The initial conditions are satisfied if

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \cdot \sin\left(\frac{n\pi x}{L}\right)$$

$$g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi c}{L} \cdot \sin\left(\frac{n\pi x}{L}\right).$$

Thus, writing the (ICs) $f(x)$ and $g(x)$ as Fourier sine series, we obtain for the coefficients

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$B_n \cdot \frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\Rightarrow B_n = \frac{2}{n\pi c} \cdot \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

<u>Interpretation of the results for a musical stringed instrument:</u>

- The vertical displacement is a linear combination of the simple product solutions

$$\sin\left(\frac{n\pi x}{L}\right)\left(A_n \cdot \cos\left(\frac{n\pi c}{L} \cdot t\right) + B_n \cdot \sin\left(\frac{n\pi c}{L} \cdot t\right)\right),$$

called the <u>normal modes of vibration</u>.

- The intensity of the produced sound depends on the amplitude $\sqrt{A_n^2 + B_n^2}$:

We can write

$$A_n \cos\left(\frac{n\pi c}{L} \cdot t\right) + B_n \cdot \sin\left(\frac{n\pi c}{L} \cdot t\right)$$

$$= \sqrt{A_n^2 + B_n^2} \cdot \sin\left(\frac{n\pi c}{L} \cdot t + \theta\right)$$

where $\theta = \arctan\left(\frac{A_n}{B_n}\right)$

<u>Derivation:</u>

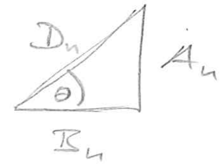
$$A_n \cos\left(\frac{n\pi c}{L} \cdot t\right) + B_n \cdot \sin\left(\frac{n\pi c}{L} \cdot t\right)$$

$$= D_n \sin(\theta) \cos\left(\frac{n\pi c}{L} \cdot t\right) + D_n \cos(\theta) \sin\left(\frac{n\pi c}{L} \cdot t\right)$$

$$= D_n \sin\left(\frac{n\pi c}{L} \cdot t + \theta\right)$$

using that

$$\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) = \sin(\alpha+\beta).$$



$\tan(\theta) = \frac{A_n}{B_n}$, $D_n = \sqrt{A_n^2 + B_n^2}$

$A_n = D_n \sin(\theta)$

$B_n = D_n \cos(\theta)$

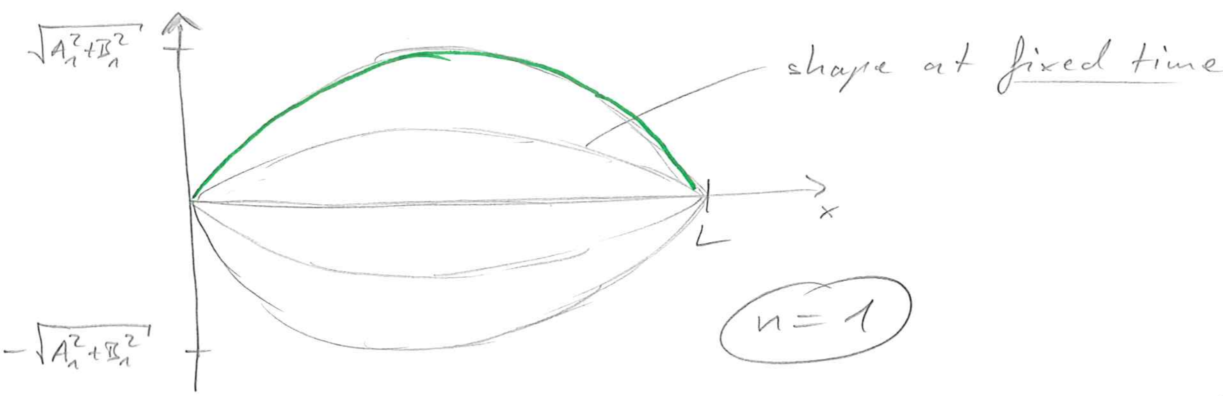
- The time dependence is just periodic with <u>circular frequency</u> given by $\frac{n\pi c}{L}$ (number of oscillations per 2π unit time).

Thus, the sound produced consists of these (possibly infinite) number of <u>natural frequencies</u>.

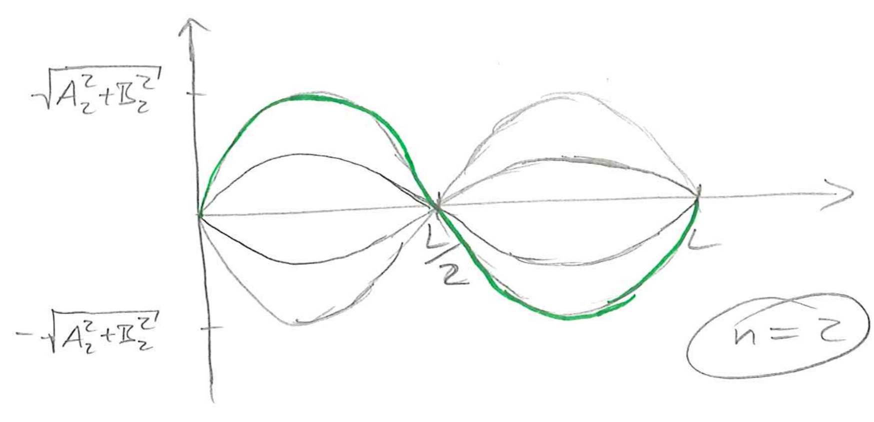
The normal mode $n=1$ is called the first harmonic or fundamental frequency.

⇝ Recall that $c = \sqrt{\frac{T_0}{\varrho_0}}$: For a stringed instrument, the mass density ϱ_0 cannot be changed, but a desired fundamental frequency can be produced by changing the length L or the tension T_0 of the string.

Motion of normal modes



standing waves



Standing Waves and Travelling Waves

Let's take another look at the expression for the normal modes of vibration

$$\sin\left(\frac{n\pi x}{L}\right)\cdot\left(A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n \cdot \sin\left(\frac{n\pi c}{L}\cdot t\right)\right)$$

$$= A_n \cdot \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi c}{L}\cdot t\right)$$

$$+ B_n \cdot \sin\left(\frac{n\pi}{L}\cdot x\right)\sin\left(\frac{n\pi c}{L}\cdot t\right)$$

Using the trigonometric identity

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$

we can write

$$A_n \cdot \sin\left(\frac{n\pi}{L}\cdot x\right)\cos\left(\frac{n\pi c}{L}\cdot t\right)$$

$$= \underbrace{\frac{A_n}{2}\cdot \sin\left(\frac{n\pi}{L}(x+ct)\right)}_{\substack{\text{wave travelling} \\ \text{to the left} \\ \text{(with velocity } -c)}} + \underbrace{\frac{A_n}{2}\cdot \sin\left(\frac{n\pi}{L}(x-ct)\right)}_{\substack{\text{wave travelling} \\ \text{to the right} \\ \text{(with velocity } c)}}$$

→ These two travelling waves must cancel out (by superposition principle) in such a way to produce a <u>standing wave</u>!

Similarly, using the trigonometric identity

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\cdot\Big(-\cos(\alpha+\beta) + \cos(\alpha-\beta)\Big),$$

we can write

$$B_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi c}{L}t\right)$$

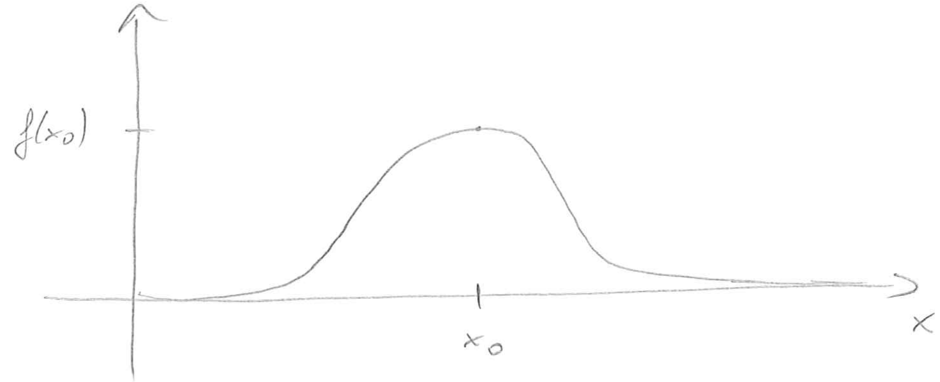
$$= \underbrace{-\frac{B_n}{2}\cos\left(\frac{n\pi}{L}(x+ct)\right)}_{\substack{\text{wave travelling}\\ \text{to the left}\\ \text{(with velocity } -c)}} + \underbrace{\frac{B_n}{2}\cdot\cos\left(\frac{n\pi}{L}(x-ct)\right)}_{\substack{\text{wave travelling}\\ \text{to the right}\\ \text{(with velocity } +c)}}$$

More generally, if you start from a "nice" (=twice differentiable) function $f(x)$ and define the time-dependent function

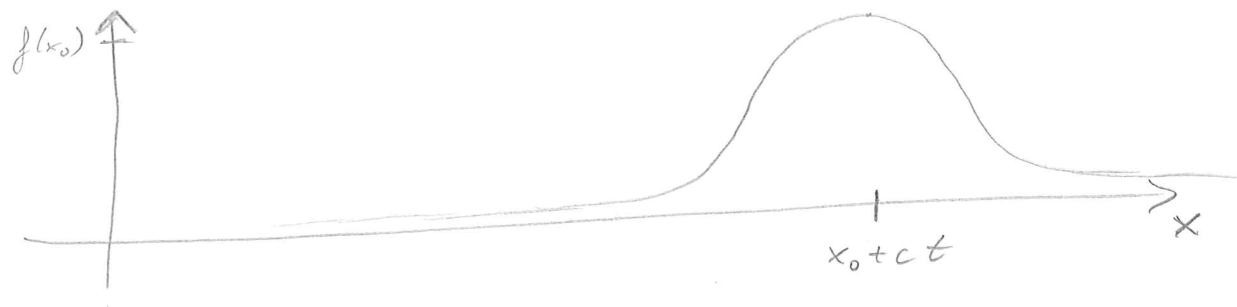
$$u(x,t) := f(x-ct),$$

then if you plot $x \mapsto u(x,t_n)$ at several consecutive times t_n, you will see how the shape of $f(x)$ travels to the right at speed c.

$u(x,0) = f(x)$



$u(x,t) = f(x-ct)$



Moreover, it turns out that this function $u(x,t)$ solves the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \cdot \frac{\partial^2 u}{\partial x^2}$$

$$= \frac{\partial^2}{\partial t^2}\left(f(x-ct)\right) - c^2 \cdot \frac{\partial^2}{\partial x^2}\left(f(x-ct)\right)$$

$$= (-c)^2 f''(x-ct) - c^2 \cdot f''(x-ct)$$

$$= \underline{\underline{0}} \; !$$

However, one has to be careful about satisfying the boundary conditions too.

Using the above trigonometric identities, you will show in Exercise 4.4.6 that the solution

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\cdot\left(A_n \cos\left(\frac{n\pi c}{L}t\right) + B_n \sin\left(\frac{n\pi c}{L}t\right)\right)$$

to the IVP for the one-dim. wave equation

(PDE) $$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L$$

(BC) $$u(0,t) = 0$$
$$u(L,t) = 0$$

(IC) $$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

can always be written as

$$u(x,t) = \underbrace{S(x+ct)}_{\text{wave travelling to the left}} + \underbrace{R(x-ct)}_{\text{wave travelling to the right}}$$

for some functions S, R.

Comparison between 1D heat equation and 1D wave equation

We can now compute the solutions to the 1D heat equation

$$\frac{\partial v}{\partial t} = \underset{\substack{\uparrow \\ k=1}}{1} \cdot \frac{\partial^2 v}{\partial x^2}, \qquad 0 \le x \le L, \quad t > 0$$

$$v(0,t) = 0$$

$$v(L,t) = 0$$

$$v(x,0) = f(x)$$

and the 1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = \underset{\substack{\uparrow \\ c=1}}{1} \cdot \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L, \quad t > 0$$

$$u(0,t) = 0$$

$$u(L,t) = 0$$

$$u(x,0) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x).$$

Although the PDEs look relatively similar, their solutions behave <u>fundamentally different</u>:

The solution to the heat equation

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 t}$$

<u>decays exponentially</u> in time

(to the steady-state solution, for these BCs just zero) while the solution to the wave equation

$$u(x,t) = \sum_{n=1}^{\infty} \left(\sin\left(\frac{n\pi x}{L}\right) \left(A_n \cos\left(\frac{n\pi c}{L}\cdot t\right) + B_n \cdot \sin\left(\frac{n\pi c}{L}\cdot t\right)\right)\right)$$

oscillates periodically in time (and in particular its amplitude does <u>not</u> decay).

This is also reflected by the fact that the wave equation admits a conserved energy, while such conserved quantities are not available for the heat equation.

For the solution $u(x,t)$ to the above wave equation with fixed ends note that we must also have that

$$\frac{\partial u}{\partial t}(0,t) = 0 = \frac{\partial u}{\partial t}(L,t).$$

Then we can compute that

$$\frac{\partial}{\partial t} \int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$= \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x}\underbrace{\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)}_{=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)} dx$$

integrate by parts

$$= \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial u}{\partial t}\, dx + \underbrace{\left[\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t}\right]_{x=0}^{x=L}}_{=0}$$

$$= \int_0^L \frac{\partial u}{\partial t}\underbrace{\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right)}_{=0} dx$$

$$= 0$$

u solves the wave equation

Thus, the energy

$$E(u) = \int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$$

is conserved (constant in time).

In contrast, for the solution $v(x,t)$ to the above heat equation we find that

$$\frac{\partial}{\partial t}\int_0^L v(x,t)^2\,dx$$

$$= \int_0^L 2v\frac{\partial v}{\partial t}\,dx$$

insert heat equation

$$= \int_0^L 2v\cdot\frac{\partial^2 v}{\partial x^2}\,dx$$

integrate by parts

$$= \underbrace{\left[2\cdot v\cdot\frac{\partial v}{\partial x}\right]_{x=0}^{x=L}}_{=0} - 2\underbrace{\int_0^L\left(\frac{\partial v}{\partial x}\right)^2 dx}_{\geq 0}$$

$$= -2\int_0^L\left(\frac{\partial v}{\partial x}\right)^2 dx \leq 0$$

Thus, $\int_0^L v(x,t)^2\,dx$ is decreasing as time goes by! Integrating in time from $t=0$ to $t=T$ we find

$$\underbrace{\int_0^L v(x,T)^2\,dx}_{\substack{\text{decreasing} \\ \text{(as } T \text{ grows)}}} + 2\underbrace{\int_0^T\int_0^L\left(\frac{\partial v}{\partial x}(x,t)\right)^2 dx\,dt}_{\substack{\text{increasing} \\ \text{(as } T \text{ grows)}}} = \underbrace{\int_0^L v(x,0)^2\,dx}_{\text{initial condition}}$$