Chapter 3: Fourier Series

As we developed the method of separation of variables in the previous chapter, we relied on the fact that "any reasonable" initial condition can be written as an infinite series of sines or cosines or both, known as a <u>Fourier series</u>:

Given a function $f(x)$ on an interval $-L \leq x \leq +L$, can we write

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right), \quad -L \leq x \leq +L$$

for suitable coefficients a_n, b_n?

In which sense does such an infinite series converge to $f(x)$? What conditions on $f(x)$ are needed?

We try to answer these questions in this chapter. To this end we first need to introduce some definitions.

Definition:

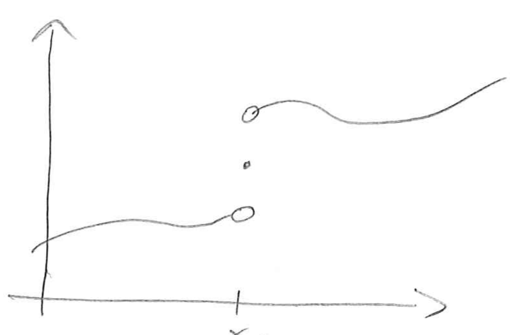
Let $[c,d]$ be an interval. A function $f: [c,d] \to \mathbb{R}$ is piecewise smooth on $[c,d]$ if the interval $[c,d]$ can be partitioned into finitely many consecutive subintervals such that inside each subinterval $f(x)$ and its derivative $f'(x)$ are continuous.

Moreover, $f(x)$ is only allowed to have jump discontinuities at the intersections of the subintervals.

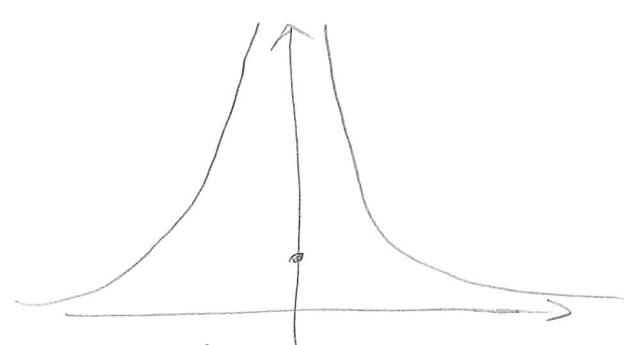
$\rightsquigarrow$ For us a "reasonable" function is piecewise smooth.

Recall:

A function $f(x)$ has a jump discontinuity at a point $x = x_0$ if the one-sided limits $f(x_0-) = \lim\limits_{x \uparrow x_0} f(x)$ and $f(x_0+) = \lim\limits_{x \downarrow x_0} f(x)$ exist and are unequal.

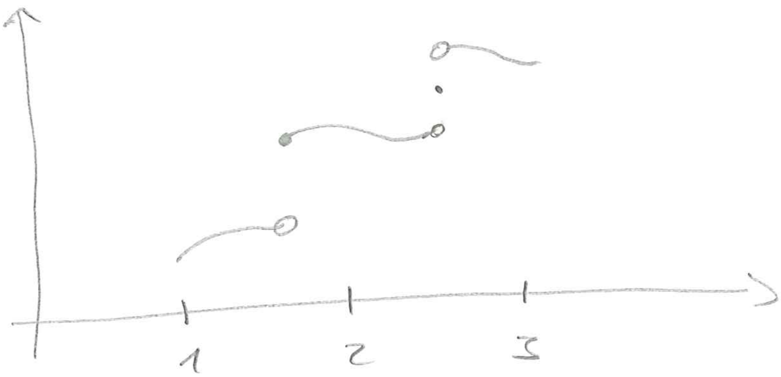


Example of a function with a jump discontinuity at $x = x_0$



The function $f(x) = \begin{cases} \frac{1}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ does not have a jump discontinuity at $x_0 = 0$

Example of a piecewise smooth function on $[1, 3]$:



Definition:

A function $f: \mathbb{R} \to \mathbb{R}$ (defined on the whole line) is periodic with period P if

$$f(x+P) = f(x) \qquad \text{for all } x \in \mathbb{R}.$$

Example:

The functions $\cos\left(\frac{n\pi x}{L}\right)$, $n = 0, 1, 2, \ldots$, and $\sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, 3, \ldots$, are periodic with period $2L$.

(Hence, the series

$$a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right)$$

is periodic with period $2L$)

Given a function $f(x)$ on an interval $-L \leq x \leq +L$, we introduce the *periodic extension* $f_{per}(x)$ of $f(x)$ to the whole line by

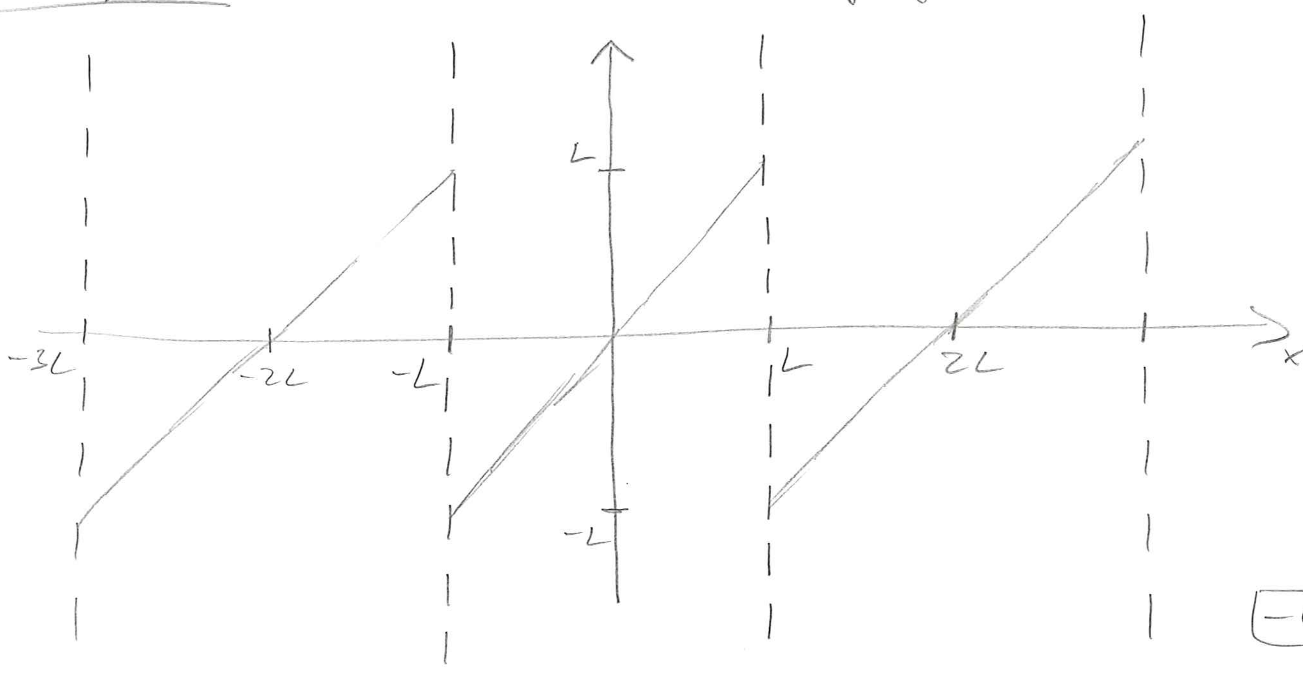
$$f_{per}(x \pm m \cdot 2L) := f(x) \quad \text{for } -L < x \leq +L \text{ and any integer } m \in \mathbb{Z}.$$

In other words, in order to sketch the periodic extension of $f(x)$, simply sketch $f(x)$ for $-L < x \leq +L$ and then keep repeating the pattern by translating the original sketch.

Note:

Unless $f(-L) = f(+L)$, the periodic extension of $f(x)$ from the interval $-L \leq x \leq +L$ to the whole line has jump discontinuities at $x = L + m \cdot 2L$, $m \in \mathbb{Z}$.

Example: Periodic extension of $f(x) = x$, $-L \leq x \leq +L$



Definition: (Fourier series)

Given a function $f(x)$ over the interval $-L \leq x \leq +L$, we define its Fourier series $(Sf)(x)$ to be the infinite series

$$(Sf)(x) := a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right),$$

["S" for "sum"]

where the Fourier coefficients a_n, b_n are defined by

$$a_0 := \frac{1}{2L} \cdot \int_{-L}^{+L} f(x)\, dx,$$

$$a_n := \frac{1}{L} \int_{-L}^{+L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx,$$

$$b_n := \frac{1}{L} \int_{-L}^{+L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Note:

- For any piecewise smooth function on $[-L, L]$, the Fourier coefficients are well-defined. (In contrast, for the function $f(x) = \frac{1}{x^2}$, the integral $a_0 := \frac{1}{2L} \int_{-L}^{L} \frac{1}{x^2}\, dx$ does not exist due to the singularity at $x=0$; but $f(x) = \frac{1}{x^2}$ is not piecewise smooth on any interval $[-L, L]$).
- The Fourier series $(Sf)(x)$ is defined on the whole line, i.e. for any $x \in \mathbb{R}$.

The following fundamental theorem describes in which sense the Fourier series $(Sf)(x)$ of a piecewise smooth function converges and what its relation to the original function $f(x)$ is:

Fourier's theorem:

Let $f: [-L, +L] \to \mathbb{R}$ be piecewise smooth.
Then for any $x \in \mathbb{R}$, the Fourier series $(Sf)(x)$ converges (pointwise)

(i) to the periodic extension $f_{per}(x)$ of f, where the periodic extension is continuous;

(ii) to the average of the two one-sided limits

$$\frac{1}{2}\left(f_{per}(x+) + f_{per}(x-)\right)$$

where the periodic extension has a jump discontinuity.

Recall:

$$f_{per}(x_0+) := \lim_{x \downarrow x_0} f(x), \quad f_{per}(x_0-) := \lim_{x \uparrow x_0} f(x).$$

$\rightsquigarrow$ We will not discuss the proof of this theorem, but I'll give you some intuition later.

We conclude that for a piecewise smooth function $f: [-L, L] \to \mathbb{R}$, it holds that

- for $-L < x < +L$ (inside the interval)

$$(Sf)(x) = \frac{1}{2}\left(f(x+) + f(x-)\right)$$

- for $-L < x < +L$ where $f(x)$ is continuous [then $f(x-) = f(x+)$]

$$(Sf)(x) = f(x)$$

- at the endpoints $x = -L$ and $x = +L$

$$(Sf)(+L) = (Sf)(-L) = \frac{1}{2} \cdot \left(f((-L)+) + f((+L)-)\right)$$

- outside the interval $-L \leq x \leq +L$, the values of $(Sf)(x)$ can be determined from the above using that the Fourier series $(Sf)(x)$ is periodic with period $2L$.

<u>Example:</u>

Let

$$f(x) = \begin{cases} 0, & -L \leq x < \frac{L}{2} \\ 1, & \frac{L}{2} \leq x \leq L \end{cases}$$

Determine the Fourier series $(Sf)(x)$ of the function $f(x)$ over the interval $-L \leq x \leq +L$.

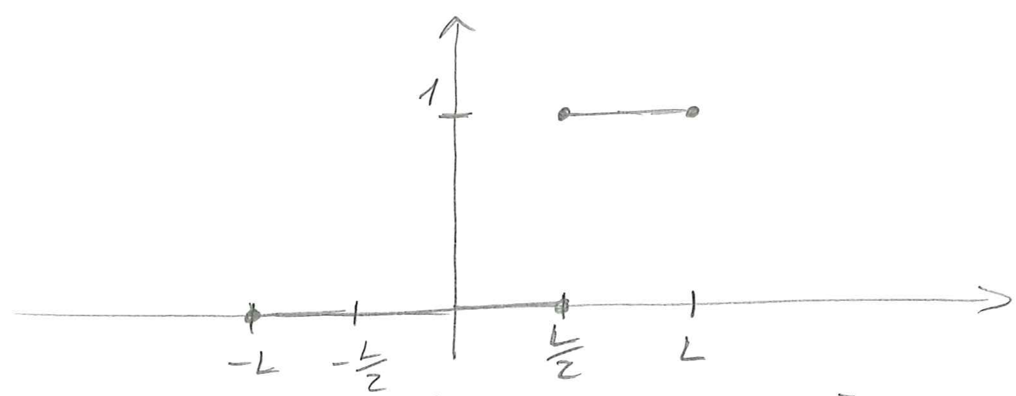
⇝ Note that $f(x)$ is piecewise smooth, so we can apply Fourier's theorem.

Strategy:

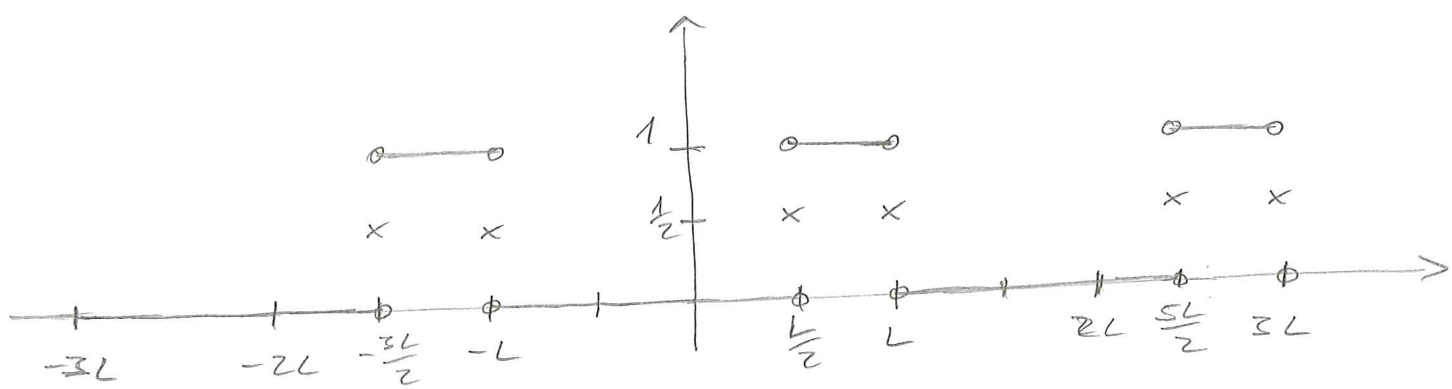
(1) Sketch $f(x)$ over $-L \le x \le +L$

(2) Sketch the periodic extension $f_{per}(x)$ of $f(x)$

(3) Determine the jump discontinuities of the periodic extension $f_{per}(x)$ and determine the average of the one-sided limits of $f_{per}(x)$ there.
[Mark those average values with an $\times$ on the graph]



graph of $f(x)$ on $[-L,+L]$



Values of $(Sf)(x)$ on $[-L,+L]$:

$$(Sf)(x) = \begin{cases} \frac{1}{2} & , x = L \\ 1 & , \frac{L}{2} < x < L \\ \frac{1}{2} & , x = \frac{L}{2} \\ 0 & , -L < x < \frac{L}{2} \\ \frac{1}{2} & , x = -L \end{cases}$$

Fourier coefficients

Thanks to Fourier's theorem and the above strategy, we actually do not have to compute the Fourier coefficients a_n, b_n in order to sketch the graph of the Fourier series of a piecewise smooth function!
However, it is still important to know how to calculate Fourier coefficients.
Sometimes this can be a tricky exercise in evaluating integrals (where integration by parts is often helpful).

Example: (continued)

Fourier coefficients of

$$f(x) = \begin{cases} 0, & -L \leq x < \frac{L}{2} \\ 1, & \frac{L}{2} \leq x \leq L \end{cases}$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\,dx = \frac{1}{2L} \int_{\frac{L}{2}}^{L} 1\,dx = \frac{1}{2L} \cdot \frac{L}{2} = \underline{\underline{\frac{1}{4}}}$$

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L} \int_{\frac{L}{2}}^{L} \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L} \left[\frac{L}{n\pi} \cdot \sin\left(\frac{n\pi x}{L}\right) \right]_{x=\frac{L}{2}}^{x=L} \\ &= \frac{1}{n\pi} \Big(\underbrace{\sin(n\pi)}_{=0} - \sin\left(\frac{n\pi}{2}\right) \Big) = \underline{\underline{-\frac{1}{n\pi} \sin\left(\frac{n\pi}{2}\right)}} \end{aligned}$$

By carefully evaluating $\sin\left(\frac{n\pi}{2}\right)$, one could further simplify this to

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 \ , & n = 4m \\ 1 \ , & n = 4m+1 \\ 0 \ , & n = 4m+2 \\ -1 \ , & n = 4m+3 \ , \quad m \in \mathbb{Z}. \end{cases}$$

Similarly,

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L} \int_{\frac{L}{2}}^{L} \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L} \left[-\frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right) \right]_{x=\frac{L}{2}}^{x=L} \\ &= \frac{1}{n\pi} \left(-\cos(n\pi) + \cos\left(\frac{n\pi}{2}\right) \right). \end{aligned}$$

Interlude: (Honors Material)

The convergence of the Fourier series of a function $f(x)$ can be a delicate matter. Different types of convergence are possible and this all crucially depends on the smoothness of $f(x)$. [pointwise convergence, uniform convergence, L^2 convergence]

Generally speaking, one can say that the smoother $f(x)$, the stronger the convergence of its Fourier series:

To illustrate this, assume that $f(x)$ is periodic and continuously differentiable. Then we find for its Fourier coefficients a_n by integration by parts:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{= \frac{L}{n\pi}\frac{d}{dx}\left(\sin\left(\frac{n\pi x}{L}\right)\right)} dx$$

integration by parts

$$= \frac{1}{L}\cdot\frac{L}{n\pi}\cdot \underbrace{\left[f(x) \sin\left(\frac{n\pi x}{L}\right)\right]_{x=-L}^{x=L}}_{=0}$$

$$- \frac{1}{L}\cdot\frac{L}{n\pi}\int_{-L}^{L} f'(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$= -\frac{1}{n\pi}\int_{-L}^{L} f'(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

Thus, for all $n \geq 1$ (since $\left|\sin\left(\frac{n\pi x}{L}\right)\right| \leq 1$)

$$|a_n| \leq \frac{1}{n\cdot\pi}\int_{-L}^{L} |f'(x)|\, dx \implies |a_n| \leq \frac{C(f)}{n}$$

If f was twice continuously differentiable, we could integrate by parts once more and infer that

$$|a_n| \leq \frac{C(f)}{n^2}$$

stronger decay!

Hence, more smoothness for $f(x)$ implies more decay of the Fourier coefficients a_n, which in turn implies stronger convergence of the Fourier series.