Method of eigenfunction expansion with homogeneous boundary conditions

We now seek to solve the following nonhomogeneous heat equation (with homogeneous boundary conditions)

$$(*)\begin{cases} \text{(PDE)} & \dfrac{\partial v}{\partial t} = k \cdot \dfrac{\partial^2 v}{\partial x^2} + \overline{Q}(x,t) \qquad \begin{matrix} 0 < x < L \\ t > 0 \end{matrix} \\ \text{(BC)} & \begin{matrix} v(0,t) = 0 \\ v(L,t) = 0 \end{matrix} \\ \text{(IC)} & v(x,0) = g(x) \end{cases}$$

Recall:

If $\overline{Q}(x,t) = 0$, then

$$v(x,t) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right) e^{-k\cdot\left(\frac{n\pi}{L}\right)^2 t}$$

$$= \sum_{n=1}^{\infty} \underbrace{\left(B_n \cdot e^{-k\cdot\left(\frac{n\pi}{L}\right)^2 t}\right)}_{=a_n(t)} \cdot \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\phi_n(x)}$$

where the related homogeneous boundary value problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

$$\phi(0) = 0$$

$$\phi(L) = 0$$

and

$$B_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

has the eigenvalues

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad , \quad n = 1, 2, 3, \ldots$$

with associated eigenfunctions

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad , \quad n = 1, 2, 3, \ldots$$

In order to solve the nonhomogeneous heat equation $(*)$, the key idea is to write the unknown solution $v(x,t)$ at every time t as an infinite sine series; in other words, to expand the unknown solution $v(x,t)$ at every time t in terms of the eigenfunctions $\phi_n(x)$ of the associated (homogeneous) boundary value problem

$$(**) \; v(x,t) = \sum_{n=1}^{\infty} a_n(t) \cdot \phi_n(x) \qquad \text{for } t \geq 0$$

method of eigenfunction expansion

time-dependent (not equal to $B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}$)

$\leadsto$ For $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ this is just the Fourier sine series of $v(x,t)$ at time t; but we use this more general notation because the underlying idea works much more generally.

In $(**)$ the homogeneous boundary conditions are automatically satisfied and for the initial condition we need

$$g(x) = \sum_{n=1}^{\infty} a_n(0) \cdot \phi_n(x).$$

By the orthogonality of the eigenfunctions $\phi_n(x)$ we can determine the coefficients

$$a_n(0) = \frac{\int_0^L g(x)\, \phi_n(x)\, dx}{\int_0^L \phi_n(x)^2\, dx} \quad \left(= \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx \right)$$

To determine $a_n(t)$ we plug the expansion $(**)$ for $v(x,t)$ at every $t \geq 0$ into the PDE for v:

$$\frac{\partial v}{\partial t} = \sum_{n=1}^{\infty} \frac{da_n}{dt}(t) \cdot \phi_n(x),$$

$$\frac{\partial^2 v}{\partial x^2} = \sum_{n=1}^{\infty} a_n(t) \cdot \frac{d^2 \phi_n}{dx^2}(x)$$

$$= - \sum_{n=1}^{\infty} a_n(t) \cdot \lambda_n\, \phi_n(x)$$

Then our PDE

$$\frac{\partial v}{\partial t} - k \cdot \frac{\partial^2 v}{\partial x^2} = \overline{Q}(x,t)$$

reads

$$\sum_{n=1}^{\infty}\left(\frac{da_n}{dt}(t)+k\cdot\lambda_n\, a_n(t)\right)\phi_n(x) = \overline{Q}(x,t)$$

We can of course also expand the given (and thus known) source term $\overline{Q}(x,t)$ in terms of the eigenfunctions $\phi_n(x)$ (at every $t\geq 0$)

$$\overline{Q}(x,t) = \sum_{n=1}^{\infty} \overline{q}_n(t)\cdot\phi_n(x)$$

with

$$\overline{q}_n(t) = \frac{\int_0^L \overline{Q}(x,t)\,\phi_n(x)\,dx}{\int_0^L \phi_n(x)^2\,dx} \quad \left(= \frac{2}{L}\int_0^L \overline{Q}(x,t)\sin\left(\frac{n\pi x}{L}\right)dx\right)$$

Hence we have

$$\sum_{n=1}^{\infty}\left(\frac{da_n}{dt}(t)+k\cdot\lambda_n\cdot a_n(t)\right)\phi_n(x) = \sum_{n=1}^{\infty}\overline{q}_n(t)\;\phi_n(x)$$

and it follows that we must have for every $n\in\mathbb{N}$

$$\frac{da_n}{dt}(t)+k\cdot\lambda_n\, a_n(t) = \overline{q}_n(t),$$

which is a 1^{st} order linear nonhomogeneous ODE!

Multiplying it by the integrating factor $e^{k\lambda_n t}$ we obtain

$$\underbrace{e^{k\lambda_n t}\left(\frac{da_n}{dt} + k\lambda_n a_n\right)}_{} = e^{k\lambda_n t}\,\overline{q}_n(t)$$

$$= \frac{d}{dt}\left(e^{k\lambda_n t} a_n(t)\right)$$

Integrating from 0 to t yields

$$e^{k\lambda_n t} a_n(t) - a_n(0) = \int_0^t e^{k\lambda_n \tau}\,\overline{q}_n(\tau)\,d\tau$$

$$\Rightarrow \boxed{a_n(t) = a_n(0)\,e^{-k\lambda_n t} + e^{-k\lambda_n t}\int_0^t e^{k\lambda_n \tau}\,\overline{q}_n(\tau)\,d\tau}$$

With this choice of $a_n(t)$ the solution to $(*)$ is then given by

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\cdot \phi_n(x).$$

Note as a "sanity check" that if $\overline{Q}(x,t) = 0$ we obtain

$$v(x,t) = \sum_{n=1}^{\infty} a_n(0)\,e^{-k\lambda_n t}\,\phi_n(x),$$

Fourier sine coefficients of the initial condition $f(x)$

which is what we'd had previously derived with the method of separation of variables.

Example:

Determine the solution to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-t}\sin(3x) \qquad \begin{array}{l} 0 \le x \le \pi \\ t > 0 \end{array}$$

$$u(0,t) = 0$$

$$u(\pi,t) = 1$$

$$u(x,0) = f(x)$$

Step 1: We make the boundary conditions homogeneous by passing to the variable (displacement from equilibrium)

$$v(x,t) = u(x,t) - \frac{x}{\pi},$$

which has to satisfy

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + e^{-t}\sin(3x)$$

$$v(0,t) = 0$$

$$v(L,t) = 0$$

$$v(x,0) = f(x) - \frac{x}{\pi}$$

Step 2: Solve the nonhomogeneous heat equation for $v(x,t)$ (with homogeneous BCs!)

The eigenfunctions are

$$\phi_n(x) = \sin\left(\frac{n\pi x}{\pi}\right) = \sin(nx), \quad n=1,2,\ldots$$

with eigenvalues

$$\lambda_n = \left(\frac{n\pi}{\pi}\right)^2 = n^2, \quad n=1,2,\ldots$$

So we write

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t) \cdot \sin(nx).$$

Substituting into the PDE yields

$$\sum_{n=1}^{\infty} \left(\frac{da_n}{dt} + n^2 a_n \right) \sin(nx) = e^{-t} \sin(3x).$$

Thus,

$$\frac{da_n}{dt} + n^2 a_n = \begin{cases} 0 & , \; n \neq 3 \\ e^{-t} & , \; n = 3 \end{cases}$$

which has the solutions

$\underline{n \neq 3:}$ $\quad a_n(t) = a_n(0)\, e^{-n^2 t}$

$\underline{n = 3:}$ $\quad a_3(t) = a_3(0)\, e^{-9t} + e^{-9t} \int_0^t \underbrace{e^{+9\tau} \cdot e^{-\tau}}_{= e^{8\tau}} \, d\tau$

$$= a_3(0)\, e^{-9t} + \frac{1}{8}\left(e^{-t} + e^{-9t}\right)$$

where

$$a_n(0) = \frac{2}{\pi} \int_0^{\pi} \left(f(x) - \frac{x}{\pi} \right) \sin(nx)\, dx.$$