# Fourier coefficients

Thanks to Fourier's theorem and the above strategy, we actually do not have to compute the Fourier coefficients $a_n, b_n$ in order to sketch the graph of the Fourier series of a piecewise smooth function!

However, it is still important to know how to calculate Fourier coefficients.

Sometimes this can be a tricky exercise in evaluating integrals (where integration by parts is often helpful).

Example: (continued)

Fourier coefficients of

$$f(x) = \begin{cases} 0, & -L \leq x < \frac{L}{2} \\ 1, & \frac{L}{2} \leq x \leq L \end{cases}$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\,dx = \frac{1}{2L} \int_{\frac{L}{2}}^{L} 1\,dx = \frac{1}{2L} \cdot \frac{L}{2} = \underline{\underline{\frac{1}{4}}}$$

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L} \int_{\frac{L}{2}}^{L} \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L} \left[ \frac{L}{n\pi} \cdot \sin\left(\frac{n\pi x}{L}\right) \right]_{x=\frac{L}{2}}^{x=L} \\ &= \frac{1}{n\pi} \Big( \underbrace{\sin(n\pi)}_{=0} - \sin\left(\frac{n\pi}{2}\right) \Big) = \underline{\underline{-\frac{1}{n\pi} \sin\left(\frac{n\pi}{2}\right)}} \end{aligned}$$

By carefully evaluating $\sin\left(\frac{n\pi}{2}\right)$, one could find that further simplify this to

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & , \quad n = 4m \\ 1 & , \quad n = 4m+1 \\ 0 & , \quad n = 4m+2 \\ -1 & , \quad n = 4m+3 \end{cases} \quad , \quad m \in \mathbb{Z}.$$

Similarly,

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L} \int_{\frac{L}{2}}^{L} \sin\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L} \left[ -\frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right) \right]_{x=\frac{L}{2}}^{x=L} \\ &= \frac{1}{n\pi} \left( -\cos(n\pi) + \cos\left(\frac{n\pi}{2}\right) \right). \end{aligned}$$

Interlude: (Honors Material)

The convergence of the Fourier series of a function $f(x)$ can be a delicate matter. Different types of convergence are possible and this all crucially depends on the smoothness of $f(x)$. [pointwise convergence, uniform convergence, $L^2$ convergence]

Generally speaking, one can say that the smoother $f(x)$, the stronger the convergence of its Fourier series:

To illustrate this, assume that $f(x)$ is periodic and continuously differentiable. Then we find for its Fourier coefficients $a_n$ by integration by parts:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{= \frac{L}{n\pi}\frac{d}{dx}\left(\sin\left(\frac{n\pi x}{L}\right)\right)} dx$$

integration by parts

$$= \underbrace{\frac{1}{L}\cdot\frac{L}{n\pi}\cdot\left[f(x)\underbrace{\sin\left(\frac{n\pi x}{L}\right)}\right]_{x=-L}^{x=L}}_{=0}$$

$$- \frac{1}{L}\cdot\frac{L}{n\pi}\int_{-L}^{L} f'(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= -\frac{1}{n\pi}\int_{-L}^{L} f'(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Thus, for all $n \geq 1$ (since $|\sin(\frac{n\pi x}{L})| \leq 1$)

$$|a_n| \leq \frac{1}{n\cdot\pi}\int_{-L}^{L}|f'(x)|\,dx \Rightarrow |a_n| \leq \frac{C(f)}{n}$$

If $f$ was twice continuously differentiable, we could integrate by parts once more and infer that

$$|a_n| \leq \frac{C(f)}{n^2}$$

stronger decay!

Hence, more smoothness for $f(x)$ implies more decay of the Fourier coefficients $a_n$, which in turn implies stronger convergence of the Fourier series.

# Fourier Sine and Cosine Series

We will see in the following that the series of sines only and the series of cosines only are just special cases of a Fourier series.

## Fourier Sine Series

### Definition:

An odd function $f(x)$ satisfies the equation

$$f(-x) = -f(x).$$

### Examples

$\sin(x)$, $\sin(5x)$, $x$, $x^3$, ...

### Note:

- The sketch of an odd function for $x < 0$ is 'minus the mirror image for $x > 0$
- The integral of an odd function over a symmetric interval is zero

$$\int_{-L}^{+L} f(x)\,dx = 0.$$

## Fourier series of odd functions

Let $f(x)$, $-L \leq x \leq +L$, be odd. Then

$$a_0 = \frac{1}{2L} \cdot \int_{-L}^{+L} \underbrace{f(x)}_{\text{odd}}\, dx = 0$$

$$a_n = \frac{1}{L} \cdot \int_{-L}^{+L} \underbrace{f(x) \cdot \cos\left(\frac{n\pi x}{L}\right)}_{\substack{\text{odd function!} \\ \text{(the product of an odd an} \\ \text{even function is odd)}}} dx = 0$$

Thus, the Fourier series $(Sf)(x)$ of an odd function $f(x)$ is an infinite series of sines only

$$(Sf)(x) = \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right),$$

where the coefficients $b_n$ can be simplified a bit

$$(*1) \quad b_n := \frac{1}{L} \int_{-L}^{+L} f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

use change of variables $x \mapsto -x$ on $[-L, 0]$ and use ~~that~~ $f(x)$ and $\sin\left(\frac{n\pi x}{L}\right)$ are odd

## Fourier sine series

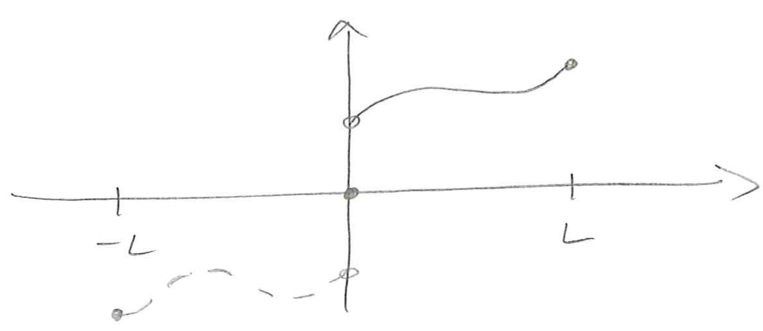
When we used the method of separation of variables to solve the heat equation on a rod of length $L$ with zero boundary conditions, we needed to express a given initial condition $f(x)$ on $0 \le x \le L$ as an infinite sine series

$$f(x) \overset{?}{=} \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right).$$

This looks like the Fourier series of an odd function, but $f(x)$ is only defined for $0 \le x \le L$ (so it does not make sense to think of $f(x)$ as odd)!

<u>Idea:</u> Define the <u>odd extension</u> $f_{odd}(x)$ of a given function $f(x)$ on $0 \le x \le L$ by

$$f_{odd}(x) := \begin{cases} f(x), & 0 < x \le L \\ -f(-x), & -L \le x < 0 \\ 0, & x = 0 \end{cases}$$



Since the odd extension $f_{odd}(x)$ is by construction an odd function, its Fourier series just consists of sines

$$S(f_{odd})(x) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right),$$

where from (*1) we have that

$$B_n = \frac{2}{L} \int_0^L \underbrace{f_{odd}(x)}_{= f(x) \text{ on } 0 \le x \le L\,!} \sin\left(\frac{n\pi x}{L}\right) dx$$

Since $f_{odd}(x)$ is identical to $f(x)$ on $0 \le x \le L$, we can restrict the Fourier series of $f_{odd}(x)$ to the interval $0 \le x \le L$ to obtain the <u>Fourier sine series of $f(x)$</u>

$$f(x) \sim \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right), \qquad 0 \le x \le L$$

with

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

$\rightsquigarrow$ We use the convention from our textbook: "$\sim$" means that $f(x)$ is on the left-hand side and the Fourier (sine) series is on the right-hand side, but that the two functions may be quite different!

Example:

Sketch the Fourier sine series of

$$f(x) = 1 - x \quad , \quad 0 \le x \le 1$$

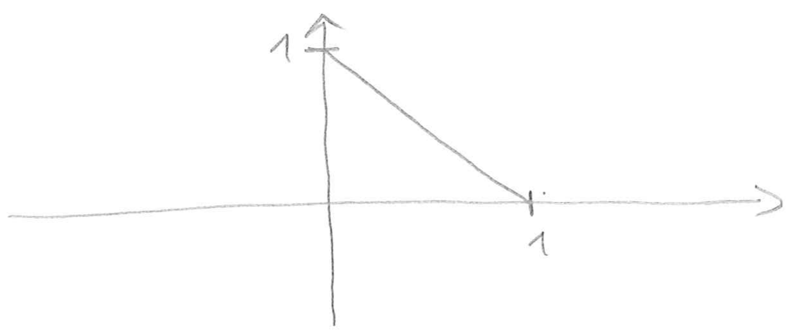
$\rightsquigarrow$ Strategy:

(1) Sketch $f(x)$ (for $0 \le x \le 1$)

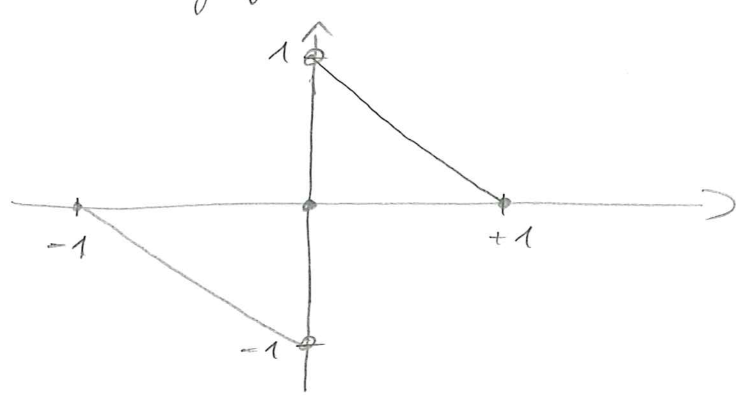
(2) Sketch the odd extension $f_{odd}(x)$ of $f(x)$

(3) Extend $f_{odd}$ as a periodic function to the whole line

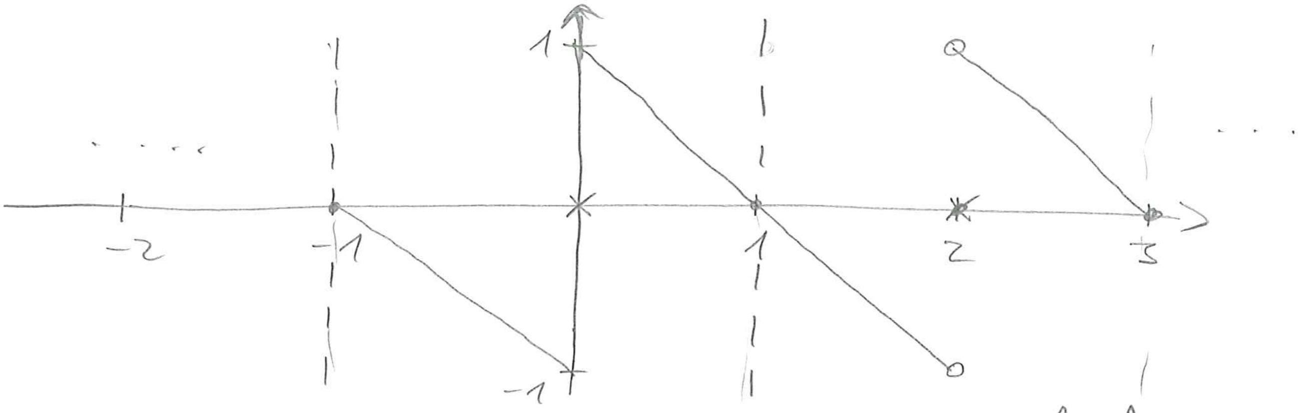
(4) Mark an $\times$ at the average of the one-sided limits where the periodic extension of $f_{odd}$ has a jump discontinuity



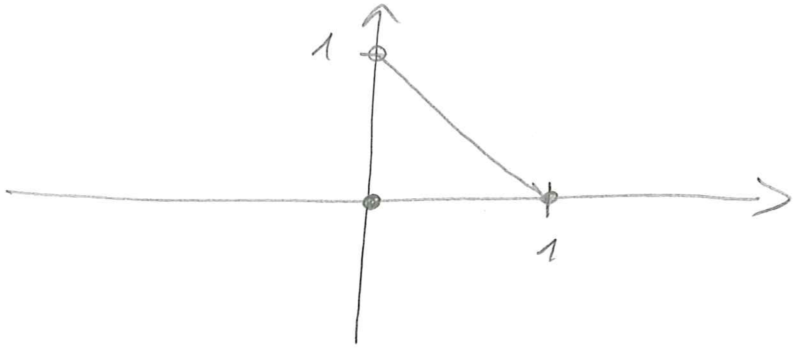

sketch of $f(x) = 1 - x$, $0 \le x \le 1$




sketch of $f_{odd}(x)$, $-1 \le x \le 1$




sketch of periodic extension of $f_{odd}$

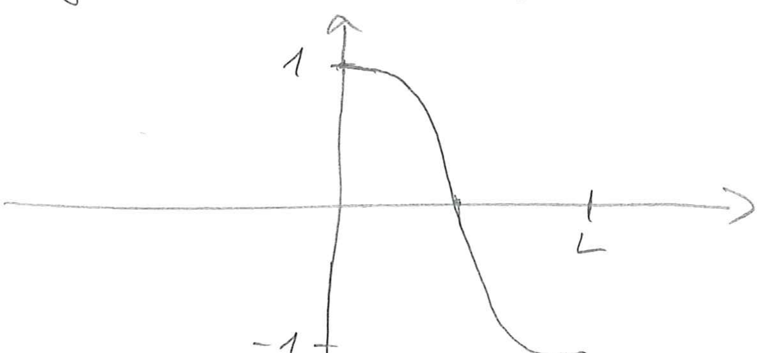



sketch of Fourier sine series of $f(x)$ on $0 \le x \le 1$.

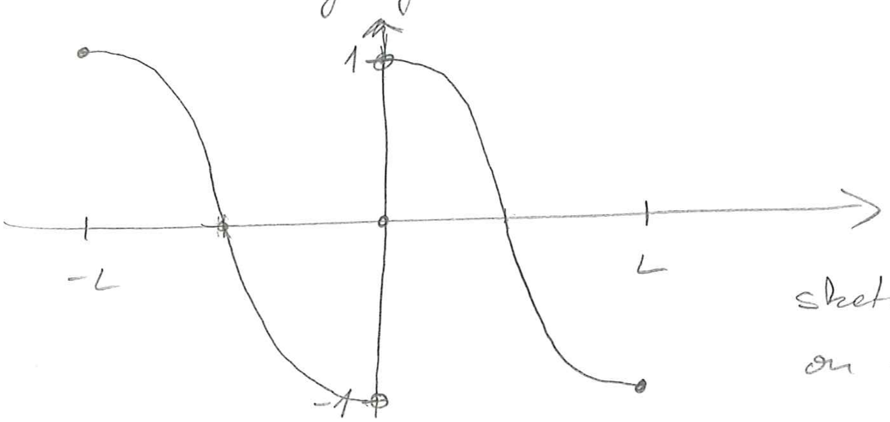
Example:

Sketch the Fourier sine series of

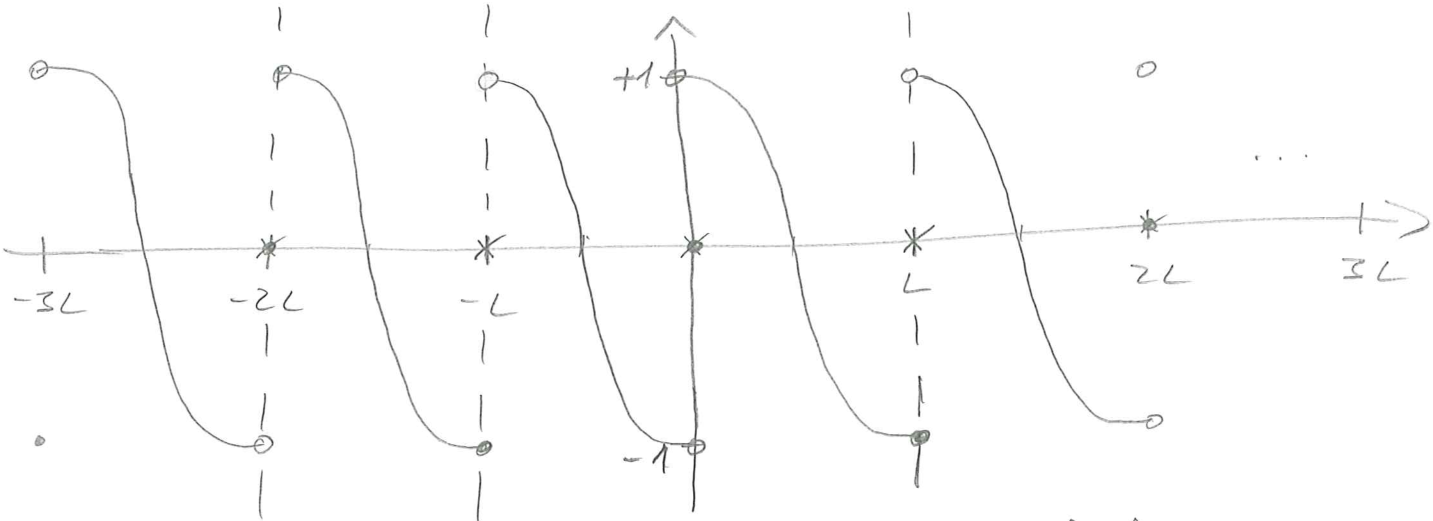
$$f(x) = \cos\left(\frac{\pi x}{L}\right) \quad , \quad 0 \le x \le L$$




sketch of $f(x)$ on $0 \le x \le L$




sketch of $f_{odd}(x)$ on $-L \le x \le +L$




sketch of periodic extension of $f_{odd}(x)$