Fourier Sine and Cosine Series

We will see in the following that the series of sines only and the series of cosines only are just special cases of a Fourier series.

Fourier Sine Series

Definition:

An odd function $f(x)$ satisfies the equation

$$f(-x) = -f(x).$$

Examples

$\sin(x)$, $\sin(5x)$, x, x^3, ...

Note:

- The sketch of an odd function for $x<0$ is minus the mirror image for $x>0$
- The integral of an odd function over a symmetric interval is zero

$$\int_{-L}^{+L} f(x)\,dx = 0.$$

Fourier series of odd functions

Let $f(x)$, $-L \leq x \leq +L$, be odd. Then

$$a_0 = \frac{1}{2L} \cdot \int_{-L}^{+L} \underbrace{f(x)}_{\text{odd}} \, dx = 0$$

$$a_n = \frac{1}{L} \cdot \int_{-L}^{+L} \underbrace{f(x) \cdot \cos\left(\frac{n\pi x}{L}\right)}_{\substack{\text{odd function!} \\ \text{(the product of an odd and an} \\ \text{even function is odd)}}} dx = 0$$

Thus, the Fourier series $(Sf)(x)$ of an odd function $f(x)$ is an infinite series of sines only.

$$(Sf)(x) = \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right),$$

where the coefficients b_n can be simplified a bit

$$(*1) \quad b_n := \frac{1}{L} \int_{-L}^{+L} f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

use change of variables $x \mapsto -x$ on $[-L, 0]$ and use that $f(x)$ and $\sin\left(\frac{n\pi x}{L}\right)$ are odd

Fourier sine series

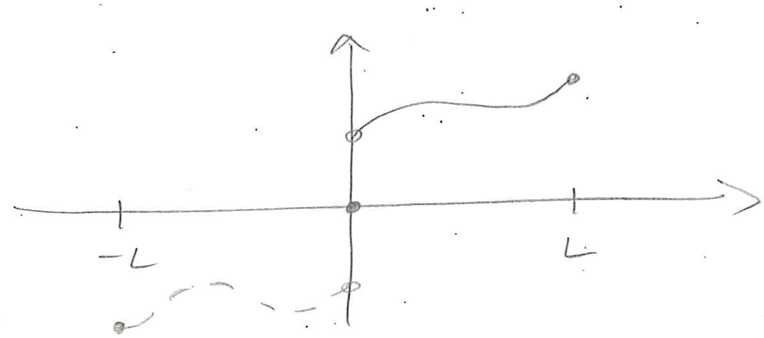
When we used the method of separation of variables to solve the heat equation on a rod of length L with zero boundary conditions, we needed to express a given initial condition $f(x)$ on $0 \le x \le L$ as an infinite sine series

$$f(x) \overset{?}{=} \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right).$$

This looks like the Fourier series of an odd function, but $f(x)$ is only defined for $0 \le x \le L$ (so it does not make sense to think of $f(x)$ as odd)!

Idea: Define the odd extension $f_{odd}(x)$ of a given function $f(x)$ on $0 \le x \le L$ by

$$f_{odd}(x) := \begin{cases} f(x), & 0 < x \le L \\ -f(-x), & -L \le x < 0 \\ 0, & x = 0 \end{cases}$$



Since the odd extension $f_{odd}(x)$ is by construction an odd function, its Fourier series just consists of sines

$$S(f_{odd})(x) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right),$$

where from (*1) we have that

$$B_n = \frac{2}{L} \int_0^L \underbrace{f_{odd}(x)}_{= f(x) \text{ on } 0 \le x \le L\,!} \sin\left(\frac{n\pi x}{L}\right) dx$$

Since $f_{odd}(x)$ is identical to $f(x)$ on $0 \le x \le L$, we can restrict the Fourier series of $f_{odd}(x)$ to the interval $0 \le x \le L$ to obtain the <u>Fourier sine series of $f(x)$</u>

$$f(x) \sim \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L$$

with

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

$\rightsquigarrow$ We use the convention from our textbook: "$\sim$" means that $f(x)$ is on the left-hand-side and the Fourier (sine) series is on the right-hand side, but that the two functions may be quite different!

Example:

Sketch the Fourier sine series of

$f(x) = 1 - x \quad , \quad 0 \le x \le 1$

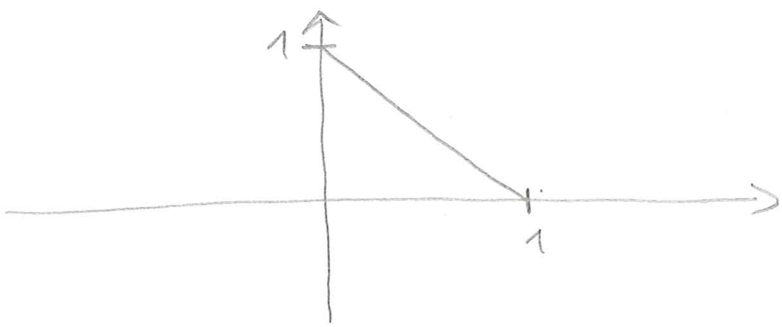
⇝ Strategy:

(1) Sketch $f(x)$ (for $0 \le x \le 1$)

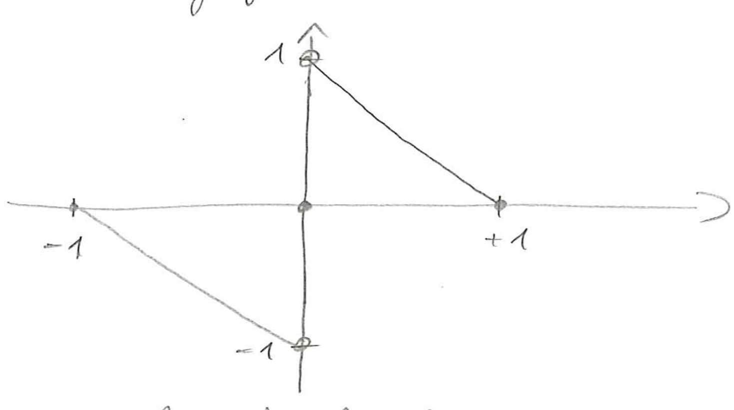
(2) Sketch the odd extension $f_{odd}(x)$ of $f(x)$

(3) Extend f_{odd} as a periodic function to the whole line

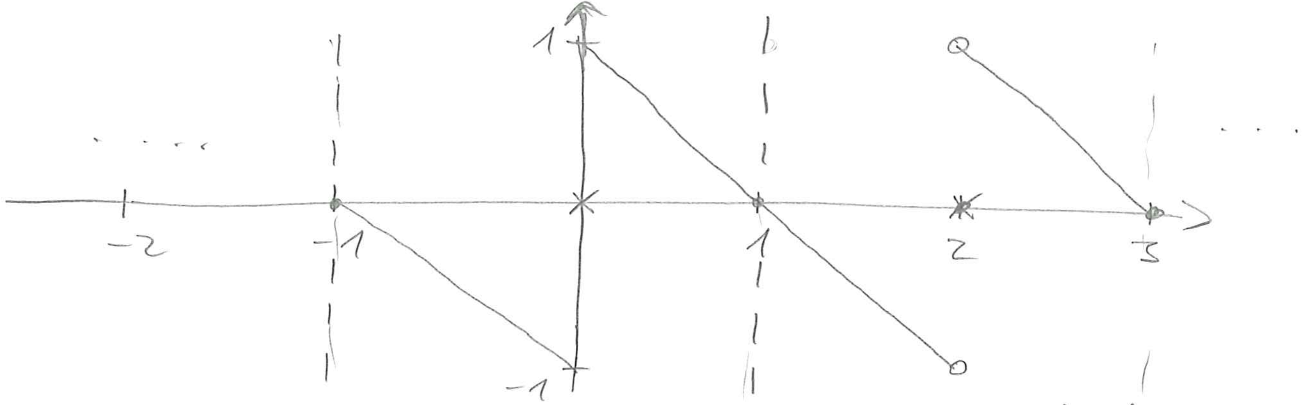
(4) Mark an $\times$ at the average of the one-sided limits where the periodic extension of f_{odd} has a jump discontinuity



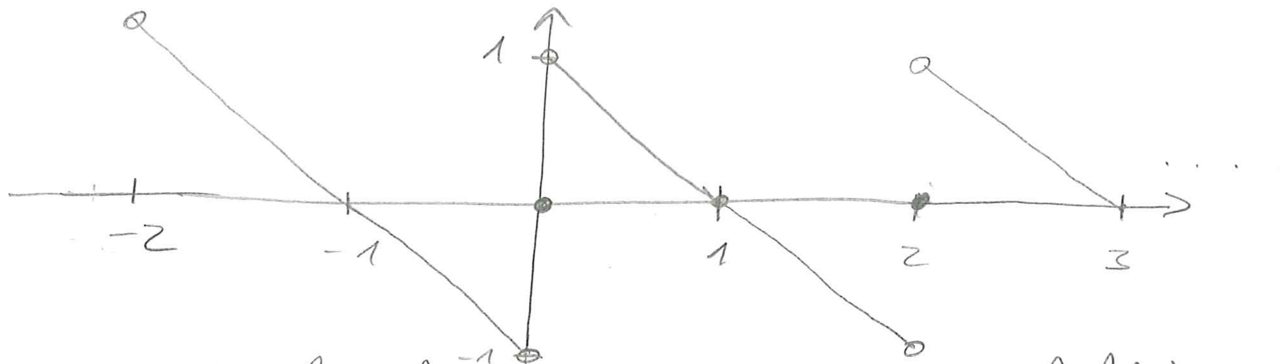
sketch of $f(x) = 1 - x$; $0 \le x \le 1$



sketch of $f_{odd}(x)$, $-1 \le x \le 1$



sketch of periodic extension of f_{odd}

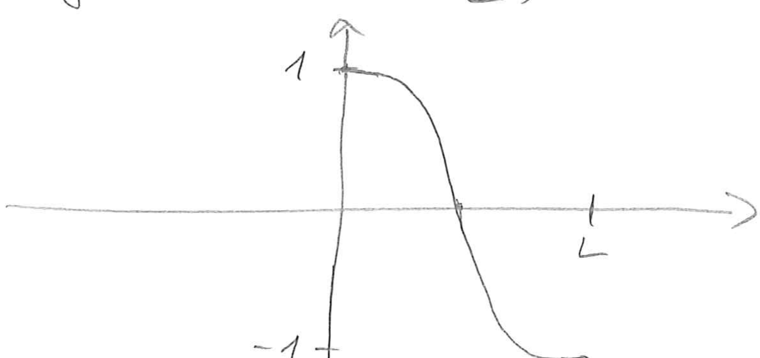


sketch of Fourier sine series of $f(x)$

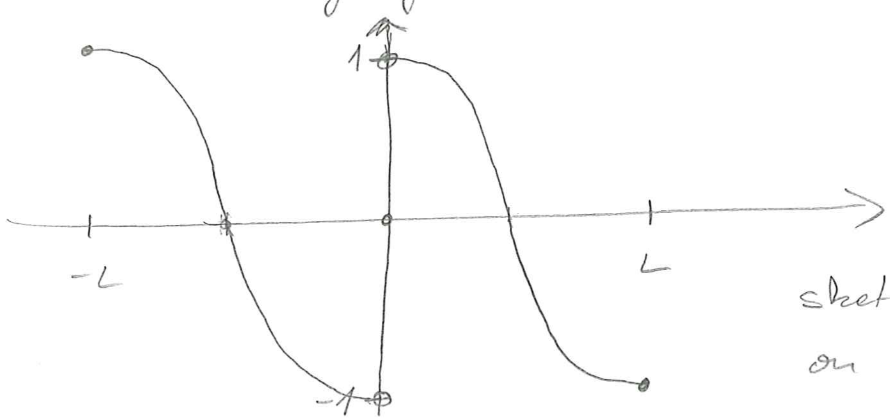
<u>Example:</u>

Sketch the Fourier sine series of

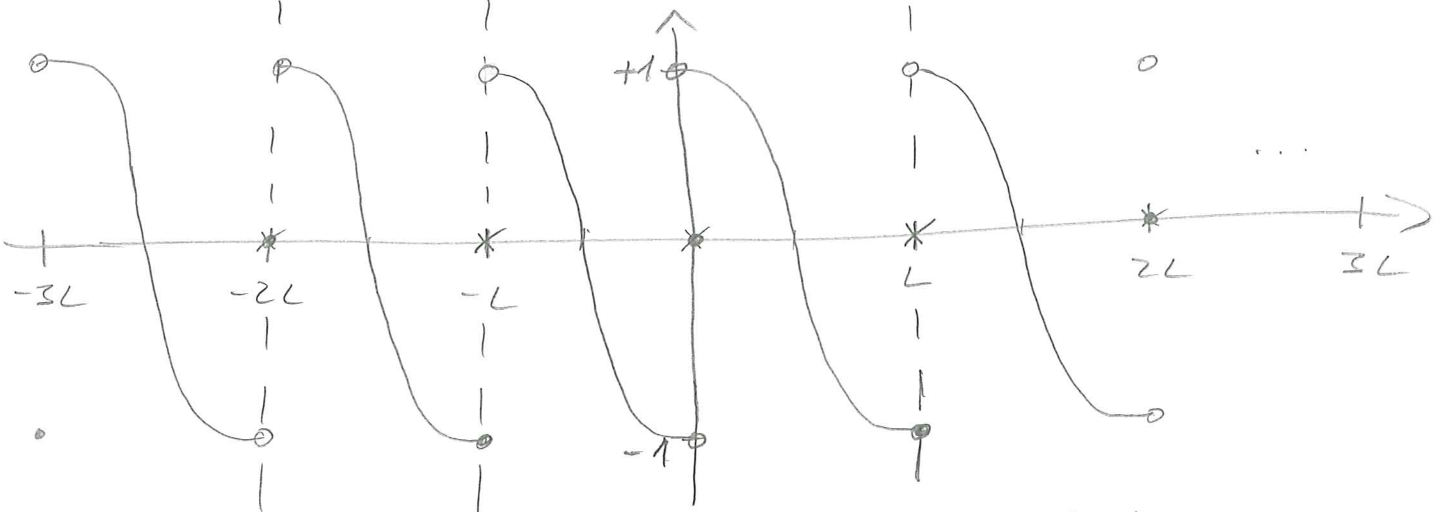
$$f(x) = \cos\left(\frac{\pi x}{L}\right) \quad , \quad 0 \le x \le L$$



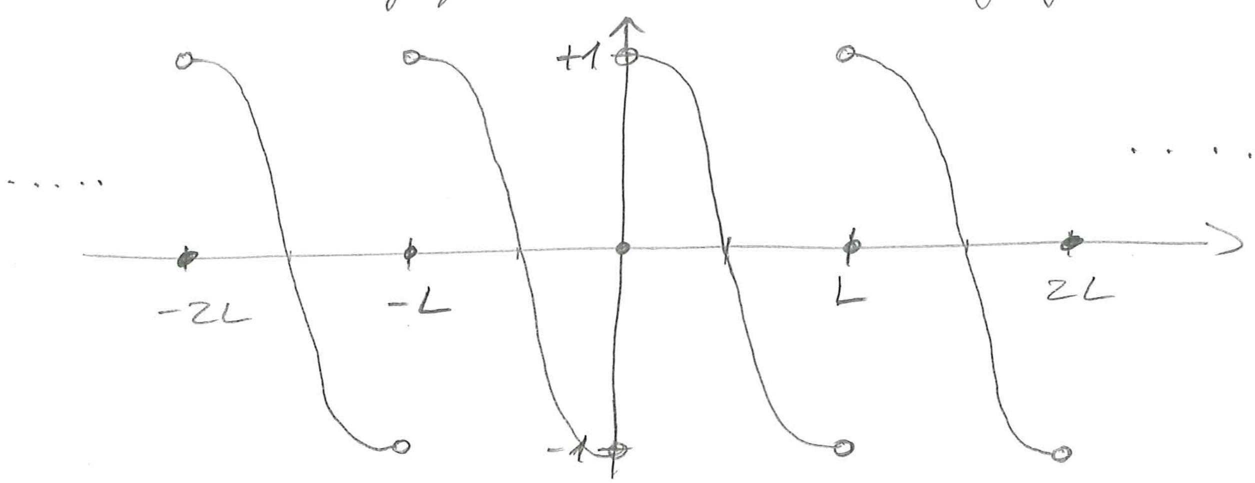
sketch of $f(x)$ on $0 \le x \le L$



sketch of $f_{odd}(x)$ on $-L \le x \le +L$



sketch of periodic extension of $f_{odd}(x)$



Sketch of Fourier sine series of $\cos\left(\frac{\pi x}{L}\right)$

Fourier cosine series

Definition:

An even function $f(x)$ satisfies the equation

$$f(-x) = f(x)$$

Examples

x^2, x^4, $\cos(x)$, $\cos(3x)$, ...

Note:

The sketch of an even function for $x < 0$ is the mirror image of that for $x > 0$

Fourier series of even functions

Let $f(x)$ on $-L \leq x \leq +L$ be even. Then

$$b_n = \frac{1}{L} \int_{-L}^{+L} \underbrace{f(x) \cdot \sin\left(\frac{n\pi x}{L}\right)}_{\text{odd function}} dx = 0$$

(even times odd gives odd function)

Thus, the Fourier series $(Sf)(x)$ of an even function $f(x)$ is an infinite series of cosines only

$$(Sf)(x) = \sum_{n=0}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right),$$

where the formulas for the coefficients a_n can be simplified a bit

$$a_0 = \frac{1}{2L} \cdot \underbrace{\int_{-L}^{L} f(x)\,dx}_{= 2 \cdot \int_0^L f(x)\,dx \text{ since } f(x) \text{ even}} = \underline{\underline{\frac{1}{L} \int_0^L f(x)\,dx}},$$

(*2)

$$a_n = \frac{1}{L} \underbrace{\int_{-L}^{L} f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx}_{\substack{\text{even function} \\ \text{(even times even} \\ \text{gives even function)}}} = \underline{\underline{\frac{2}{L} \cdot \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx}}$$

In our applications of the method of separation of variables it is sometimes necessary to write a given function $f(x)$ on an interval $0 \le x \le L$ as an infinite cosine series

$$f(x) \overset{?}{=} \sum_{n=0}^{\infty} \cos\left(\frac{n\pi x}{L}\right).$$

Analogously to the case of an infinite sine series, this looks like the Fourier series of an even function, but $f(x)$ is only defined for $0 \le x \le L$.

Correspondingly, we introduce the <u>even extension</u> of a given function $f(x)$ on $0 \le x \le L$ by

$$f_{even}(x) := \begin{cases} f(x), & 0 \le x \le L \\ f(-x), & -L \le x \le 0. \end{cases}$$

Then

$$(S f_{even})(x) = \sum_{n=0}^{\infty} A_n \cdot \cos\left(\frac{n\pi x}{L}\right), \quad -L \le x \le +L,$$

where from $(*2)$ we have

$$A_0 = \frac{1}{L} \cdot \int_0^L f(x)\, dx,$$

$$A_n = \frac{2}{L} \cdot \int_0^L f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx$$

In this manner we obtain the <u>Fourier cosine series</u> of a function $f(x)$ on the interval $0 \le x \le L$:

$$f(x) \sim \sum_{n=0}^{\infty} A_n \cdot \cos\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L,$$

with

$$A_0 = \frac{1}{L} \int_0^L f(x)\, dx,$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx.$$

Example: Sketch the Fourier cosine series of

$$f(x) = x \,, \quad 0 \leq x \leq L$$

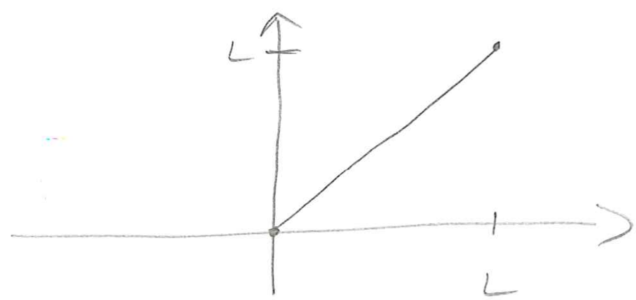
$\rightsquigarrow$ Strategy:

(1) Sketch $f(x)$ over the interval $0 \leq x \leq L$

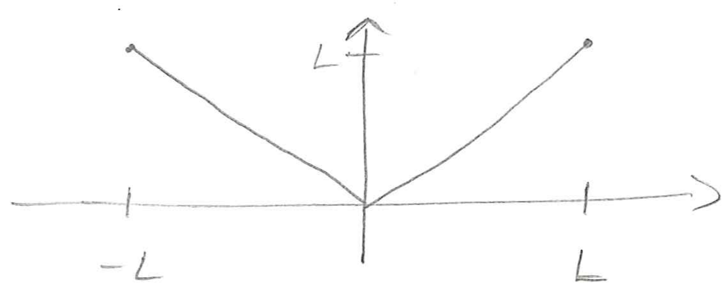
(2) Sketch the even extension of $f(x)$ over the interval $-L \leq x \leq +L$

(3) Extend as a periodic function on the whole line (with period $2L$)

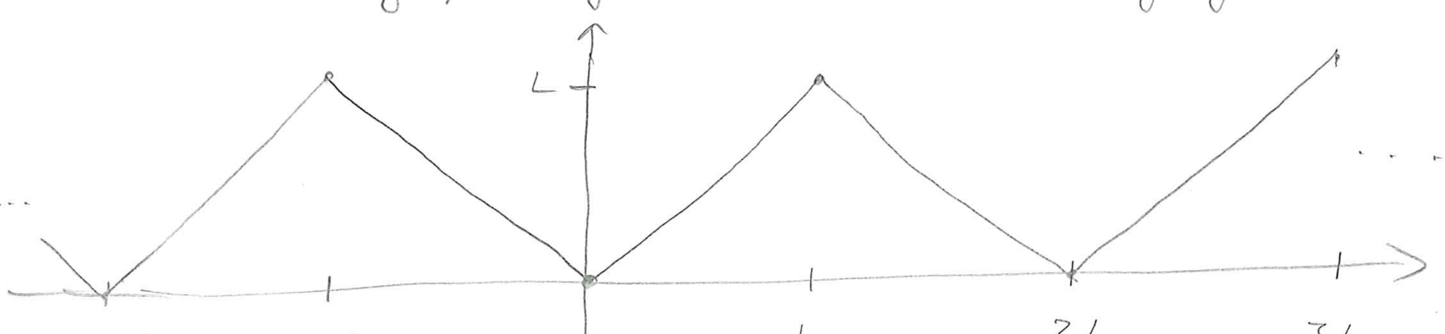
(4) Mark $\times$ at points of discontinuity



graph of $f(x)$ over $0 \leq x \leq L$



graph of even extension of $f(x)$



graph of Fourier cosine series of $f(x)$

Even and odd parts

An arbitrary function $f(x)$ on $-L \le x \le +L$ (which is neither even nor odd) can be decomposed into its even and odd parts

$$f(x) = \underbrace{\frac{1}{2}\left(f(x) + f(-x)\right)}_{\substack{\text{even part } f_e(x) \\ \text{of } f(x)}} + \underbrace{\frac{1}{2}\left(f(x) - f(-x)\right)}_{\substack{\text{odd part } f_o(x) \\ \text{of } f(x)}}$$

Then the Fourier series of $f(x)$

$$(Sf)(x) = \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right)}_{\substack{\text{Fourier cosine} \\ \text{series of } f_e(x)}} + \underbrace{\sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right)}_{\substack{\text{Fourier sine} \\ \text{series of } f_o(x)}}$$

equals the Fourier cosine series of $f_e(x)$ plus the Fourier sine series of $f_o(x)$.