Nonhomogeneous Problems

(following Sections 8.1 – 8.3)

So far we have only been able to use the method of separation of variables to solve the linear and homogeneous heat equation (with linear and homogeneous boundary conditions). Here we develop the additional tools that will enable us to also solve the linear nonhomogeneous heat equation with linear nonhomogeneous boundary conditions of the form

(PDE) $\dfrac{\partial u}{\partial t} = k \cdot \dfrac{\partial^2 u}{\partial x^2} + Q(x,t)$, $\quad 0 \le x \le L$, $t > 0$

source term

(BC) $u(0,t) = A(t)$
$u(L,t) = B(t)$

(IC) $u(x,0) = f(x)$

We proceed step by step, eventually dealing with the above most general case.

Time-independent boundary conditions

(with no sources)

$$\text{(PDE)} \quad \frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,\ t > 0$$

$$\text{(BC)} \quad \begin{aligned} u(0,t) &= A \\ u(L,t) &= B \end{aligned}$$

$$\text{(IC)} \quad u(x,0) = f(x)$$

for some fixed (temperatures) $A, B \in \mathbb{R}$.

To solve this problem, first determine the equilibrium temperature distribution $u_E(x)$ satisfying

$$\frac{d^2 u_E}{dx^2} = 0$$

$$u_E(0) = A$$

$$u_E(L) = B$$

$$\Rightarrow \quad u_E(x) = A + \frac{B-A}{L} \cdot x$$

[we computed this in the first week of our course]

Then define the new variable

$$v(x,t) := u(x,t) - u_E(x)$$

(which can be interpreted as the temperature displacement from the equilibrium temperature distribution $u_E(x)$)

Then since $\frac{\partial u_E}{\partial t} = 0$ and $\frac{d^2 u_E}{dx^2} = 0$,
v must satisfy

$$\text{(PDE)} \quad \frac{\partial v}{\partial t} = k \cdot \frac{\partial^2 v}{\partial x^2}$$

$$\text{(BC)} \quad \begin{aligned} v(0,t) &= \underline{\underline{0}} \\ v(L,t) &= \underline{\underline{0}} \end{aligned}$$

$$\text{(IC)} \quad v(x,0) = f(x) - u_E(x)$$ ← modified initial condition!

This is a linear homogeneous heat equation with zero BCs (!). We have already derived that the solution is given by

$$v(x,t) = \sum_{n=1}^{\infty} a_n \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 t},$$

where the coefficients a_n are determined from the IC for $v(x,0) = f(x) - u_E(x)$:

$$f(x) - u_E(x) = v(x,0) = \sum_{n=1}^{\infty} a_n \cdot \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \quad \underline{\underline{a_n = \frac{2}{L} \int_0^L \left(f(x) - u_E(x)\right) \sin\left(\frac{n\pi x}{L}\right) dx}}$$

Hence, the solution to the original problem is

$$u(x,t) = u_E(x) + \sum_{n=1}^{\infty} a_n \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 t}.$$

In particular, we see that

$$u(x,t) \longrightarrow u_E(x) \quad \text{as} \quad t \to \infty.$$

Steady (time-independent) nonhomogeneous source terms

Consider

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2} + Q(x) \qquad \begin{array}{c} 0 \le x \le L \\ t > 0 \end{array}$$

time-independent

$$u(0,t) = A$$

$$u(L,t) = B$$

$$u(x,0) = f(x)$$

We proceed analogously as above by first determining the steady-state temperature distribution $u_E(x)$

$$0 = k \cdot \frac{d^2 u_E}{dx^2} + Q(x)$$

$$u_E(0) = A$$

$$u_E(L) = B$$

Then

$$v(x,t) := u(x,t) - u_E(x)$$

again has to solve a linear homogeneous heat equation with zero BCs, which we know how to deal with.

Time-dependent nonhomogeneous terms

Let's now up the ante and try to deal with time-dependent nonhomogeneous terms:

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad \begin{matrix} 0 \le x \le L \\ t > 0 \end{matrix}$$

$$u(0,t) = A(t)$$

$$u(L,t) = B(t)$$

$$u(x,0) = f(x)$$

Here the first step is to transform the problem into a linear (possibly still) nonhomogeneous heat equation but at least with homogeneous boundary conditions.

To this end we take any reference temperature distribution $r(x,t)$ satisfying

$$\left.\begin{matrix} r(0,t) = A(t) \\ r(L,t) = B(t) \end{matrix}\right. \quad \text{for all } t > 0.$$

For instance,

$$r(x,t) = A(t) + \frac{x}{L}\left(B(t) - A(t)\right)$$

does the job (There are many candidates!).

$\rightsquigarrow$ Note that $r(x,\underline{\underline{t}})$ is time-dependent.

Then define

$$v(x,t) := u(x,t) - r(x,t).$$

Observe that

$$v(0,t) = 0 \qquad \leftarrow \text{homogeneous BCs!}$$
$$v(L,t) = 0$$

and

$$\frac{\partial v}{\partial t} - k\cdot\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial t}(u-r) - k\cdot\frac{\partial^2}{\partial x^2}(u-r)$$

$$= \underbrace{\frac{\partial u}{\partial t} - k\cdot\frac{\partial^2 u}{\partial x^2}}_{=\,Q(x,t)} - \frac{\partial r}{\partial t} + k\cdot\frac{\partial^2 r}{\partial x^2}$$

using the PDE for u

$$\Rightarrow \frac{\partial v}{\partial t} = k\cdot\frac{\partial^2 v}{\partial x^2} + \underbrace{\left(Q(x,t) - \frac{\partial r}{\partial t}(x,t) + k\cdot\frac{\partial^2 r}{\partial x^2}(x,t)\right)}_{=:\,\overline{Q}(x,t)}$$

<u>new</u> source term

and for the initial condition

$$v(x,0) = u(x,0) - r(x,0) =: g(x).$$

Hence, $v(x,t)$ has to satisfy a linear heat equation with a <u>nonhomogeneous</u> source term, but at least with zero boundary conditions. Thus, we have slightly reduced the complexity of our problem and can now focus on how to deal with a time-dependent source term.