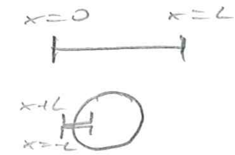
Review

PDEs

Linear heat equation (1D)

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}, \quad t > 0,$$



Laplace's equation (2D)

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

inside some domain $R \subset \mathbb{R}^2$

Notation: $\Delta \equiv \nabla^2$

no "physics" questions on exam

linear $\longleftrightarrow$ nonlinear

homogeneous $\longleftrightarrow$ inhomogeneous

superposition principle (!)

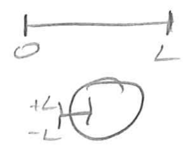
Key techniques

- method of separation of variables to solve
  - linear heat equation on an interval subject to various boundary conditions
  - Laplace's equation inside 
- sketch / determine Fourier series, Fourier cosine series, or Fourier sine series of a given (piecewise smooth) function

discussed in the last few lectures

Linear heat equation : $u = u(x,t)$

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2} \quad , \quad t > 0$$



Initial condition:

$$u(x,0) = f(x)$$ ← initial temperature distribution

Boundary conditions:

zero temperature	insulated boundary	periodic $(-L \le x \le +L)$
$u(0,t) = 0$	$\frac{\partial u}{\partial x}(0,t) = 0$	$u(-L,t) = u(+L,t)$
$u(L,t) = 0$	$\frac{\partial u}{\partial x}(L,t) = 0$	$\frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(+L,t)$

⇝ Heuristic strategy of method of separation of variables:

- first "cook up" special product solutions
- then use superposition principle to assemble them to obtain solutions (to linear heat equation) given by infinite series of product solutions
- invoke Fourier series results to treat any piecewise smooth function

Product solution ansatz:

$$u(x,t) = \phi(x) \cdot G(t)$$

Substitute into $\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2}$:

$$\phi \cdot \frac{dG}{dt} = k \cdot \frac{d^2\phi}{dx^2} \cdot G$$

$$\Rightarrow \quad \frac{1}{k \cdot G} \cdot \frac{dG}{dt} = \frac{1}{\phi} \frac{d^2\phi}{dx^2} = -\lambda$$

separation constant

<u>t-dependent ODE:</u>

$$\frac{dG}{dt} = -\lambda k \cdot G$$

$$\Rightarrow \quad G(t) = C \cdot e^{-\lambda k t}$$

<u>x-dependent ODE / boundary value problem:</u>

$$\frac{d^2\phi}{dx^2} = -\lambda \phi$$

Determine <u>eigenvalues</u> and <u>eigenfunctions</u>

I <u>zero temp.:</u> $(0 \le x \le L)$

$\phi(0) = 0$

$\phi(L) = 0$

II <u>insulated boundary</u> $(0 \le x \le L)$

$\frac{d\phi}{dx}(0) = 0$

$\frac{d\phi}{dx}(L) = 0$

III <u>periodic</u> $(-L \le x \le L)$

$\phi(-L) = \phi(+L)$

$\frac{d\phi}{dx}(-L) = \frac{d\phi}{dx}(+L)$

Memorize these three cases

I: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

$n = 1, 2, 3, \ldots$

$\phi_n(x) = \sin(nx)$

II: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

$n = 0, 1, 2, 3, \ldots$

$\phi_n(x) = \cos(nx)$

III: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

$n = 0, 1, 2, 3, \ldots$

$\phi_n(x) = \cos(nx), \sin(nx)$

Recall how to determine eigenvalues by distinguishing cases for λ:

general solutions

$\lambda > 0$: $\quad c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

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$\lambda = 0$: $\quad c_1 x + c_2$

$\lambda < 0$: $\quad c_1 \cdot \cosh(\sqrt{-\lambda}\, x) + c_2 \cdot \sinh(\sqrt{-\lambda}\, x)$

or $\quad c_1 \cdot e^{+\sqrt{-\lambda}\, x} + c_2 \cdot e^{-\sqrt{-\lambda}\, x}$

Use superposition principle to assemble the product solutions:

(I) $$u(x,t) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 t}$$

$$\Rightarrow f(x) \overset{?}{=} u(x,0) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L$$

possible to write any piecewise smooth $f(x)$, $0 \le x \le L$, as such an infinite sine series (by Fourier's theorem...)

From orthogonality of sines $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

not necessary to memorize Fourier coefficients formulas for exam!

Example:

$$f(x) = 9 \cdot \sin\left(\frac{2019\pi x}{L}\right) + 5 \sin\left(\frac{7\pi x}{L}\right)$$

$$\Rightarrow u(x,t) = 9 \cdot \sin\left(\frac{2019\pi x}{L}\right) e^{-k \cdot \left(\frac{2019\pi}{L}\right)^2 t} + 5 \cdot \sin\left(\frac{7\pi x}{L}\right) \cdot e^{-k \cdot \left(\frac{7\pi}{L}\right)^2 t}$$

(II) $$u(x,t) = \sum_{n=0}^{\infty} A_n \cdot \cos\left(\frac{n\pi x}{L}\right) \cdot e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 t} \quad , \quad 0 \leq x \leq L$$

$$f(x) = \sum_{n=0}^{\infty} A_n \cdot \cos\left(\frac{n\pi x}{L}\right)$$

Fourier cosine series

$$A_0 = \frac{1}{L} \int_0^L f(x)\, dx$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

(III) $$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-k \cdot \left(\frac{n\pi}{L}\right)^2 t} \quad , \quad -L \leq x \leq +L$$

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{L}\right)$$

Fourier series

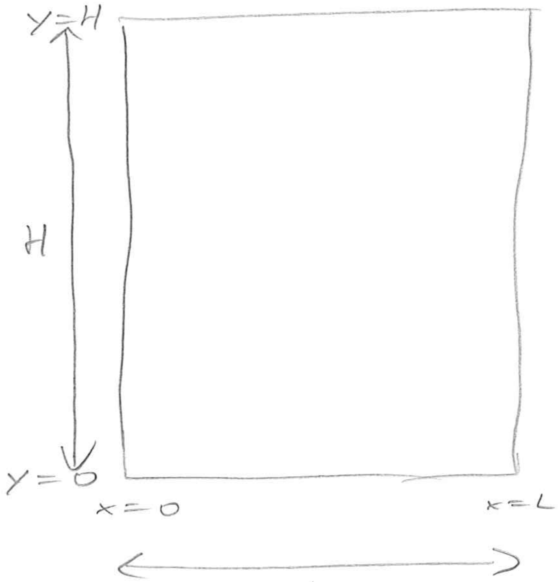
$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{+L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \quad , \quad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L} \int_{-L}^{+L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad , \quad n = 1, 2, 3, \ldots$$

Laplace's equation inside a rectangle

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \begin{array}{l} 0 \le x \le L \\ 0 \le y \le H \end{array}$$



with one BC for each side of rectangle

By superposition principle, we can write

$$u(x,y) = u_1(x,y) + u_2(x,y) + \ldots + u_4(x,y)$$

so that each u_j, $j = 1,2,3,4$, satisfies $\Delta u_j = 0$ where u_j has <u>zero</u> BCs on three sides (homogeneous!) and <u>only one</u> non-trivial BC.

Product solution ansatz:

$$u(x,y) = h(x) \cdot \phi(y)$$

$$\Rightarrow \quad \frac{1}{h}\frac{d^2 h}{dx^2} = -\frac{1}{\phi}\frac{d^2\phi}{dy^2} = \text{const.}\ (\lambda)$$

Then we obtain one BVP for either $h(x)$ or $\phi(y)$ [only one of them will have two zero BCs].

Use that BVP to determine the eigenvalues and eigenfunctions!

Then assemble product solutions to match the one non-trivial BC.

Laplace's equation inside a disk

Use polar coordinates

$u = u(r,\theta)$, $0 \le r \le a$, $-\pi \le \theta \le \pi$

$u(a,\theta) = f(\theta)$

Compatibility conditions

$$u(r,-\pi) = u(r,\pi)$$

$$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi)$$

On exam: polar coordinates related formulas would be provided!

$$|u(0,\theta)| < \infty$$

(sensible from physics point of view)

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Product ansatz:

$$u(r,\theta) = G(r)\cdot\phi(\theta)$$

$$\Rightarrow \quad \phi(-\pi) = \phi(\pi)$$

$$\frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi)$$

leads to

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\cdot\frac{d^2\phi}{d\theta^2} = \lambda$$

Boundary value problem:

$$\frac{d^2\phi}{d\theta^2} = -\lambda\phi, \quad -\pi \le \theta \le \pi$$

$$\phi(-\pi) = \phi(\pi)$$

$$\frac{d\phi}{d\theta}(-\pi) = \frac{d\phi}{d\theta}(\pi)$$

$\lambda = \left(\frac{n\pi}{\pi}\right)^2 = n^2, \; n = 0,1,2,\ldots$

$\cos(n\theta), \sin(n\theta)$

ODE for G

$$r^2 \cdot \frac{d^2G}{dr^2} + r \cdot \frac{dG}{dr} - n^2 G = 0$$

general solution:

$$n \neq 0: \quad G(r) = c_1 \cdot r^{+n} + \cancel{c_2 \cdot r^{-n}}$$

$$n = 0: \quad G(r) = \bar{c}_1 + \cancel{\bar{c}_2 \ln(r)}$$

$|u(0,\theta)| < \infty$

Assemble product solutions:

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n \cdot r^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n \cdot r^n \sin(n\theta)$$

$$f(\theta) = \sum_{n=0}^{\infty} A_n \cdot a^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n \cdot a^n \cdot \sin(n\theta)$$

with

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$A_n \cdot a^n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(\theta) \cos(n\theta)\, d\theta$$

$$B_n \cdot a^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta$$