REVIEW: First Order Differential Equations

Ahmed Kaffel

Dept of Mathematical & Statistical Sciences
Marquette University
Miwaukee, Wisconsin
Email: ahmed.kaffel@marquette.edu
https://www.mscsnet.mu.edu/ ahmed/

Outline

1. Introduction
2. Classification of Differential Equations by:
   - Type
   - Order
   - Linearity
3. Applications of Differential Equations
4. Solving first order linear Differential Equations using integrating factor.
5. Solving Separable Differential Equations.
6. Solving Bernoulli's equations.
7. Conclusions

Introduction

- Differential equations frequently arise in modeling situations
- They describe population growth, chemical reactions, heat exchange, motion, and many other applications
  - The classical example is Newton's Law of motion
    - The mass of an object times its acceleration is equal to the sum of the forces acting on that object
    - Acceleration is the first derivative of velocity or the second derivative of position
  - In biology, a differential equation describes a growth rate, a reaction rate, or the change in some physiological state

What is a Differential Equation?

What is a Differential Equation?

Definition (Differential Equation)

An equation that contains derivatives of one or more unknown functions with respect to one or more independent variables is said to be a **differential equation**.

Example : $y' = 4y + 2t - 1$

Type of a Differential Equation

- This lecture considers **Ordinary Differential Equations**, where the **unknown function and its derivatives** depend on a single **independent variable**
- Mathematical physics often needs **Partial Differential Equations**, where the **unknown function and its derivatives** depend on two or more **independent variables**
  - **Example: Heat Equation**

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2}$$

Classification of Differential Equations

Definition (Order)

The **order** of a **differential equation** matches the order of the highest derivative that appears in the equation.

Definition (Linear Differential Equation)

An n^{th} order ordinary differential equation $F(t, y, y', ..., y^{(n)}) = 0$ is said to be **linear** if it can be written in the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + ... + a_n(t)y = g(t).$$

The functions a_0, a_1, ... a_n, called the **coefficients** of the equation, can depend at most on the independent variable t. This equation is said to be **homogeneous** if the function $g(t)$ is zero for all t. Otherwise, the equation is **nonhomogeneous**.

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of change in temperature of a cooling body is proportional to the difference between the temperature of the body and the surrounding environmental temperature

- If $T(t)$ is the temperature of the body, then it satisfies the differential equation

$$\frac{dT}{dt} = -k(T(t) - T_e) \quad \text{with} \quad T(0) = T_0$$

- The parameter k is dependent on the specific properties of the particular object (body in this case)
- T_e is the environmental temperature
- T_0 is the initial temperature of the object

Falling Cat

Model for the Falling Cat: Newton's law of motion

- **Mass times acceleration is equal to the sum of all the forces** acting on the object
    - Equation for **Falling Cat**

$$ma = -mg \qquad \text{or} \qquad a = -g$$

    - m is the mass of the cat
    - a is the acceleration
    - $-mg$ is the force of gravity (assuming up is positive)
    - Ignore other forces (air resistance)
- g is a constant ($g = 980.7$ cm/sec^2)

Falling Cat

Height, Velocity, and Acceleration

- Let $h(t)$ be the height (or position) of the cat at any time t
- **Velocity** and **Acceleration** satisfy:

$$\frac{dh}{dt} = v(t) \qquad \text{and} \qquad \frac{d^2h}{dt^2} = \frac{dv}{dt} = a$$

- The **initial conditions** for falling off a limb:

$$h(0) = h_0 > 0 \qquad \text{and} \qquad v(0) = 0$$

Solution of Linear Growth and Decay Models

For **Malthusian growth** or **Radioactive decay** the ***linear differential equation***:

$$\frac{dy}{dt} = a\,y \qquad \text{with} \qquad y(0) = y_0,$$

has the solution:

$$y(t) = y_0 e^{at}.$$

More generally, we have the following solution:

Method (General Solution to Linear Growth and Decay Models)

Consider

$$\frac{dy}{dt} = a\,y \qquad \text{with} \qquad y(t_0) = y_0.$$

The solution is

$$y(t) = y_0 e^{a(t-t_0)}.$$

Example: Linear Decay Model

Example: Linear Decay Model: Consider

$$\frac{dy}{dt} = -0.3\,y \qquad \text{with} \qquad y(4) = 12$$

The solution is

$$y(t) = 12\,e^{-0.3(t-4)}$$

This solution shows a substance decaying at a rate $k = 0.3$ starting with 12 units of substance y.

However, the solution is ***shifted (horizontally)*** by 4 units of time.

Linear Differential Equation with Only Time Varying

Definition (Differential Equation with Time Varying Function)

The simplest first order (linear) differential equation has only a time varying nonhomogeneous function, $f(t)$,

$$\frac{dy}{dt} = f(t). \tag{1}$$

Theorem (Solution)

Consider the differential equation with only a time varying nonhomogeneous function, (1). Provided $f(t)$ is integrable, the solution satisfies:

$$y(t) = \int f(t)\,dt.$$

Linear Differential Equation Example

DE Example: Initial Value Problem

$$\frac{dy}{dt} = 2\,t - \sin(t), \qquad y(0) = 3$$

Solution:

$$y(t) = \int (2\,t - \sin(t))\, dt = t^2 + \cos(t) + C$$

$$y(0) = 1 + C = 3, \quad \text{so} \quad C = 2$$

$$y(t) = t^2 + \cos(t) + 2$$

First Order Linear ODEs

General Linear Differential Equation

There is no general method that solves every first order differential equation

Definition (General Linear Differential Equation)

A differential equation that is in the form

$$\frac{dy}{dt} + p(t)y = g(t) \tag{2}$$

is said to be a **first order linear differential equation** with dependent variable, y, and independent variable, t.

Integrating Factor

Definition (Integrating Factor)

Consider an undetermined function $\mu(t)$ with

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y.$$

The function $\mu(t)$ is an **integrating factor** for (2) if it satisfies the differential equation

$$\frac{d\mu(t)}{dt} = p(t)\mu(t).$$

First Order Linear ODEs

General Integrating Factor

The differential equation for the **integrating factor** is

$$\frac{d\mu(t)}{dt} = p(t)\mu(t) \qquad \text{or} \qquad \frac{1}{\mu(t)}\frac{d\mu(t)}{dt} = p(t).$$

Note that $\frac{d(\ln(\mu(t)))}{dt} = \frac{1}{\mu(t)}\frac{d\mu(t)}{dt}$. It follows that

$$\ln(\mu(t)) = \int p(t)dt.$$

The **general integrating factor** satisfies

$$\mu(t) = e^{\int p(t)dt}.$$

1^{st} Order Linear DE Solution

Thus, the 1^{st} **Order Linear DE Solution**

$$\frac{dy}{dt} + p(t)y = g(t) \qquad \text{with} \qquad \mu(t) = e^{\int p(t)dt}$$

is integrated to produce

$$\mu(t)y(t) = \int \mu(t)g(t)\,dt + C.$$

Theorem (Solution of 1^{st} Order Linear DE)

With the 1^{st} Order Linear DE given above and assuming integrability of $p(t)$ and $g(t)$, then the solution is given by

$$y(t) = e^{-\int p(t)dt}\left[\int e^{\int p(t)dt} g(t)\,dt + C\right].$$

First Order Linear ODEs

The Integrating factor method

1. Put the Differential Equation in the form $y' + p(t)y = g(t)$.
2. Identify $p(t)$ and find the integrating factor $\mu(t)$.
3. Multiply the Differential Equation by $\mu(t)$ and simplify the equation.
4. Integrate the simplified Differential Equation and solve for y.

Solving a Linear DE

Consider the **Linear Differential Equation**

$$\frac{dy}{dt} - 2y = 4 - t.$$

Multiply the equation by the undetermined function, $\mu(t)$, so

$$\mu(t)\frac{dy}{dt} - 2\mu(t)y = \mu(t)(4 - t).$$

If $\mu(t)$ is an integrating factor, then

$$\frac{d\mu(t)}{dt} = -2\mu(t) \qquad \text{or} \qquad \mu(t) = e^{-2t}$$

Solving a Linear DE

With the **integrating factor**, our example can be write

$$e^{-2t}\frac{dy}{dt} - 2e^{-2t}(t)y = \frac{d}{dt}\left[e^{-2t}y\right] = (4-t)e^{-2t}.$$

The quantity $\frac{d}{dt}\left[e^{-2t}y\right]$ is a total derivative, so we integrate both sides giving:

$$e^{-2t}y(t) = \int (4-t)e^{-2t}dt + C = \frac{1}{4}(2t-7)e^{-2t} + C,$$

so

$$y(t) = \frac{1}{4}(2t-7) + Ce^{2t}.$$

Linear DE –Example

Consider the **Linear DE Solution**

$$t\frac{dy}{dt} - y = 3t^2\sin(t).$$

1. Put this equation into standard form, so divide by t and obtain

$$\frac{dy}{dt} - \left(\frac{1}{t}\right)y = 3t\sin(t). \tag{3}$$

2. Observe $p(t) = -\frac{1}{t}$, so find integrating factor

$$\mu(t) = e^{\int(-1/t)dt} = e^{-\ln(t)} = \frac{1}{t}.$$

3. Multiply (3) by $\mu(t)$ giving

$$\left(\frac{1}{t}\right)\frac{dy}{dt} - \left(\frac{1}{t^2}\right)y = \frac{d}{dt}\left(\frac{y}{t}\right) = 3\sin(t).$$

Linear DE –Example

The previous slide showed the transformation of

$$t\frac{dy}{dt} - y = 3t^2 \sin(t)$$

with the integrating factor $\mu(t) = \frac{1}{t}$ to

$$\frac{d}{dt}\left(\frac{y}{t}\right) = 3\sin(t).$$

4. Integrate this equation

$$\left(\frac{1}{t}\right) y(t) = 3\int \sin(t)dt + C = -3\cos(t) + C,$$

which gives the **solution**

$$y(t) = -3t\cos(t) + Ct.$$

Separable Differential Equations

Separation of Variables

Definition (Separable Differential Equation)

Consider the differential equation

$$\frac{dy}{dt} = f(t, y),$$

and suppose that $f(t, y)$ can be written as the product of a function, $p(t)$, that only depends on t and another function, $q(y)$, that depends only on y. The differential equation

$$\frac{dy}{dt} = f(t, y) = p(t)q(y),$$

is called **separable**.

Separation of Variables

Theorem (Solution of a Separable Differential Equation)

Consider the ***separable differential equation***

$$\frac{dy}{dt} = p(t)q(y),$$

and assume that $q(y)$ is nonzero for y values of interest. The solution of this differential equation satisfies

$$\int q^{-1}(y)dy = \int p(t)dt.$$

Example 1 - Separable Differential Equation

Example - Separable Differential Equation Consider the differential equation

$$\frac{dy}{dt} = 2ty^2$$

Solution:

- Separate the variables t and y
  - Put only $2t$ and dt on the right hand side
  - And only y^2 and dy are on the left hand side
- The integral form is

$$\int \frac{dy}{y^2} = \int 2t\, dt$$

Example 1 - Separable Differential Equation

Solution (cont) The two integrals are

$$\int \frac{dy}{y^2} = \int 2t\, dt$$

- The two integrals are easily solved

$$-\frac{1}{y} = t^2 + C$$

- **Note** that you only need to put **one arbitrary constant**, despite solving two integrals
- This is easily rearranged to give the solution in explicit form

$$y(t) = -\frac{1}{t^2 + C}$$

Example 2 - Separable Differential Equation

Example 2: Consider the initial value problem

$$\frac{dy}{dt} = \frac{4\,\sin(2t)}{y} \quad \text{with} \quad y(0) = 1$$

Solution: Begin by separating the variables, so

$$\int y\,dy = 4\int \sin(2t)dt$$

Solving the integrals gives

$$\frac{y^2}{2} = -2\,\cos(2t) + C$$

Example 2 - Separable Differential Equation

Solution (cont) Since

$$\frac{y^2}{2} = -2\cos(2t) + C$$

We write

$$y^2(t) = 2C - 4\cos(2t) \qquad \text{or} \qquad y(t) = \pm\sqrt{2C - 4\cos(2t)}$$

From the initial condition

$$y(0) = 1 = \sqrt{2C - 4\cos(0)} = \sqrt{2C - 4}$$

Thus, $2C = 5$, and

$$y(t) = \sqrt{5 - 4\cos(2t)}$$

Example 3 - Separable Differential Equation

Example 3: Consider the initial value problem

$$\frac{dy}{dt} = -y\frac{(1+2\,t^2)}{t} \quad \text{with} \quad y(1) = 2$$

Solution: Begin by separating the variables, so

$$\int \frac{dy}{y} = -\int \frac{(1+2\,t^2)}{t}dt = -\int \frac{dt}{t} - 2\int t\,dt$$

Solving the integrals gives

$$\ln(y) = -\ln(t) - t^2 + C$$

Example 3 - Separable Differential Equation

Solution (cont): Since

$$\ln(y) = -\ln(t) - t^2 + C$$

Exponentiate both sides to give

$$y(t) = e^{-\ln(t)-t^2+C} = e^{-\ln(t)}e^{-t^2}e^C = \frac{A}{t}e^{-t^2}$$

where $A = e^C$
With the initial condition

$$y(1) = 2 = A\,e^{-1} \quad \text{or} \quad A = 2\,e^1$$

The solution is

$$y(t) = \frac{2}{t}e^{1-t^2}$$

Bernoulli's Equation

Definition

A differential equation of the form

$$\frac{dy}{dt} + q(t)y = r(t)y^n,$$

where n is any real number, is called a **Bernoulli's equation**

Define $u = y^{1-n}$, so

$$\frac{du}{dt} = (1-n)y^{-n}\frac{dy}{dt}$$

Bernoulli's Equation

The substitution $u = y^{1-n}$ suggests multiply by $(1-n)y^{-n}$, changing **Bernoulli's Equation** to

$$(1-n)y^{-n}\frac{dy}{dt} + (1-n)q(t)y^{1-n} = (1-n)r(t),$$

which results in the new equation

$$\frac{du}{dt} + (1-n)q(t)u = (1-n)r(t)$$

This is a 1^{st} **order linear differential equation**, which is easy to solve

Example: Bernoulli's Equation

Example: Consider the Bernoulli's equation:

$$3t\frac{dy}{dt} + 9y = 2ty^{5/3}$$

Solution: Rewrite the equation

$$\frac{dy}{dt} + \frac{3}{t}y = \frac{2}{3}y^{5/3}$$

and use the substitution $u = y^{1-5/3} = y^{-2/3}$ with $\frac{du}{dt} = -\frac{2}{3}y^{-5/3}\frac{dy}{dt}$

Multiply equation above by $-\frac{2}{3}y^{-5/3}$ and obtain

$$\frac{du}{dt} - \frac{2}{t}u = -\frac{4}{9},$$

which is a **linear differential equation**

Example: Bernoulli's Equation

Example (cont): The **linear differential equation** in $u(t)$ is

$$\frac{du}{dt} - \frac{2}{t}u = -\frac{4}{9},$$

which has an integrating factor

$$\mu(t) = e^{-2\int \frac{dt}{t}} = e^{-2\ln(t)} = \frac{1}{t^2}$$

This gives

$$\frac{d}{dt}\left(\frac{u}{t^2}\right) = -\frac{4}{9t^2},$$

which integrating gives

$$\frac{u}{t^2} = \frac{4}{9t} + C \qquad \text{or} \qquad u(t) = \frac{4t}{9} + Ct^2$$

Example: Bernoulli's Equation

Example (cont): However, $u(t) = y^{-2/3}(t)$, so if

$$u(t) = \frac{4t}{9} + Ct^2, \qquad \text{then} \qquad y^{-2/3}(t) = \frac{4t}{9} + Ct^2$$

The explicit solution is

$$y(t) = \left(\frac{9}{4t + 9Ct^2} \right)^{\frac{3}{2}}$$

Conclusions

Things to Remember

1. Classification of Differential Equations by:
   - Type
   - Order
   - Linearity
2. Solving first order linear Differential Equations using integrating factor.
3. Solving Separable Differential Equations.
4. Solving Bernoulli's Equations.

These concepts can be used in future to

- Solve System of Differential Equations.
- Solve second and higher order Partial Differential Equations.

References

Blanchard,Devaney, and Hall (2012)
Differential Equations, 3rd edition, by *ISBN 0-495-01265-3*.

Richard Haberman (2013)
Applied Partial Differential Equations (with Fourier Series and Boundary Value Problems). Pearson 2013. ISBN 9780321797056

Thank You!