Qualitative properties of Laplace's equation

[Honors Material]

Mean-value theorem

If we evaluate the solution to Laplace's equation on a circular disk at the origin $r=0$, we find that

$$u(0,\theta) = A_0 = \frac{1}{2\pi}\cdot\int_{-\pi}^{+\pi} f(\theta)\, d\theta$$

Thus, the temperature at the origin is the average value of the temperature on the boundary of the circle!

This mean-value property for Laplace's equation holds more generally:

Theorem:

Let $R \subset \mathbb{R}^2$ be a region and let u be a solution to $\Delta u = 0$ in R.

For any $x_0 \in R$ and $r > 0$ such that

$$\mathbb{B}(x_0,r) := \{ y \in \mathbb{R}^2 \mid |y-x_0| < r \} \subset R,$$

it holds that

$$u(x_0) = \frac{1}{2\pi r} \int_{\partial\mathbb{B}(x_0,r)} u(y)\, dS(y)$$

surface measure on boundary $\partial\mathbb{B}(x_0,r)$

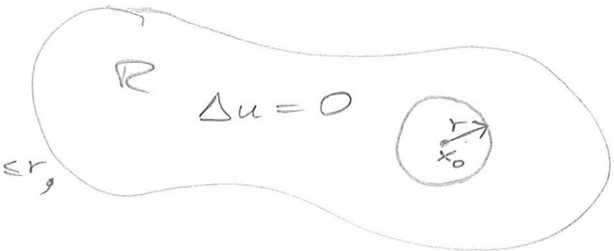
boundary of disk $\mathbb{B}(x_0,r)$

$$\partial\mathbb{B}(x_0,r) = \{ y \in \mathbb{R}^2 \mid |y-x_0| = r \}.$$

Proof:

⇝ We could just use our previous analysis for the disk $\mathbb{B}(x_0, r)$ via explicit computation, but we can also argue indirectly as follows:

Define



$$\phi(s) := \frac{1}{2\pi s} \int_{\partial \mathbb{B}(x_0, s)} u(y)\, dS(y), \quad 0 < s \leq r,$$

We first show that $\phi(s)$ is constant.
Once we know that, we just observe that then

$$\frac{1}{2\pi r} \int_{\partial \mathbb{B}(x_0, r)} u(y)\, dS(y) = \phi(r) \underset{\substack{\uparrow \\ \phi \text{ constant}}}{=} \lim_{s \to 0} \phi(s) = u(x_0)$$

and we are done!

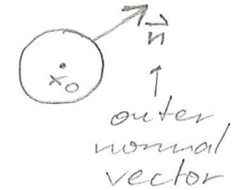
To conclude that $\phi(s)$ is constant, we compute

$$\phi'(s) = \frac{d\phi}{ds} = \frac{d}{ds}\left(\frac{1}{2\pi \cancel{s}} \int_{\partial \mathbb{B}(x_0, 1)} u(x_0 + sz)\, \cancel{s}\, dS(z) \right)$$

surface measure on $\partial \mathbb{B}(x_0, 1)$

$$= \frac{1}{2\pi} \int_{\partial \mathbb{B}(x_0, 1)} (\nabla u)(x_0 + sz) \cdot z\, dS(z)$$

$$= \frac{1}{2\pi s} \int_{\partial \mathbb{B}(x_0, s)} (\nabla u)(y) \cdot \vec{n}\, dS(y)$$



divergence theorem

$$\overset{\text{divergence theorem}}{=} \frac{1}{2\pi s} \int_{\mathbb{B}(x_0, s)} \underbrace{\nabla \cdot (\nabla u)(y)}_{= (\Delta u)(y) = 0}\, dy$$

u solves Laplace's equation!

$$= \underline{\underline{0}} \quad ☺$$

Maximum principle

A solution to Laplace's equation in a region $R \subset \mathbb{R}^2$ cannot attain its maximum in the interior of the region R (unless the temperature is constant everywhere).

Proof: (by contradiction)

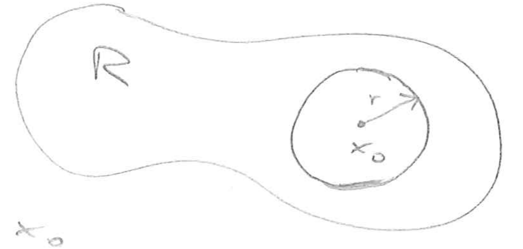
Let $R \subset \mathbb{R}^2$ be a region and let u be a solution to Laplace's equation

$$\Delta u = 0 \quad \text{in } R.$$

Let $x_0 \in R$ be a point in the interior of R and assume that

$$u(x_0) = M := \max_{x \in R} u(x),$$

i.e. u assumes its maximum at x_0.



Now for any

$0 < r < \operatorname{dist}(x_0, \partial R)$

distance of x_0 to the boundary ∂R

we have by the mean-value property

$$M = u(x_0) = \frac{1}{2\pi r} \int_{\partial B(x_0,r)} \underbrace{u(y)}_{\leq M\,!} \, dS(y) \leq M$$

not possible unless u is constant everywhere!

We arrive at a contradiction; thus the statement must be true.

Analogously, we can show that the minimum cannot be attained in the interior of R. (Minimum principle)

Thus, a solution to Laplace's equation $\Delta u = 0$ in R must assume its maximum and its minimum on the boundary of the region R.

Well-posedness and uniqueness

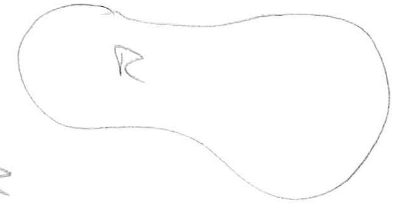
Definition:

We say that a PDE is well-posed if there exists a unique solution that depends continuously on the (nonhomogeneous) data, i.e. the solution only varies a small amount if the data are slightly changed.

⇝ This is an important property for a PDE to be relevant for problems in physics because in nature things usually only change by small degrees.

We can use the maximum and minimum principles to show that Laplace's equation is well-posed:

Let $R \subset \mathbb{R}^2$ be a region and let $f(x)$ and $g(x)$ be boundary data (for the boundary of R).



Let $\Delta u = 0$ in R with $u(x) = f(x)$ on the boundary of R
and let $\Delta v = 0$ in R with $v(x) = g(x)$ on the boundary of R.

$\Rightarrow$ $\Delta(u-v) = 0$ in R with
$(u-v)(x) = f(x) - g(x)$ on the boundary of R

$\Rightarrow$ By maximum / minimum principle:

$$\min_{bdry}(u-v) = \min(f-g) \le u - v \le \max_{bdry}(f-g) = \max_{bdry}(u-v)$$

Hence, if $f(x)$ and $g(x)$ differ only a little bit, then the corresponding solutions u and v to Laplace's equation also only differ a little bit!

Moreover, we can prove that the solution to Laplace's equation is unique:

Proof: (by contradiction)

Suppose there are two solutions u and v ($\Delta u = 0$ and $\Delta v = 0$) with the same boundary data ($u = f(x)$ and $v(x) = f(x)$ on the boundary).

By the maximum/minimum principle

$$0 = \underbrace{\min(f-g)}_{\substack{\min(u-v) \\ \text{boundary}}} \leq u - v \leq \underbrace{\max(f-f)}_{\substack{\max(u-v) \\ \text{boundary}}} = 0$$

$$\Rightarrow \quad u - v = 0 \quad \Rightarrow \quad \underline{\underline{u = v}} \qquad \boxtimes$$

Chapter 3: Fourier Series

As we developed the method of separation of variables in the previous chapter, we relied on the fact that "any reasonable" initial condition can be written as an infinite series of sines or cosines or both, known as a <u>Fourier series</u>:

Given a function $f(x)$ on an interval $-L \leq x \leq +L$, can we write

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right), \quad -L \leq x \leq +L$$

for suitable coefficients a_n, b_n?

In which sense does such an infinite series converge to $f(x)$? What conditions on $f(x)$ are needed?

We try to answer these questions in this chapter. To this end we first need to introduce some definitions.

Definition:

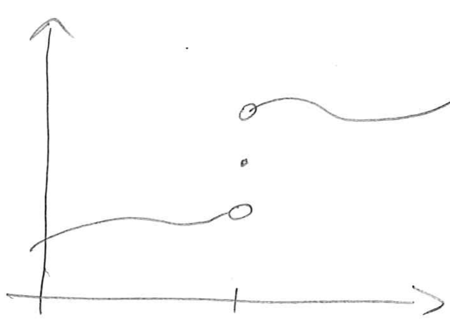
Let $[c,d]$ be an interval. A function $f:[c,d]\to\mathbb{R}$ is *piecewise smooth* on $[c,d]$ if the interval $[c,d]$ can be partitioned into finitely many consecutive subintervals such that inside each subinterval $f(x)$ and its derivative $f'(x)$ are continuous.

Moreover, $f(x)$ is only allowed to have jump discontinuities at the intersections of the subintervals.

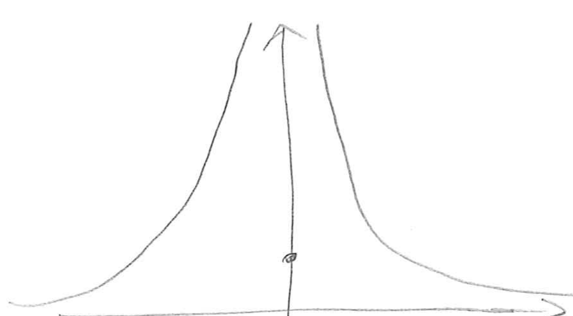
→ For us a "reasonable" function is piecewise smooth.

Recall:

A function $f(x)$ has a *jump discontinuity* at a point $x=x_0$ if the one-sided limits $f(x_0^-)=\lim\limits_{x\uparrow x_0} f(x)$ and $f(x_0^+)=\lim\limits_{x\downarrow x_0} f(x)$ <u>exist</u> and are <u>unequal</u>.

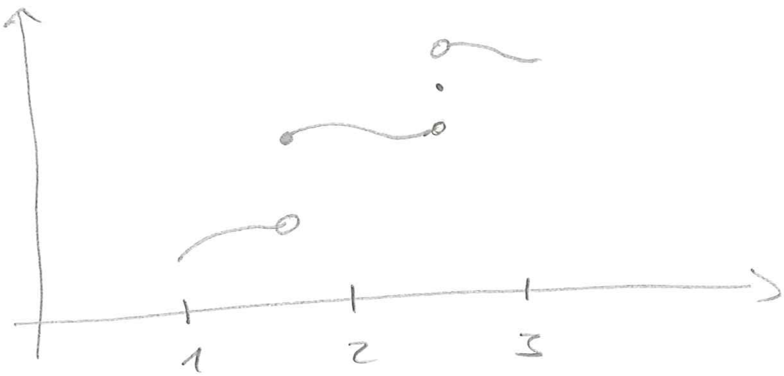


Example of a function with a jump discontinuity at $x=x_0$



The function
$$f(x)=\begin{cases}\frac{1}{x^2}, & x\neq 0\\ 1, & x=0\end{cases}$$
does <u>not</u> have a jump discontinuity at $x_0=0$

Example of a piecewise smooth function on $[1,3]$:



<u>Definition:</u>

A function $f: \mathbb{R} \to \mathbb{R}$ (defined on the whole line) is <u>periodic</u> with period P if

$$f(x+P) = f(x) \qquad \text{for all } x \in \mathbb{R}.$$

<u>Example:</u>

The functions $\cos\left(\frac{n\pi x}{L}\right)$, $n = 0, 1, 2, \ldots$, and $\sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, 3, \ldots$, are periodic with period $2L$.

[Hence, the series

$$a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right)$$

is periodic with period $2L$]

Given a function $f(x)$ on an interval $-L \le x \le +L$, we introduce the periodic extension $f_{per}(x)$ of $f(x)$ to the whole line by

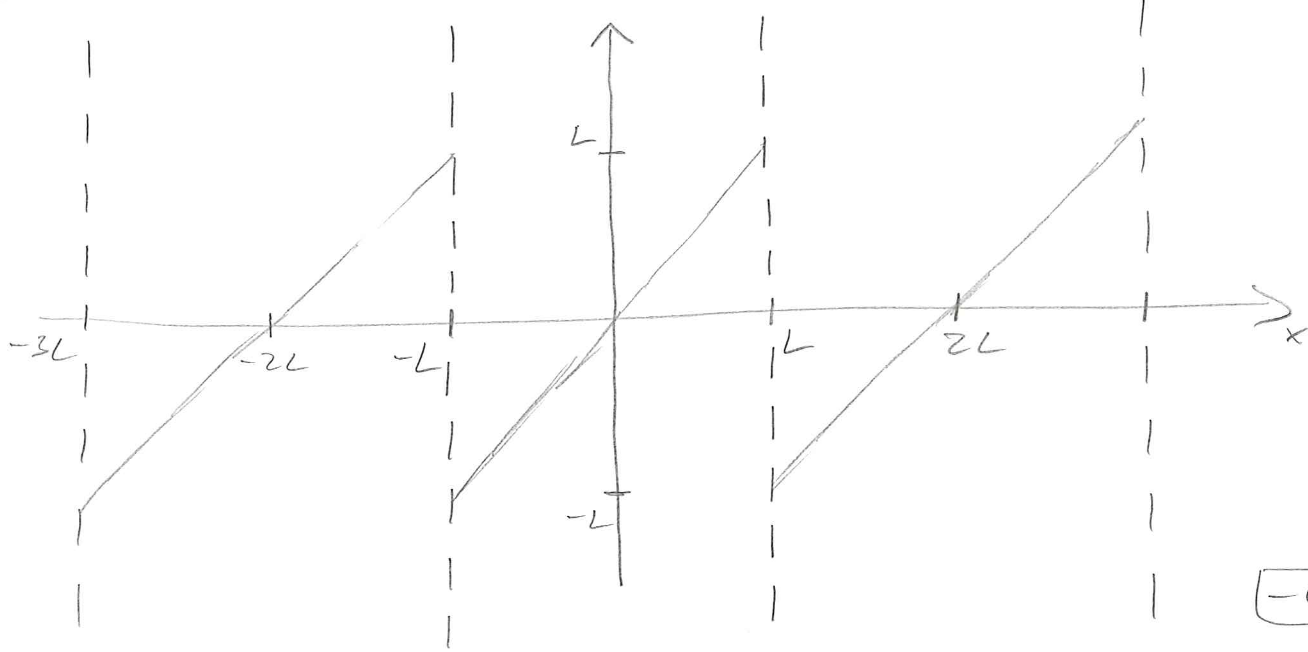
$$f_{per}(x \pm m \cdot 2L) := f(x) \quad \text{for } -L < x \le +L \text{ and any integer } m \in \mathbb{Z}.$$

In other words, in order to sketch the periodic extension of $f(x)$, simply sketch $f(x)$ for $-L < x \le +L$ and then keep repeating the pattern by translating the original sketch.

Note:

Unless $f(-L) = f(+L)$, the periodic extension of $f(x)$ from the interval $-L \le x \le +L$ to the whole line has jump discontinuities at $x = L + m \cdot 2L$, $m \in \mathbb{Z}$.

Example: Periodic extension of $f(x) = x$, $-L \le x \le +L$



Definition: (Fourier series)

Given a function $f(x)$ over the interval $-L \leq x \leq +L$, we define its Fourier series $(Sf)(x)$ to be the infinite series

$$(Sf)(x) := a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right),$$

["S" for "sum"]

where the Fourier coefficients a_n, b_n are defined by

$$a_0 := \frac{1}{2L} \cdot \int_{-L}^{+L} f(x)\, dx,$$

$$a_n := \frac{1}{L} \int_{-L}^{+L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx,$$

$$b_n := \frac{1}{L} \int_{-L}^{+L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Note:

- For any piecewise smooth function on $[-L, L]$, the Fourier coefficients are well-defined. (In contrast, for the function $f(x) = \frac{1}{x^2}$, the integral $a_0 := \frac{1}{2L} \int_{-L}^{L} \frac{1}{x^2} dx$ does not exist due to the singularity at $x=0$; but $f(x) = \frac{1}{x^2}$ is not piecewise smooth on any interval $[-L, L]$).
- The Fourier series $(Sf)(x)$ is defined on the whole line, i.e. for any $x \in \mathbb{R}$.

The following fundamental theorem describes in which sense the Fourier series $(Sf)(x)$ of a piecewise smooth function converges and what its relation to the original function $f(x)$ is:

Fourier's theorem:

Let $f: [-L, +L] \to \mathbb{R}$ be piecewise smooth.

Then for any $x \in \mathbb{R}$, the Fourier series $(Sf)(x)$ converges (pointwise)

(i) to the periodic extension $f_{per}(x)$ of f, where the periodic extension is continuous;

(ii) to the average of the two one-sided limits

$$\frac{1}{2}\left(f_{per}(x+) + f_{per}(x-)\right)$$

where the periodic extension has a jump discontinuity.

Recall:

$$f_{per}(x_0+) :\equiv \lim_{x \downarrow x_0} f(x), \quad f_{per}(x_0-) := \lim_{x \uparrow x_0} f(x).$$

$\rightsquigarrow$ We will not discuss the proof of this theorem, but I'll give you some intuition later.