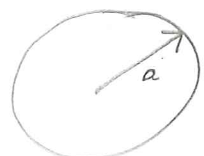
Laplace's equation for a circular disk

Consider Laplace's equation on a circular disk of radius a with prescribed temperature on the boundary of the disk:



Due to the geometry of the problem use polar coordinates (r, θ):

$$u = u(r, \theta) \,, \quad \begin{array}{l} 0 \le r \le a \\ -\pi \le \theta \le \pi \end{array}$$

(PDE) $$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2} = 0$$

Laplace operator Δ in polar coordinates

(BC) $$u(a, \theta) = f(\theta)$$

The use of polar coordinates requires some additional compatibility conditions:

- polar coordinates are singular at $r = 0$; for physical reasons require

$$|u(0, \theta)| < \infty$$

- periodicity condition: $\theta = -\pi$ and $\theta = \pi$ correspond to the same point!

$$u(r, -\pi) = u(r, \pi)$$

$$\frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi)$$

<u>Note:</u> The compatibility conditions are all <u>linear</u> and homogeneous.

we try to use the method of separation of variables and make the product ansatz:

$$u(r,\theta) = \phi(\theta) \cdot G(r),$$

where we need to satisfy the three compatibility conditions, but ignore the nonhomogeneous BC $u(a,\theta) = f(\theta)$ for the moment:

$$\phi(-\pi) = \phi(\pi)$$

$$\frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi)$$

Inserting the product ansatz into the PDE

$$\frac{1}{r}\frac{d}{dr}\left(r \cdot \frac{dG}{dr}\right) \cdot \phi + \frac{1}{r^2} G(r) \cdot \frac{d^2\phi}{d\theta^2} = 0$$

Multiply by r^2 and divide by $G(r) \cdot \phi(\theta)$ to get

$$\frac{r}{G} \cdot \frac{d}{dr}\left(r \cdot \frac{dG}{dr}\right) + \frac{1}{\phi} \cdot \frac{d^2\phi}{d\theta^2} = 0$$

$$\Rightarrow \underbrace{\frac{r}{G} \cdot \frac{d}{dr}\left(r \cdot \frac{dG}{dr}\right)}_{\text{only } r\text{-dependent}} = \underbrace{-\frac{1}{\phi} \cdot \frac{d^2\phi}{d\theta^2}}_{\text{only } \theta\text{-dependent}} = \lambda$$

separation constant $\lambda \in \mathbb{R}$

We first solve the BVP for ϕ:

$$\frac{d^2\phi}{d\theta^2} = -\lambda \cdot \phi$$

$$\phi(-\pi) = \phi(\pi)$$

$$\frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi)$$

$\Rightarrow$ By analogy to the problem for the circular wire with $\underline{L = \pi}$:

Eigenvalues : $\lambda = \left(\frac{n\pi}{L}\right)^2 = n^2$, $n = \underline{\underline{0}}, 1, 2, \dots$

Eigenfunctions : $\sin(n\theta)$, $\cos(n\theta)$

Now we turn to the ODE for $G(r)$:

$$\frac{r}{G} \cdot \underbrace{\frac{d}{dr}\left(r \cdot \frac{dG}{dr}\right)}_{= r \cdot \frac{d^2G}{dr^2} + \frac{dG}{dr}} = \lambda = n^2$$

$$\Rightarrow r^2 \cdot \frac{d^2G}{dr^2} + r \cdot \frac{dG}{dr} - n^2 \cdot G = 0$$

From $|u(0,\theta)| < \infty$ we inherit the requirement

$$|G(0)| < \infty.$$

This is a 2^{nd} order linear ODE with variable coefficients, which are in general very difficult to solve explicitly.

Here we are lucky though, because the linear operator $r^2 \frac{d^2}{dr^2} + r \cdot \frac{d}{dr} - n^2$. has the property that any power $G(r) = r^n$ "reproduces" itself!

Substitute $G(r) = r^\mu$ to get

$$\mu\cdot(\mu-1)\cdot r^\mu + \mu\cdot r^\mu - n^2\cdot r^\mu = 0$$

$$\Rightarrow \quad (\mu(\mu-1) + \mu - n^2)\cdot r^\mu = 0$$

$$\Rightarrow \quad (\mu^2 - n^2)\cdot r^\mu = 0$$

$$\Rightarrow \quad \mu = \pm n$$

Thus, for $n \neq 0$ the general solution is

$$G(r) = c_1\cdot r^n + c_2\cdot r^{-n}, \qquad c_1, c_2 \in \mathbb{R}.$$

For $n = 0$, one solution is $r^0 = 1$ (constant sol.).

To find another independent solution, note that for $n = 0$

$$\frac{d}{dr}\left(r\cdot\frac{dG}{dr}\right) = 0$$

$$\Rightarrow \quad \frac{dG}{dr} = \frac{\text{const.}}{r}$$

$$\Rightarrow \quad G(r) = c\cdot\ln(r)$$

Hence, for $n = 0$ the general solution is

$$G(r) = \bar{c}_1 + \bar{c}_2\cdot\ln(r), \qquad c_1, c_2 \in \mathbb{R}.$$

The condition $|G(0)| < \infty$ rules out the $\ln(r)$ solution as well as the r^{-n} solution for $n \geq 1$. Thus,

$$G(r) = c\cdot r^n, \qquad n \geq 0.$$

Using the superposition principle we may put together all product solutions to arrive at the solution(s)

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n \cdot r^n \cdot \cos(n\theta) + \sum_{n=1}^{\infty} B_n \cdot r^n \cdot \sin(n\theta),$$

$$0 \le r \le a$$
$$-\pi \le \theta \le \pi$$

To satisfy the nonhomogeneous BC $u(a,\theta) = f(\theta)$ we must have

$$u(a,\theta \quad f(\theta) = \sum_{n=0}^{\infty} \underbrace{A_n \cdot a^n}_{\text{modified coefficient}} \cdot \cos(n\theta) + \sum_{n=1}^{\infty} \underbrace{B_n \cdot a^n}_{\text{modified coefficient}} \cdot \sin(n\theta).$$

Using the orthogonality formulas for the family of $\cos(n\theta)$, $\sin(n\theta)$ (as in the treatment of the circular wire with $\underline{L=\pi}$) we conclude that

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta,$$

$$A_n \cdot a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta, \quad n \geq 1,$$

$$B_n \cdot a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta, \quad n \geq 1.$$

Qualitative properties of Laplace's equation

[Honors Material]

Mean-value theorem

If we evaluate the solution to Laplace's equation on a circular disk at the origin $r=0$, we find that

$$u(0,\theta) = A_0 = \frac{1}{2\pi} \cdot \int_{-\pi}^{+\pi} f(\theta)\, d\theta$$

Thus, the temperature at the origin is the <u>average value</u> of the temperature on the boundary of the circle!

This <u>mean-value property</u> for Laplace's equation holds more generally:

<u>Theorem</u>:

Let $R \subset \mathbb{R}^2$ be a region and let u be a solution to $\Delta u = 0$ in R.

For any $x_0 \in R$ and $r > 0$ such that

$$\mathbb{B}(x_0, r) := \{ y \in \mathbb{R}^2 \mid |y - x_0| < r \} \subset R,$$

it holds that

$$u(x_0) = \frac{1}{2\pi r} \int_{\partial \mathbb{B}(x_0,r)} u(y)\, dS(y)$$

surface measure on boundary $\partial\mathbb{B}(x_0,r)$

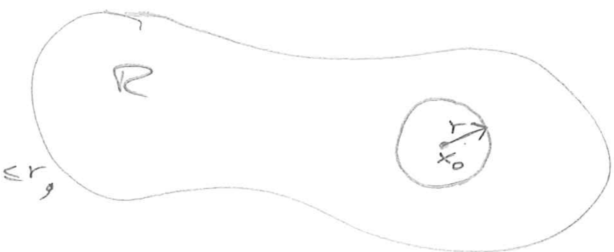
boundary of disk $\mathbb{B}(x_0,r)$

$$\partial\mathbb{B}(x_0,r) = \{ y \in \mathbb{R}^2 \mid |y - x_0| = r \}.$$

Proof:

⇝ We could just use our previous analysis for the disk $\mathbb{B}(x_0,r)$ via explicit computation, but we can also argue indirectly as follows:

Define



$$\phi(s) := \frac{1}{2\pi s}\int_{\partial\mathbb{B}(x_0,\underline{\underline{s}})} u(y)\, dS(y), \quad 0<s\le r,$$

we first show that $\phi(s)$ is constant.

Once we know that, we just observe that then

$$\frac{1}{2\pi r}\int_{\partial\mathbb{B}(x_0,r)} u(y)\, dS(y) = \phi(r) \underset{\substack{\uparrow \\ \phi \text{ constant}}}{=} \lim_{s\to 0}\phi(s) = u(x_0)$$

and we are done!

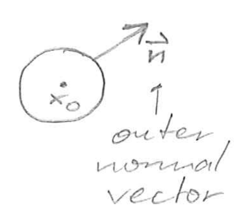
To conclude that $\phi(s)$ is constant, we compute

$$\phi'(s) = \frac{d\phi}{ds} = \frac{d}{ds}\left(\frac{1}{2\pi \cancel{s}}\int_{\partial\mathbb{B}(x_0,\underline{\underline{1}})} u(x_0+sz)\,\cancel{s}\, dS(z)\right)$$

surface measure on $\partial\mathbb{B}(x_0,\underline{\underline{1}})$

$$= \frac{1}{2\pi}\int_{\partial\mathbb{B}(x_0,\underline{\underline{1}})} (\nabla u)(x_0+sz)\cdot z\, dS(z)$$

$$= \frac{1}{2\pi \underline{\underline{s}}}\int_{\partial\mathbb{B}(x_0,s)} (\nabla u)(y)\cdot \vec{n}\, dS(y)$$



divergence theorem

$$\overset{\text{div. thm.}}{=} \frac{1}{2\pi s}\int_{\mathbb{B}(x_0,s)} \underbrace{\nabla\cdot(\nabla u)(y)}_{=(\Delta u)(y)=0}\, dy$$

u solves Laplace's equation!

$$= \underline{\underline{0}} \quad ☺$$

Maximum principle

A solution to Laplace's equation in a region $R \subset \mathbb{R}^2$ cannot attain its maximum in the interior of the region R (unless the temperature is constant everywhere).

Proof: (by contradiction)

Let $R \subset \mathbb{R}^2$ be a region and let u be a solution to Laplace's equation

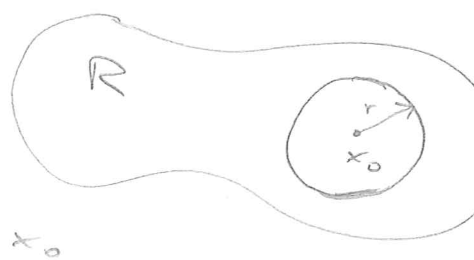
$$\Delta u = 0 \quad \text{in } R.$$

Let $x_0 \in R$ be a point in the interior of R and assume that

$$u(x_0) = M := \max_{x \in R} u(x),$$

i.e. u assumes its maximum at x_0.

Now for any



$0 < r < \operatorname{dist}(x_0, \partial R)$

distance of x_0 to the boundary ∂R

we have by the mean-value property

$$M = u(x_0) = \frac{1}{2\pi r} \int_{\partial B(x_0,r)} \underbrace{u(y)}_{\leq M !} \, dS(y) \leq M$$

We arrive at a contradiction; thus the statement must be true.

not possible unless u is constant everywhere!

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Analogously, we can show that the minimum cannot be attained in the interior of R. (Minimum principle)

Thus, a solution to Laplace's equation $\Delta u = 0$ in R must assume its maximum and its minimum on the boundary of the region R.

Well-posedness and uniqueness

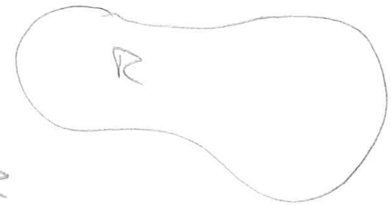
Definition:

We say that a PDE is well-posed if there exists a unique solution that depends continuously on the (nonhomogeneous) data, i.e. the solution only varies a small amount if the data are slightly changed.

→ This is an important property for a PDE to be relevant for problems in physics because in nature things usually only change by small degrees.

We can use the maximum and minimum principles to show that Laplace's equation is well-posed:

Let $R \subset \mathbb{R}^2$ be a region and let $f(x)$ and $g(x)$ be boundary data (for the boundary of R).



Let $\Delta u = 0$ in R with $u(x) = f(x)$ on the boundary of R and let $\Delta v = 0$ in R with $v(x) = g(x)$ on the boundary of R.

$\Rightarrow$ $\Delta(u-v) = 0$ in R with
$(u-v)(x) = f(x) - g(x)$ on the boundary of R

$\Rightarrow$ By maximum / minimum principle:

$$\min_{\text{bdry}}(u-v) = \min(f-g) \le u - v \le \max(f-g) = \max_{\text{bdry}}(u-v)$$

Hence, if $f(x)$ and $g(x)$ differ only a little bit, then the corresponding solutions u and v to Laplace's equation also only differ a little bit!

Moreover, we can prove that the solution to Laplace's equation is unique:

Proof: (by contradiction)

Suppose there are two solutions u and v ($\Delta u = 0$ and $\Delta v = 0$) with the same boundary data ($u = f(x)$ and $v(x) = f(x)$ on the boundary).

By the maximum/minimum principle

$$0 = \underbrace{\min(f-g)}_{\substack{\min(u-v) \\ \text{boundary}}} \leq u - v \leq \underbrace{\max(f-f)}_{\substack{\max(u-v) \\ \text{boundary}}} = 0$$

$$\Rightarrow \quad u - v = 0 \quad \Rightarrow \quad \underline{\underline{u = v}}$$

$\boxtimes$