Even and odd parts

An arbitrary function $f(x)$ on $-L \leq x \leq +L$ (which is neither even nor odd) can be decomposed into its even and odd parts

$$f(x) = \underbrace{\frac{1}{2}\left(f(x) + f(-x)\right)}_{\substack{\text{even part } f_e(x) \\ \text{of } f(x)}} + \underbrace{\frac{1}{2}\left(f(x) - f(-x)\right)}_{\substack{\text{odd part } f_o(x) \\ \text{of } f(x)}}$$

Then the Fourier series of $f(x)$

$$(Sf)(x) = \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right)}_{\substack{\text{Fourier (cosine)} \\ \text{series of } f_e(x)}} + \underbrace{\sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right)}_{\substack{\text{Fourier (sine)} \\ \text{series of } f_o(x)}}$$

equals the Fourier cosine series of $f_e(x)$ plus the Fourier sine series of $f_o(x)$.

Continuous Fourier series

Q: Given a piecewise smooth function $f(x)$ on an interval $-L \leq x \leq +L$:
Under which conditions on $f(x)$ is its Fourier series $(Sf)(x)$ a continuous function?

By Fourier's theorem, the Fourier series $(Sf)(x)$ assumes the values of the periodic extension $f_{per}(x)$ of $f(x)$, where $f_{per}(x)$ is continuous, and otherwise the average of the one-sided limits $\frac{1}{2}(f_{per}(x+) + f_{per}(x-))$.
For $f_{per}(x)$ to be continuous, and hence $(Sf)(x)$ to be continuous, the function $f(x)$ has to be continuous on $-L \leq x \leq +L$ and must satisfy $f(-L) = f(+L)$.

We summarize:

Given a piecewise smooth function $f(x)$ on an interval $-L \leq x \leq +L$, then its Fourier series $(Sf)(x)$ is continuous and equals $f(x)$ for $-L \leq x \leq +L$ if and only if $f(x)$ is continuous and $f(-L) = f(+L)$.

From this observation we can also quickly deduce the following results about the continuity of the Fourier cosine and Fourier sine series of a function $f(x)$ given on an interval $0 \le x \le L$:

- Given a piecewise smooth function $f(x)$ on an interval $0 \le x \le L$, the Fourier cosine series of $f(x)$ is <u>continuous</u> and <u>equals</u> $f(x)$ for $0 \le x \le L$ if and only if $f(x)$ is continuous.

- Given a piecewise smooth function $f(x)$ on an interval $0 \le x \le L$, the Fourier sine series of $f(x)$ is <u>continuous</u> and <u>equals</u> $f(x)$ for $0 \le x \le L$ if and only if $f(x)$ is <u>continuous</u> and both $f(0) = 0$ and $f(L) = 0$.

Complex form of Fourier series

The Fourier series of a function $f(x)$ on an interval $-L \leq x \leq +L$

$$(Sf)(x) = \sum_{n=0}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right),$$

with

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\, dx,$$

$$a_n = \frac{1}{L} \int_{-L}^{+L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{+L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx,$$

can also be written in terms of complex exponentials instead of sines and cosines.

To this end recall Euler's formula

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

or

$$\cos(\theta) = \frac{1}{2}\left(e^{+i\theta} + e^{-i\theta}\right),$$

$$\sin(\theta) = \frac{1}{2i}\left(e^{+i\theta} - e^{-i\theta}\right).$$

Thus,

$$(Sf)(x) = a_0 + \sum_{n=1}^{\infty} \frac{1}{2}(a_n - ib_n)\, e^{+i\cdot\frac{n\pi x}{L}}$$

$$+ \sum_{n=1}^{\infty} \frac{1}{2}(a_n + ib_n)\, e^{-i\cdot\frac{n\pi x}{L}}.$$

To have only $e^{-i\frac{n\pi x}{L}}$ terms, we change the dummy index in the first summation, replacing n by $-n$, to get

$$(Sf)(x) = a_0 + \sum_{n=-1}^{-\infty} \frac{1}{2}\left(a_{(-n)} - ib_{(-n)}\right) e^{+i\frac{n\pi x}{L}}$$

$$+ \sum_{n=1}^{\infty} \frac{1}{2}(a_n + ib_n)\, e^{-i\cdot\frac{n\pi x}{L}}.$$

Note that from the definition

$$a_n = \frac{1}{L}\int_{-L}^{+L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

we have $a_{(-n)} = a_n$ for any $n \in \mathbb{Z}$.

and from

$$b_n = \frac{1}{L}\int_{-L}^{+L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

we have

$$b_{(-n)} = -b_n \quad \text{for any } n \in \mathbb{Z}$$

$$\left(\begin{array}{lll} \text{since} & \cos\left(\frac{(-n)\pi x}{L}\right) = \cos\left(\frac{n\pi x}{L}\right) \\ \text{and} & \sin\left(\frac{(-n)\pi x}{L}\right) = -\sin\left(\frac{n\pi x}{L}\right) \end{array}\right).$$

Thus, if we define

$$c_0 = a_0,$$

$$c_n = \frac{a_n + i b_n}{2}, \quad n \in \mathbb{Z} \setminus \{0\},$$

then $(Sf)(x)$ simply becomes

$$\boxed{(Sf)(x) = \sum_{n=-\infty}^{+\infty} c_n \cdot e^{-i \cdot \frac{n\pi x}{L}}.}$$

complex form of the Fourier series $(Sf)(x)$

In this form the complex Fourier coefficients are

$$c_n = \frac{1}{2}(a_n + i b_n) \qquad \text{for } n \in \mathbb{Z} \setminus \{0\}$$

$$= \frac{1}{2L} \int_{-L}^{+L} f(x) \underbrace{\left(\cos\left(\frac{n\pi x}{L}\right) + i \sin\left(\frac{n\pi x}{L}\right)\right)}_{= e^{+i \frac{n\pi x}{L}} \text{ Euler's formula}} dx$$

$$= \frac{1}{2L} \int_{-L}^{+L} f(x)\, e^{+i \cdot \frac{n\pi x}{L}} dx.$$

Since $c_0 = a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x) \cdot \underbrace{1}_{= e^{+i \cdot \frac{0 \cdot \pi x}{L}}} dx$,

we have the simple unified formula

$$\boxed{c_n = \frac{1}{2L} \int_{-L}^{+L} f(x)\, e^{+i \cdot \frac{n\pi x}{L}}, \quad n \in \mathbb{Z}}$$

→ The complex form of the Fourier series of a function will be a useful analogy when we introduce the Fourier transform.

Complex orthogonality

Definition:

A complex-valued function $\phi(x)$ is orthogonal to a complex-valued function $\psi(x)$ over the interval $a \le x \le b$ if

$$\int_a^b \overline{\phi(x)} \cdot \psi(x)\, dx = 0.$$

Observe that the functions $\left\{ e^{+i \cdot \frac{n\pi x}{L}} \right\}_{n \in \mathbb{Z}}$ are orthogonal over $-L \le x \le +L$:

$$\int_{+L}^{+L} \overline{e^{+i \cdot \frac{n\pi x}{L}}} \cdot e^{+i\frac{m\pi x}{L}}\, dx$$

$$= \int_{-L}^{+L} e^{+i \cdot \frac{(-n+m)\cdot \pi x}{L}}\, dx$$

$$= \begin{cases} 0, & n \neq m, \\ 2L, & n = m. \end{cases}$$

Similarly to using the orthogonality of sines and cosines, we can use the orthogonality of $\left\{ e^{+i \frac{n \pi x}{L}} \right\}_{n \in \mathbb{Z}}$ to directly determine the formula for the Fourier coefficients

$$c_n = \frac{1}{2L} \int_{-L}^{+L} f(x) \, e^{+i \cdot \frac{n \pi x}{L}} \, dx$$

by multiplying the complex form of the Fourier series

$$(Sf)(x) = \sum_{m=-\infty}^{+\infty} c_m \cdot e^{-i \cdot \frac{m \pi x}{L}}$$

by $e^{+i \cdot \frac{n \pi x}{L}}$ and then integrate over $-L \leq x \leq L$.