Solving the PDE for a vibrating string with fixed ends

We solve the one-dimensional wave equation for a uniform vibrating string of length L without external forces and fixed ends with zero displacement:

$$\text{(PDE)} \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L, \qquad t > 0$$

$$\text{(BCs)} \qquad \begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}$$

$$\text{(ICs)} \qquad \begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x) \end{aligned}$$

Note that the wave equation contains second-order time derivatives; we therefore need two initial conditions, one for $u(x,0)$ and the other one for the first time derivative $\frac{\partial u}{\partial t}(x,0)$.

(Think of initial position and initial velocity; informally, one has to "integrate twice in time" to obtain the unknown $u(x,t)$ from the PDE involving $\frac{\partial^2 u}{\partial t^2}$; along the way two integration constants will come up that therefore have to be specified)

The (PDE) and the (BCs) are linear and homogeneous. Hence, it is reasonable to try to use the method of separation of variables to solve it:

Ansatz:

$$u(x,t) = \phi(x) \cdot h(t).$$

Plugging the ansatz into the (PDE):

$$\phi(x) \cdot \frac{d^2 h}{dt^2} = c^2 \cdot h(t) \cdot \frac{d^2 \phi}{dx^2}$$

$$\Rightarrow \frac{1}{c^2} \frac{1}{h(t)} \frac{d^2 h}{dt^2} = \frac{1}{\phi(x)} \frac{d^2 \phi}{dx^2} = -\lambda$$

minus sign for convenience

for some separation constant $\lambda \in \mathbb{R}$.

We obtain the following ~~t~~ t-dependent ODE for $h(t)$

$$\frac{d^2 h}{dt^2} = -\lambda c^2 h \quad , \quad t > 0$$

and the following familiar (x-dependent) boundary value problem for $\phi(x)$:

$$\begin{cases} \dfrac{d^2\phi}{dx^2} = -\lambda \cdot \phi \quad , \quad 0 \leq x \leq L \\ \phi(0) = 0 \\ \phi(L) = 0 \end{cases}$$

We know that the boundary value problem has the eigenvalues

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 , \quad n = 1, 2, 3, \dots$$

with eigenfunctions

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right) , \quad n = 1, 2, 3, \dots$$

Then the t-dependent ODE

$$\frac{d^2h}{dt^2} = -c^2 \cdot \left(\frac{n\pi}{L}\right)^2 h(t) , \quad n = 1, 2, 3, \dots$$

has the general solution

$$h(t) = c_1 \cdot \cos\left(\frac{n\pi c}{L} \cdot t\right) + c_2 \cdot \sin\left(\frac{n\pi c}{L} \cdot t\right)$$

$\rightsquigarrow$ oscillating in time!

Using the superposition principle to put together the product solutions

$$\sin\left(\frac{n\pi x}{L}\right) \cdot \left(c_1 \cdot \cos\left(\frac{n\pi c}{L} \cdot t\right) + c_2 \cdot \sin\left(\frac{n\pi c}{L} \cdot t\right)\right)$$

as infinite linear combinations, we obtain solutions to the wave equation (PDE) of the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cdot \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi c}{L} \cdot t\right)$$

$$+ \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi c}{L} \cdot t\right)$$

The initial conditions are satisfied if

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \cdot \sin\left(\frac{n\pi x}{L}\right)$$

$$g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi c}{L} \cdot \sin\left(\frac{n\pi x}{L}\right).$$

Thus, writing the (ICs) $f(x)$ and $g(x)$ as Fourier sine series, we obtain for the coefficients

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$B_n \cdot \frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Interpretation of the results for a musical stringed instrument:

- The vertical displacement is a linear combination of the simple product solutions

$$\sin\left(\frac{n\pi x}{L}\right)\left(A_n \cdot \cos\left(\frac{n\pi c}{L}\cdot t\right) + B_n \cdot \sin\left(\frac{n\pi c}{L}\cdot t\right)\right),$$

called the <u>normal modes of vibration</u>.

- The intensity of the produced sound depends on the amplitude $\sqrt{A_n^2+B_n^2}$:

We can write

$$A_n \cos\left(\frac{n\pi c}{L}\cdot t\right) + B_n \cdot \sin\left(\frac{n\pi c}{L}\cdot t\right)$$

$$= \sqrt{A_n^2+B_n^2}\cdot \sin\left(\frac{n\pi c}{L}\cdot t + \theta\right)$$

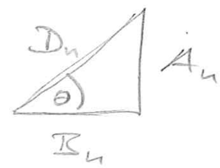
where $\theta = \arctan\left(\frac{A_n}{B_n}\right)$

<u>Derivation:</u>

$$A_n \cos\left(\frac{n\pi c}{L}\cdot t\right) + B_n \cdot \sin\left(\frac{n\pi c}{L}\cdot t\right)$$

$$= D_n \sin(\theta) \cos\left(\frac{n\pi c}{L}\cdot t\right) + D_n \cos(\theta) \sin\left(\frac{n\pi c}{L}\cdot t\right)$$

$$= D_n \sin\left(\frac{n\pi c}{L}\cdot t + \theta\right)$$



$\tan(\theta) = \frac{A_n}{B_n}$, $D_n = \sqrt{A_n^2+B_n^2}$

$A_n = D_n \sin(\theta)$

$B_n = D_n \cos(\theta)$

using that

$$\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) = \sin(\alpha+\beta).$$

- The time dependence is just periodic with <u>circular frequency</u> given by $\frac{n\pi c}{L}$ (number of oscillations per 2π unit time).

Thus, the sound produced consists of these (possibly infinite) number of <u>natural frequencies</u>.

The normal mode $n=1$ is called the <u>first harmonic</u> or <u>fundamental frequency</u>.

⇝ recall that $c = \sqrt{\frac{T_0}{\rho_0}}$: For a stringed instrument, the mass density ρ_0 cannot be changed, but a desired fundamental frequency can be produced by changing the length L or the tension T_0 of the string.

Motion of normal modes