Chapter 4 : Wave Equation

We live in a world of waves:

- "hearing": our ears detect waves of compression in the air
- "seeing": our eyes detect waves of electromagnetic radiation
- radio, television, mobile telephone networks use waves of electromagnetic radiation
- ocean waves
- earthquakes
- gravitational waves

(...)

Here we consider one of the simplest, yet extremely important wave equations, which for instance describes the vibrations of a musical stringed instrument (like a violin or guitar).

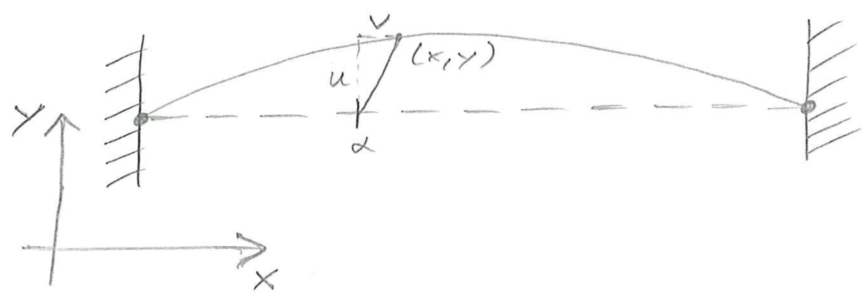
Vibrating strings: derivation of the governing equation

Consider a horizontally (tightly) stretched string whose ends are fixed and tied down (Think of a string in a violin).

If you slightly pull the string vertically upwards and then let it loose, we expect the string to start vibrating.

We now want to derive the governing equations describing such vibrations.

tightly stretched string in equilibrium



perturbed string

v: horizontal displacement

u: vertical displacement

Let α be the x-coordinate of a particle when the string is in equilibrium.

Assume that the slope of the string is small. Then the horizontal displacement v can be neglected and the motion is (approximately) entirely vertical $x = \alpha$. Correspondingly, the vertical displacement u depends only on x and t:

$$y = u(x,t)$$

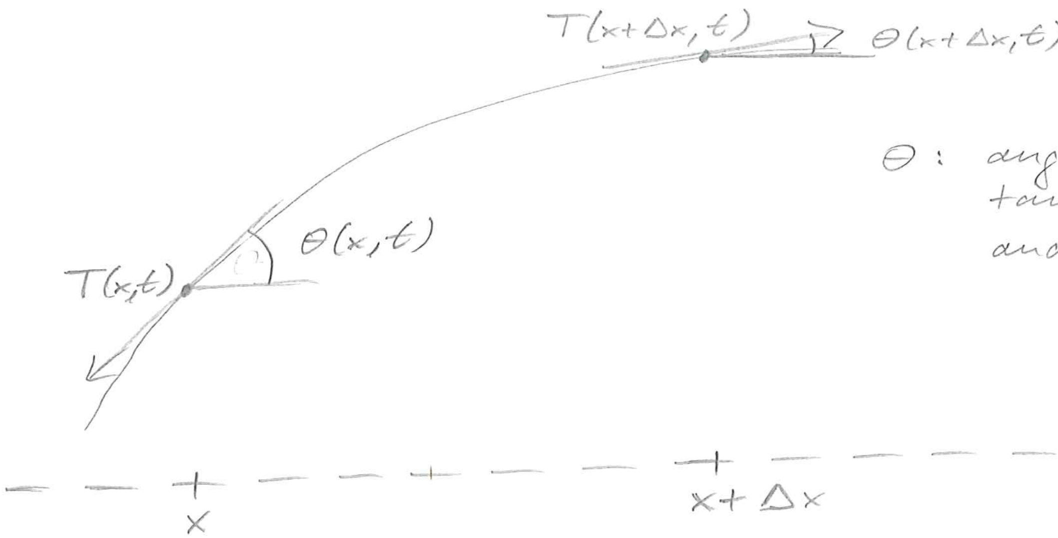
To describe the motion of the string we want to bring in Newton's law of motion

$$F = m \cdot a$$

force exerted on a body

acceleration of the body

Consider a very small segment of the string contained between x and $x + \Delta x$:
(think of this segment as a particle that is subject to several forces pulling it, which we now want to track)



θ: angle between tangent to string and horizontal

Forces acting on the string segment:

- Assume that the body forces (such as gravitational force) act only vertically

- Assume that the string is perfectly flexible (it offers no resistance to bending). Then the force exerted by the rest of the string on the endpoints of the string segment is <u>tangential</u> to the string. This tangential force is called the <u>tension</u> in the string, denoted by $T(x,t)$, and is trying to stretch the string.

Let $\rho_0(x)$ be the mass density of the string. Then the total mass of the string segment is

$$\rho_0(x) \cdot \Delta x.$$

By Newton's law of motion, we obtain for the vertical motion of the string segment:

$$\underbrace{\rho_0(x)\cdot\Delta x}_{\text{mass of string segment}}\cdot\underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{vertical acceleration of string segment}} = \left.\begin{array}{l} T(x+\Delta x,t)\cdot\sin(\theta(x+\Delta x,t)) \\ -\,T(x,t)\cdot\sin(\theta(x,t)) \end{array}\right\}\text{vertical components of tensile forces}$$

$$+\,\rho_0(x)\cdot\Delta x\cdot Q(x,t)\ \Big\}\ \text{vertical component of body force per unit mass}$$

Dividing by Δx and letting $\Delta x \to 0$ yields

$$\rho_0(x)\cdot\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\Big(T(x,t)\cdot\sin(\theta(x,t))\Big) + \rho_0(x)\cdot Q(x,t)$$

The slope of the string (at position x and time t) is given by

$$\frac{\partial u}{\partial x} = \frac{dy}{dx} = \tan(\theta(x,t))$$

For small angles θ (i.e. $|\theta| \ll 1$)

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \approx \sin(\theta)$$

$$\Rightarrow \frac{\partial u}{\partial x} \approx \sin(\theta)$$

Thus, we arrive at the equation

$$\left\| \ \rho_0(x)\cdot\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right) + \rho_0(x)\cdot Q(x,t) \ \right\|$$

For <u>perfectly elastic strings</u>, the tensile force $T(x,t)$ can be approximated by a constant T_0 (for small perturbations of the string), which leads to the PDE

$$\rho_0(x) \cdot \frac{\partial^2 u}{\partial t^2} = T_0 \cdot \frac{\partial^2 u}{\partial x^2} + Q(x,t) \cdot \rho_0(x)$$

Finally, if the only body force is gravity, then $Q(x,t) = -g$ can be neglected because the tensile forces are much larger:

$$\rho_0(x) \cdot \frac{\partial^2 u}{\partial t^2} = T_0 \cdot \frac{\partial^2 u}{\partial x^2}$$

$$\Longrightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}} \quad \text{where } c^2 = \frac{T_0}{\rho_0(x)}$$

has dimension of velocity squared

For a uniform string, c is constant.

Boundary conditions

For a vibrating string of length L, the simplest (and most common) boundary condition is to have fixed ends (usually with fixed zero displacement):

$$u(0,t) = 0, \qquad u(L,t) = 0.$$

for all $t > 0$.

One can also give physical meaning to other boundary conditions such as $\frac{\partial u}{\partial x}(0,t)$ by considering one end of the string attached to a dynamical system, but these are much less common and we won't further discuss them here.