MATH 5410 LECTURE NOTES
PRELIMINARIES:
EIGENFUNCTIONS BY EXAMPLE (FOURIER SERIES)

A. Kaffel

Related reading

Chapter 3 in Haberman reviews Fourier series. The relevant sections are 3.1 to 3.5. We'll address the complex version later, so 3.6 is not necessary (but useful to read if, for instance, you are used to the complex Fourier series from previous study).

Topics covered

- Fourier series (classical)
  - Definition
  - Convergence theory: norm, pointwise (what does equals mean?)
- Properties of Fourier series
  - Gibbs' phenomenon (oscillations around discontinuities)
  - Manipulating Fourier series; differentiation
  - Coefficient decay [[moved to supplemental notes]]
- Sine and cosine series
  - Even and odd extensions
  - Using symmetry to compute Fourier series

Main goals

- Recall why representation by an orthogonal basis of functions are useful (using Fourier series as an example), and what it means to 'converge' for such a series
- Review convergence in norm vs. pointwise convergence (and why this matters)
- Get some intuition for the effect of discontinuities on Fourier series

1. Fourier series: definitions

Before getting to the general theory, we can gain some insight by reviewing a special case. For this section, the interval will mostly be assumed to be $[-\pi, \pi]$ but this can be changed by rescaling x (see also Chapter 3 in Haberman).

1.1. **Periodic functions.** A 2π **periodic** function is a function $f(x)$ such that

$$f(x) = f(x + 2\pi) \text{ for all } x \in \mathbb{R}.$$

We can think of periodic functions as equivalent to the same function defined over one period by **restricting** the domain to $[-\pi, \pi]$. For 2π-periodic functions, we define the inner product and norm as integrals over one period, e.g.

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\, dx.$$

To go from a function on $[-\pi,\pi]$ to a 2π-periodic function, define the **periodic extension**

$$f(x+2\pi) = f(x) \text{ for all } x \in \mathbb{R}.$$

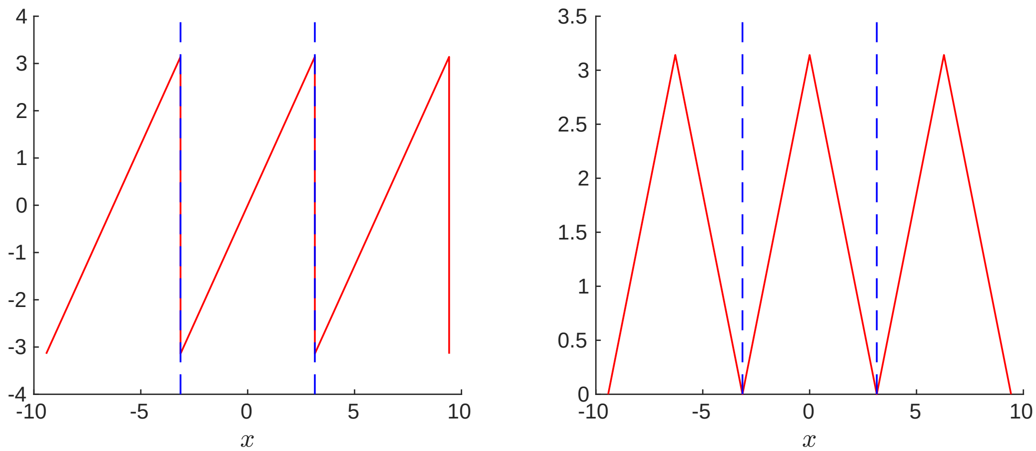
Note that if $f(-\pi) \neq f(\pi)$ one of the two values must be replaced.

Why does this matter: The periodic function 'wraps around', so it is **not continuous** if the endpoints do not match, i.e. $f(-\pi) \neq f(\pi)$. The function

$$f(x) = x, \quad x \in [-\pi,\pi]$$

is continuous as a function in $L^2[-\pi,\pi]$ but its 2π-periodic extension is discontinuous.

On the other hand, $f(x) = \pi - |x|$ (defined in the same way) is continuous.



1.2. **Fourier series.** Define the 'Fourier basis' functions

$$\phi_0 = 1/2, \qquad \phi_n = \cos nx, \quad \psi_n = \sin nx \text{ for } n \geq 1. \tag{1.1}$$

For reasons that will become clear later, we have the following result:

Theorem (Fourier series): The set of functions (1.1) is an **orthogonal basis** for the space $L^2[-\pi,\pi]$, or equivalently for the periodic version. That is, every function $f \in L^2[-\pi,\pi]$ has a unique representation (the **Fourier series**)

$$f = a_0\phi_0 + \sum_{n=1}^{\infty}(a_n\phi_n + b_n\psi_n) \tag{1.2}$$

with equality in the L^2 sense (see next section) and

$$\langle \phi_m, \psi_n \rangle = 0 \text{ for all } m, n,$$
$$\langle \phi_m, \phi_n \rangle = 0, \quad \langle \psi_m, \psi_n \rangle = 0 \text{ for } m \neq n$$

Note that the Fourier series for $f \in L^2[-\pi,\pi]$ is automatically 2π-periodic.

The orthogonality relations are straightforward to check explicitly. The fact that it is a basis is much more remarkable.[1] Because the basis is orthogonal, it is straightforward to compute

[1]Hidden behind this theorem is a linear operator L whose eigenfunctions are the ϕ's and ψ's, and this is the structure that gives us this nice orthogonal basis (but we'll get to that later).

the coefficients, which are given by

$$a_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}, \quad b_n = \frac{\langle f, \psi_n \rangle}{\langle \psi_n, \psi_n \rangle}.$$

It is not hard to generalize this to $[-\ell, \ell]$ as summarized below.

Computing the Fourier series: For $f \in L^2[-\ell, \ell]$, the Fourier basis functions are

$$\phi_0 = \frac{1}{2}, \qquad \phi_n = \cos\frac{n\pi x}{\ell}, \ \psi_n = \sin\frac{n\pi x}{\ell} \tag{1.3}$$

and the coefficients of the Fourier series (1.2) are given by

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx, \quad n \geq 0, \tag{1.4}$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx, \quad n \geq 1. \tag{1.5}$$

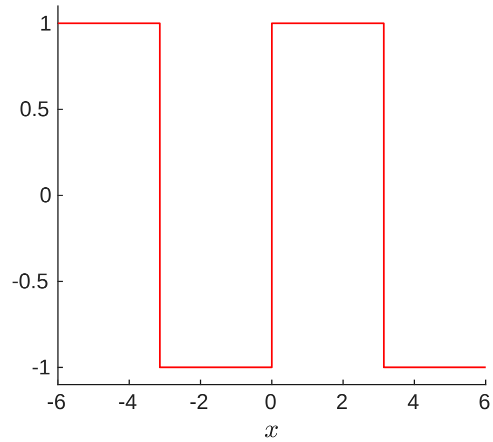
The choice of $1/2$ over some other constant (really ϕ_0 is just 'the constant function') is by convention, so the nice equation (1.4) works.

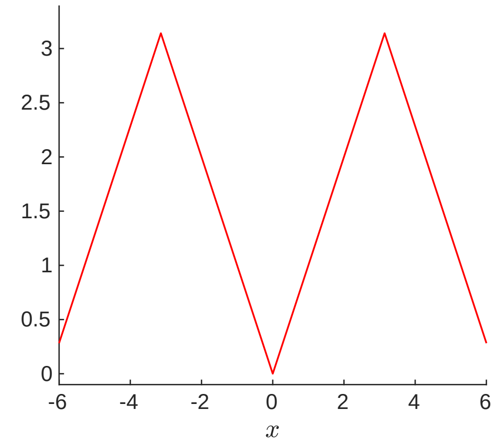
1.3. **Illustrative examples.** We review two examples here for later use:

$$\textbf{square wave:} \quad f_S(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}, \qquad x \in [-\pi, \pi] \tag{1.6}$$

$$\textbf{triangle wave:} \quad f_T(x) = |x|, \qquad x \in [-\pi, \pi] \tag{1.7}$$

with both functions 2π-periodic.





Calculations (square wave): The cosine coefficients are

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

Since $f(x)$ is an odd function, $f(x)\cos nx$ is an odd function for all n so the integral evaluates to zero. Thus $a_n = 0$ for all n.

Reminder (even and odd functions): Recall that a function is **even** if $f(x) = f(-x)$ and **odd** if $f(x) = -f(-x)$. Note that

$$\text{odd} \times \text{odd} = \text{even and odd} \times \text{even} = \text{odd}.$$

When integrating over symmetric intervals $[-\ell, \ell]$, make use of even and odd symmetry:

$$\int_{-L}^{L} f(x)\,dx = 0 \text{ if } f \text{ is odd}, \qquad \int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx \text{ if } f \text{ is even.}$$

For instance,

$$\int_{-\pi}^{\pi} 2x^3 + 4x^7\cos x + x^2\,dx = 2\int_0^{\pi} x^2\,dx$$

since the first two terms are zero. Even and odd symmetries show up often in solving DEs.

For the sine coefficients, symmetry gives us less since $f(x)\sin nx$ is even:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin nx\,dx = -\frac{2}{n\pi}\cos nx\Big|_0^{\pi} = \frac{2}{n\pi}(1-\cos n\pi).$$

Since $\cos n\pi = (-1)^n$, the result is that

$$b_n = \begin{cases} 0 & n \text{ even} \\ 4/(\pi n) & n \text{ odd} \end{cases}. \tag{1.8}$$

Thus the Fourier series is

$$f_S(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

with b_n given by (1.8). Or, if you want to plug in b_n, it can be written (two ways) as

$$f_S(x) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin nx \quad \text{or} \quad f_S(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)x).$$

How you handle writing the series is a matter of preference.

Calculations (triangle wave): The sine coefficients are

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

since $f(x)$ is even (so $f\sin nx$ is odd). For the cosines, $a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = 1$ and for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx \\ &= \frac{2}{\pi}\left(\frac{1}{n}x\sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right) \\ &= -\frac{2}{n\pi}\int_0^{\pi}\sin nx\,dx \\ &= -\frac{2}{\pi n^2}(1-\cos n\pi). \end{aligned}$$

Similar to the square wave, we get

$$a_0 = 1, \quad a_n = \begin{cases} 0 & n \text{ even} \\ -4/(\pi n^2) & n \text{ odd} \end{cases} \text{ for } n \geq 1. \tag{1.9}$$

The Fourier series is then

$$f_T(x) = \frac{1}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos((2n-1)x).$$

2. Fourier series: convergence

See last week's notes for the definitions of pointwise and norm convergence used in this section. We will also need the **partial sums**

$$S_N(x) := a_0\phi_0 + \sum_{n=1}^{N} a_n\phi_n + b_n\psi_n.$$

Each partial sum is an approximation to the Fourier series. Here we state precisely what this means, and how S_N behaves as N increases (adding more terms).

2.1. Convergence in norm.

For an L^2 function f, the expression

$$f = a_0\phi_0 + \sum_{n=1}^{\infty} (a_n\phi_n + b_n\psi_n)$$

is always true in the sense that the **partial sums converge in norm** to f:

$$\|S_N - f\|_2 \to 0 \text{ as } N \to \infty.$$

Moreover, this is enough to show that the series converges pointwise to f almost everywhere (but not at all points; see below!). To summarize:

Theorem (convergence in norm): Let $f \in L^2[-\pi, \pi]$ and let S_N be the N-th partial sum of the Fourier series

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

Then the series converges in the L^2 norm to f, i.e.

$$\|f - S_N\|_2 \to 0 \text{ as } n \to \infty$$

and converges pointwise a.e. to f, i.e.

$$\lim_{N\to\infty} S_N(x) = f(x) \text{ for almost every } x \in [-\pi, \pi].$$

To get a better sensse of this 'L^2 convergence', consider the general expression

$$f = \sum_{n=1}^{\infty} c_n\phi_n, \qquad S_N = \sum_{n=1}^{N} c_n\phi_n$$

where $\{\phi_n\}$ is an orthogonal basis. Assuming convergence, we have that

$$\|f\|^2 = \langle f, f\rangle = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_m c_n \langle \phi_m, \phi_n\rangle = \sum_{n=1}^{\infty} c_n^2 \|\phi_n\|^2$$

which is **Parseval's theorem**. Similarly, looking at the L^2 'error' between f and S_N,

$$\|f - S_N\|^2 = \langle \sum_{m=N+1}^{\infty} c_m\phi_m, \sum_{n=N+1}^{\infty} c_n\phi_n\rangle = \sum_{n=N+1}^{\infty} c_n^2\|\phi_n\|^2. \tag{2.1}$$

For instance, for the square wave $f_S(x)$, we know $c_n = 4/(\pi n)$ or zero so

$$\sum_{n=N+1}^{\infty} c_n^2\|\phi_n\|^2 \leq \sum_{n=N+1}^{\infty} \frac{8}{\pi^2 N^2} = O(1/N)$$

so the coefficients decrease fast enough to ensure convergence as $N \to \infty$.

General rule: The error (2.1) will go to zero as $N \to \infty$ if and only if the set of functions $\{\phi_n\}$ is 'complete', i.e. if it is indeed a basis. You can think of it this way: we need all the possible components of f to be in the sum, otherwise there may be one missing from the series and the error can never to go zero.

2.2. **Pointwise and uniform convergence.** This is not really the result we want, since it does **not** say that equality holds at each point x (pointwise convergence). However, with more assumptions on f, we can do better.

The Fourier series allows for some discontinuities. Call a function **piecewise continuous** if it is continuous except at a set of jump discontinuities (e.g. the square wave). For such a function, define the **left and right limits**

$$f(x^-) = \lim_{\xi \nearrow x} f(\xi), \quad f(x^+) = \lim_{\xi \searrow x} f(\xi).$$

Note that f is continuous at x if and only if $f(x^-) = f(x^+)$.

Theorem (Pointwise/uniform convergence): Let $S_N(x)$ be the N-th partial sum of the Fourier series for $f \in L^2[-\ell, \ell]$. Then the following holds:

(i) If f and f' are piecewise continuous, then

$$\lim_{N\to\infty} S_N(x) = \begin{cases} f(x) & \text{if } f \text{ is continuous at } x \\ \frac{1}{2}(f(x^-) + f(x^+)) & \text{if } f \text{ has a jump at } x. \end{cases}$$

That is, the partial sums converge **pointwise** to the average of the left and right limits.

(ii) If f is continuous as a periodic function and f' is piecewise continuous, then the partial sums converge to $f(x)$ **uniformly** as $n \to \infty$, i.e.

$$\lim_{N\to\infty}\left(\max_{x\in[-\pi,\pi]} |S_N(x) - f(x)|\right) = 0.$$

The conditions in (i) will almost always be met in practice, but (ii) may not. The key point is that while the Fourier series can represent functions with jumps, convergence is troublesome near or at the jump.

2.3. **Typical examples.** The concepts here are best illustrated by contrasting the two examples from subsection 1.3. Recall that we defined

$$\textbf{square wave:}\quad f_S(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}, \qquad x \in [-\pi, \pi],$$

$$\textbf{triangle wave:}\quad f_T(x) = |x|, \qquad x \in [-\pi, \pi],$$

and found that

$$f_S(x) = \frac{4}{\pi}\sum_{n \text{ odd}}^{\infty} \frac{1}{n}\sin(nx),$$

$$f_T(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}}^{\infty} \frac{1}{n^2}\cos(nx).$$

Triangle wave: Since $f(x) = |x|$ is continuous as a 2π-periodic function defined on $[-\pi, \pi]$, its Fourier series converges to $f(x)$ pointwise. Moreover, f' is piecewise continuous (with jumps at $x = \pm\pi$ and $x = 0$), so (ii) applies and the convergence is **uniform**.

This makes sense since the coefficients decay like $1/n^2$ (see **??** for more)

$$|f(x) - S_N(x)| \leq \frac{4}{\pi}\sum_{n=N+1}^{\infty} \frac{1}{n^2} = O(1/N).$$

The error is worst at the corners where f' has a jump (shown for $x = 0$ in figure 1).

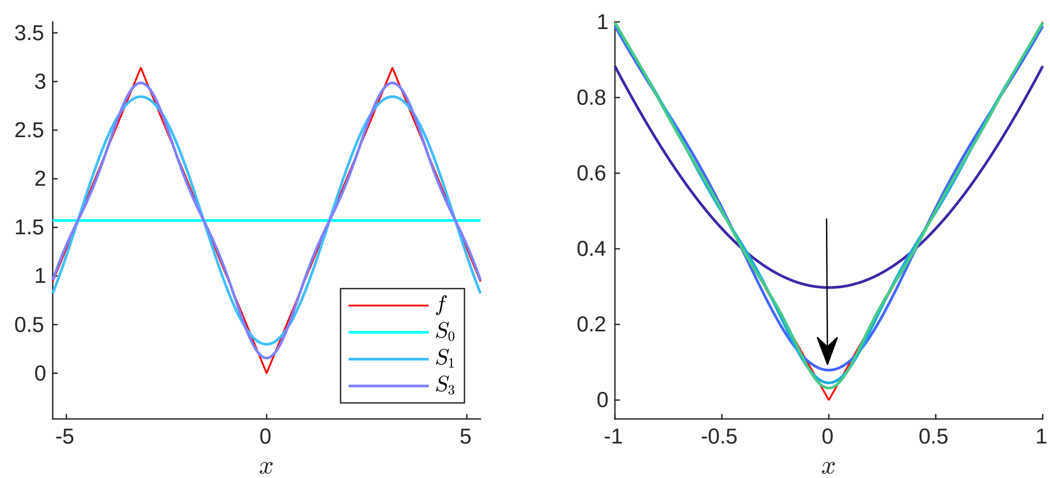


FIGURE 1. Partial sums with one, two and three terms (S_0, S_1 and S_3) for the triangle wave and zoomed in, showing the convergence at a corner ($x = 0$).

Square wave: This is more interesting since f_S has jumps at $x = 0$ and at $x = \pm\pi$. Thus (i) applies but (ii) does not. According to (i), the series converges to the function

$$\frac{1}{2}(f_S(x^-) + f_S(x^+)) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \\ 0 & x = 0 \text{ or } x = \pm\pi. \end{cases}.$$

From a plot of the partial sums (figure 2), we see that the convergence is **not uniform** in a dramatic way. We observe **Gibbs phenomenon** at the jump: the oscillations around the correct value ($f = 1$ in the figure) do not decay in magnitude as we add more terms. So although we have

$$S_N(x) \to f(x) \text{ as } n \to \infty \text{ for all } x \text{ not at jumps,}$$

the maximum error (even excluding the x-values of the jumps) does not go to zero:

$$\max_{0<x<\pi} |S_N(x) - f(x)| \approx 0.09 \cdot (\text{jump height}) \text{ as } n \to \infty.$$

This is typical behavior for Fourier series at jump discontinuities, so it deserves a box:

Gibbs' phenomenon: Suppose $f(x)$ has a jump discontinuity at x_0. Then it Fourier series overshoots the correct values on each side by about 9% of the jump height. The oscillations **always persist** but become concenrated to a **smaller and smaller** interval around the jump as $n \to \infty$. Note that the 9% is true of any function with a jump, not just the square wave!

Thus, for functions that are not continuous, we simply have to accept that the Fourier series will have these oscillations. We can 'squish' them into an arbitrarily small region around the jump by taking enough terms, but the peak will always remain.

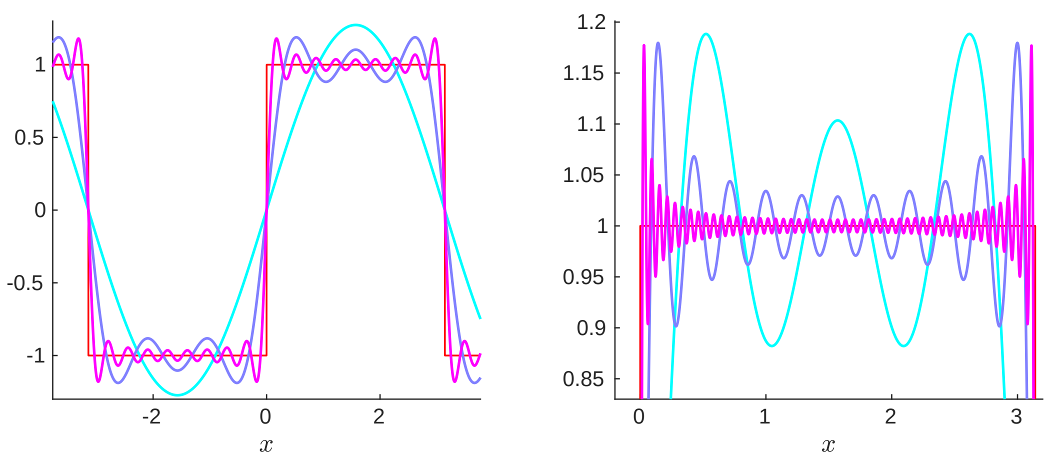


Figure 2. Top: Partial sums for the square wave (with terms up to $n = 1, 5$ and 17). Bottom: zoom in on a corner showing the effect of increasing n to a large number (with $n = 5, 21$ and $n = 101$); the oscillations never decrease in amplitude but become more concentrated near the jump.

3. Sine and cosine series

Suppose now that $f(x)$ is defined on the interval $[0,\pi]$. There are two notable ways to extend this function to $[-\pi,\pi]$:

$$\text{odd extension: } f_O(x) = \begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \end{cases}$$

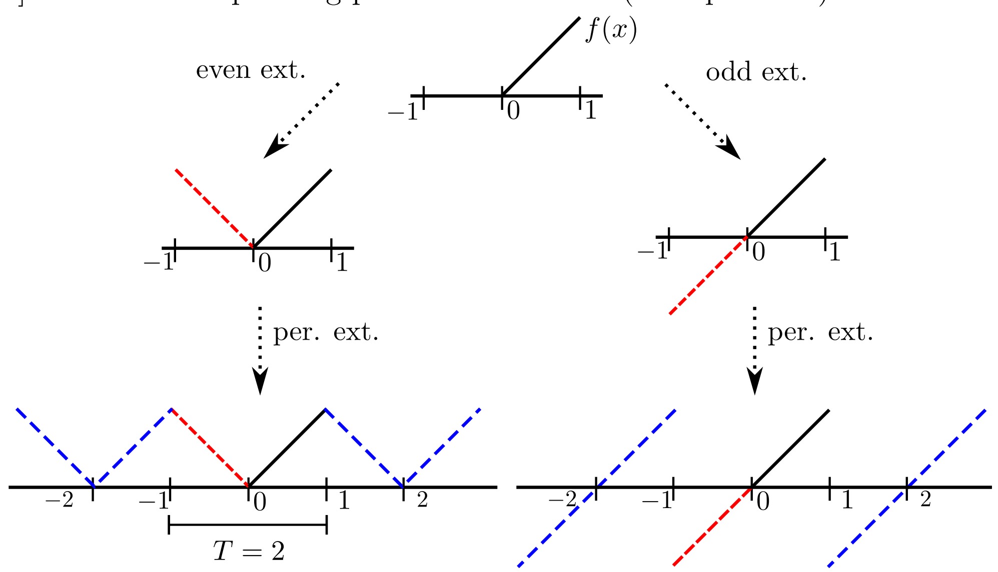
$$\text{even extension: } f_E(x) = \begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}$$

where the value at $x = 0$ in the odd case is typically taken to be zero (but does not matter). These extensions, defined on $[-\pi,\pi]$, can then be extended as 2π periodic functions (leading to the 'odd periodic' and 'even periodic' extensions). Similarly, if $f \in L^2[0,\ell]$ then it has odd/even extensions to $[-\ell,\ell]$ and 2ℓ-periodic versions.

Example (extension): function

$$f(x) = x \qquad \text{for } x \in [0,1]$$

to $[-1,1]$ and the corresponding periodic extensions (with period 2) are shown below.



Observe that the Fourier series for the **odd extension** contains **only sines** and the Fourier series for the **even extension** contains **only cosines**. Both extensions are just $f(x)$ on the original interval $[0,\pi]$, however, which gives us two representations for $f(x)$.

Fourier sine and cosine series: Consider L^2 functions defined on the interval $[0,\pi]$.

(i) The functions

$$\psi_n = \sin nx, \quad n = 1,2,3,\cdots$$

are an orthogonal basis for $L^2[0,\pi]$, and the series

$$f(x) = \sum_{n=1}^{\infty} c_n\psi_n, \qquad c_n = \frac{\langle f,\psi_n\rangle}{\langle \psi_n,\psi_n\rangle} = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

is called the **Fourier sine series** for f. This is the same as the full Fourier series for the **odd extension** of f to $[-\pi, \pi]$, with the same convergence properties.

(ii) The functions

$$\phi_0 = \frac{1}{2}, \quad \phi_n = \cos nx,\ n = 1, 2, 3, \cdots$$

are also an orthogonal basis for $L^2[0, \pi]$, and the series

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n, \qquad c_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx$$

is called the **Fourier cosine series** for f. This is the same as the full Fourier series for the **even extension** of f to $[-\pi, \pi]$,, with the same convergence properties. Note that (for applying the main convergence theorem) the odd extension $f_O(x)$ is **not continuous** as a periodic function unless $f(\pi) = 0$ (since its odd extension has $f_O(-\pi) = -f(\pi)$ and continuity requires $f_O(-\pi) = f_O(\pi)$.

This means that a function f defined on $[0, L]$ can be represented in terms of just sines or just cosines. As with the full Fourier series, both sets are eigenfunctions from a linear operator, which we'll introduce later; it is these operators that will determine which of the series is needed.

Example (sine/cosine series) Consider the function

$$f(x) = 1, \qquad x \text{ in } [0, \pi].$$

Suppose we wish to find a series representation

$$f = \sum_n c_n \phi_n.$$

We may use either the Fourier sine or cosine series (later, there will be a reason to pick one over the other; for now, we consider both just as an example).

The Fourier cosine series for $f(x)$ is trivial, since the constant function is one of the basis functions already, so

$$\langle f, \phi_n \rangle = 0 \text{ for } \phi_n = \cos nx,\ n \geq 1.$$

Thus the Fourier cosine series is

$$f(x) = 1 + \sum_{n=1}^{\infty} 0 \cdot \cos nx.$$

The even extension to $[-\pi, \pi]$ is also just $f(x) = 1$. On the other hand, the sine series is

$$f(x) = \sum_{n=1}^{\infty} c_n \psi_n, \quad c_n = \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin nx\, dx.$$

This is also the Fourier series for the odd extension, which is the square wave

$$f(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases},$$

defined in $[-\pi, \pi]$ so the series is the same as the square wave Fourier series from before,

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi/2}{n} \sin(nx).$$

The series converges to $f(x)$ for $x \in (-\pi, \pi)$ and to 0 (the average of -1 and 1 when $x = \pm\pi$. Note that the sine and cosine series agree on $(0, \pi)$ where both extensions are continuous, but**do not agree** at $x = 0$ and $x = \pi$.

4. Series operations

Consider two 'eigenfunction' series (written abstractly)

$$f = \sum_n c_n \phi_n, \quad g = \sum_n d_n \phi_n$$

such as two Fourier sine series. We may freely add series together and multiply by scalars term by term:

$$k_1 f + k_2 g = \sum_n (k_1 c_n + k_2 d_n) \phi_n.$$

We may also integrate term by term:

$$\int_0^x f(\xi)\, d\xi = \sum_n c_n \int_0^x \phi_n(\xi) d\xi.$$

For example, consider the square/triangle waves from before. We have that

$$\int_0^x f_S(\xi)\, d\xi = f_T$$

and for the series, integrating term by term gives

$$\int_0^x (\text{series for } f_S)\ dx = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi/2}{n} \int_0^x \sin n\xi\, d\xi = C + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi/2}{n^2} \cos nx = \text{series for } f_T$$

after solving for the constant C (for this, the easiest way to find the constant is by computing a_0 directly rather than using the series). Note that

$$\text{integration} \implies \text{adds a factor of } 1/n$$

which makes the series converge 'faster', which is what allows the integration to safely be performed term by term (we don't introduce any problems). Thus as long as the **original series** converges, we are okay.

However, it is **not always safe to differentiate term by term**. To see a case where it works, consider the same example. The derivative of the triangle is

$$\frac{d}{dx}(f_T) = \frac{d}{dx}|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} = f_S.$$

Differentiating term by term, we get

$$\frac{d}{dx}(\text{series for } f_T) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi/2}{n^2}(-\frac{1}{n}\sin(nx)) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi/2}{n}\sin nx = \text{series for } f_S.$$

Differentiation makes things worse since

$$\text{differentiation} \implies \text{adds a factor of } n$$

but the series for f_T converges fast enough that the **differentiated series** converges. We need coefficients to decay like $1/n^2$; from the previous section this means f must be continuous and f' must be piecewise continuous (see Haberman 3.4 for a more detailed discussion).

For a case where it fails, consider

$$f(x) = x, \quad x \in [-\pi, \pi].$$

Its Fourier series is

$$f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

You might expect its derivative to be the Fourier series for 1. However,

$$1 \neq 2\sum_{n=1}^{\infty}(-1)^{n+1}\cos nx$$

and in fact the sum on the right hand side does not even converge. Differentiating has turned a series that converges in L^2 (a real Fourier series) into nonsense. **It will be important later** to deal with such technical issues; we will need tools to manage derivatives of series without reckless operations.