MATH 5410 LECTURE NOTES
MULTI-DIMENSIONAL PDES: QUICK SUMMARY

1. Remarks: coefficients vs. eigenfunctions

Eigenfunctions for time-dependent PDEs like $u_t = \nabla^2 u$ in $\Omega \subset \mathbb{R}^N$ solve Helmholtz' equation

$$- \nabla^2 \phi = \lambda \phi, \quad \mathbf{x} \in \Omega, \qquad (\text{...+BCs...}). \qquad (1.1)$$

The solution has **coefficients** in t and **eigenfunctions** in $\mathbf{x}$ (space); $u = \sum c_{\mathbf{k}}(t)\phi_{\mathbf{k}}(\mathbf{x})$. That is, there are N-dimensional eigenfunctions and coefficients in the t-direction.

For **Laplace's equation** $0 = \nabla^2 u$ in N dimensions, the eigenfunctions are in $N - 1$ variables, e.g. $\sum c_n(r) Y_n^m(\theta, \phi)$ in a sphere with eigenfunctions in θ, ϕ. The last direction is the 'coefficient' direction, which can have inhomogeneous BCs. To use SoV, the BCs must be **homogeneous in the eigenfunction directions**.

2. A summary table

The entries for the disk (r, θ) are the same as the cylinder. A more expanded version is on the next page (with solutions, equations etc.). 'Oscillatory' solutions are used for eigenfunctions, and 'non-oscillatory' for coefficients (for Laplace).

Geometry (dir.)	oscillatory	non-oscillatory
'Flat' $(x, z, \cdot)$	$\cos \mu x, \sin \mu x$	$\cosh \mu x, \sinh \mu x$ or $e^{\pm \mu x}$
Sphere (r)	Spherical Bessel	Cauchy-Euler $(x^r, \cdots)$ or mod. Bessel
Sphere (θ)	$\cos m\theta, \sin m\theta$	-
Sphere (ϕ)	Legendre poly. $P_n^m(\cos \phi)$	*
Cylinder (r)	Bessel $J_\nu(r\sqrt{\lambda}), Y_\nu(r\sqrt{\lambda})$	Cauchy-Euler (disk) or Mod. Bessel (K_ν, I_ν)
Cylinder (θ)	$\cos m\theta, \sin m\theta$	-

Some stray remarks:

- **Positive eigenvalues:** Check using the Rayleigh quotient or use knowledge of the problem (e.g. Dirichlet, Neumann, periodic all have no negative eigenvalues).

- **Shortcut (zero terms):** If all data is in the span of some set of eigenfunctions (e.g. all $f(r) \cos \theta$) then the solution is also in this span e.g. $\sum_n c_n R_n(r) \cos \theta$.

- Eigenfunctions form an orth. basis (in L^2 inner product) for their domain. Note that this is not necessarily the PDE domain (one dim. less for Laplace). Equivalent to 1d inner products via SL theory for fully separated problems. To find coefficients:

$$f = \sum_{\mathbf{k}} c_{\mathbf{k}} \phi_{\mathbf{k}} \implies c_{\mathbf{k}} = \frac{\langle f, \phi_{\mathbf{k}} \rangle}{\langle \phi_{\mathbf{k}}, \phi_{\mathbf{k}} \rangle}.$$

3. Eigenvalue problems by geometry

In all cases, the eigenfunctions/values are for the Helmholtz equation (1.1) in the appropriate domain. Note that 'or' refers to the periodic case with $\cos/\sin$.
In all cases positive eigenvalues have been assumed (writing e.g. μ^2 for eigenvalues), **even when it is possible (with nasty BCs) to get negative eigenvalues**.
'Dependence' means later eigenvalue problems depend on earlier ones.

Rectangle. Width L and height H, $\phi = g(x)h(y)$. Dependence: (none)

$$x\text{-dir: } -g'' = \mu^2 g, \quad x \in [0, L]$$

$$y\text{-dir: } -h'' = \eta^2 h, \quad y \in [0, H]$$

Disk. Radius a. Dependence: $\theta \to r$

$$\theta\text{-dir: } -g'' = \mu^2 g, \quad \theta \in [0, 2\pi], \ (2\pi\text{-periodic})$$

$$r\text{-dir: } R'' + \frac{1}{r}R' + \left(\lambda - \frac{\mu^2}{r^2}\right)R = 0, \quad r \in [0, a], \quad (R \text{ bounded})$$

Solutions (r-dir, $\lambda > 0$): $J_\mu(r\sqrt{\lambda})$, $Y_\mu(r\sqrt{\lambda})$; $|Y(0)| = \infty$ (unbounded)

Cylinder. Radius a. height H, $\phi = R(r)g(\theta)h(z)$. Dependence: $\theta, z \to r$

$$\theta\text{-dir: } -g'' = \mu^2 g, \quad \theta \in [0, 2\pi], \ (2\pi\text{-periodic})$$

$$z\text{-dir: } -h'' = \eta^2 h, \quad y \in [0, H]$$

$$r\text{-dir: } R'' + \frac{1}{r}R' + \left(\lambda - \eta^2 - \frac{\mu^2}{r^2}\right)R = 0, \quad r \in [0, a], \quad (R \text{ bounded})$$

Solutions: As in disk, but $\sqrt{\lambda - \eta^2}$ replaces $\sqrt{\lambda}$ (assume $\lambda > \eta^2$).

Sphere. Radius a. $\phi = R(r)Y(\theta, \phi) = R(r)g(\theta)h(\phi)$. Dependence: $\theta \to \phi \to r$.

$$\theta\text{-dir: } -g'' = \eta^2 g, \quad \theta \in [0, 2\pi], \ (2\pi\text{-periodic})$$

$$\phi\text{-dir:* } \frac{1}{\sin\phi}(\sin\phi\, h')' + \left(\lambda - \frac{m^2}{\sin^2\phi}\right)h = 0$$

$$r\text{-dir: } R'' + \frac{2}{r}R' + \left(\lambda - \frac{n(n+1)}{r^2}\right)R = 0, \quad r \in [0, a], \quad (R \text{ bounded})$$

$$*\text{transformed: } ((1-\xi^2)y')' + (\lambda - \frac{m^2}{1-\xi^2})y = 0, \qquad \xi = \cos\phi, y(\xi) = h(\phi)$$

Solutions: (θ, ϕ): $\ Y_n^m = P_n^m(\cos\phi)(\cos m\theta \ \text{or} \ \sin m\theta)$, eigvals $\lambda_n = n(n+1)$ for $0 \le m \le n$

$$r\text{-dir, } \lambda > 0: \ \frac{J_{n+1/2}(r\sqrt{\lambda})}{\sqrt{r}} \ \text{and} \ \frac{Y_{n+1/2}(r\sqrt{\lambda})}{\sqrt{r}} \ (Y \text{ unbounded in } [0, a])$$

4. Coefficient equations for Laplace

ODEs to solve for coefficients with Laplace's equation. Assumes eigenfunctions obtained as in previous list. These are the 'non-oscillatory' solutions to the typical ODEs.

Rectangle. Eigenfunctions $\phi(x)$

$$y\text{-dir: } h'' = \lambda h, \quad y \in [0, H]$$

Disk (inside/outside). Eigenfunctions $\phi(\theta)$

$$r\text{-dir: } R'' + \frac{1}{r}R' - \lambda R = 0, \quad r \in [0, a] \text{ or } [a, \infty), \quad (R \text{ bounded})$$

Solution (Cauchy-Euler): $p(\alpha) = \alpha^2 - \lambda \implies r^{\alpha_1}, r^{\alpha_2}$ or other cases [see Euler procedure]

Cylinder (inside/outside). Eigenfunctions $\phi(\theta, z)$, coeffs. in r

$$r\text{-dir: } R'' + \frac{1}{r}R' - \left(\lambda + \frac{m^2}{r^2}\right)R = 0, \quad r \in [0, a] \text{ or } [a, \infty), \quad (R \text{ bounded})$$

Solution: Modified Bessel $I_m(r\sqrt{\lambda})$, $K_m(r\sqrt{\lambda})$ with $|K_m(0)| = \infty$ (unbounded)

4.1. **Cylinder.** Eigenfunctions $\phi(r, \theta)$, coeffs. in z

$$z\text{-dir: } h'' = \lambda h, \quad z \in [0, H]$$

Sphere (inside/outside). Eigenfunctions $Y(\theta, \phi)$

$$r\text{-dir: } R'' + \frac{2}{r}R' - \lambda R = 0, \quad r \in [0, a] \text{ or } [a, \infty), \quad (R \text{ bounded})$$

Solution (Cauchy-Euler): $p(\alpha) = \alpha^2 + \alpha - \lambda \implies r^{\alpha_1}, r^{\alpha_2}$ or other cases [see Euler procedure]

---

5. Solutions to ODEs, properties

For details on Bessel functions and Legendre polynomials, see special functions notes.

6. Extra: other geometries to be aware of

Not part of the course, but good to be aware of a few other common geometries:

- **parabolic coordinates:** Used e.g. for air flow past a rounded object or some problems in a half-infinite plane. $x = \sigma^2 - \tau^2$ and $y = \sigma\tau$. The 'axes $\sigma = 0$ and $\tau = 0$ are the positive/negative halves of the x-axis and other coord. = const. lines are parabolas. Eigenvalue problems require **parabolic cylinder functions**.

- **Elliptical problems:** Like a sphere/circle, but for deformed objects. Truly horrible formulas involved. If nearly circular, a 'perturbation' method using some approximation might be preferable (see e.g. Section 9.6 of the book).

- **Orthogonal/curvilinear coordinates:** General coordinates $(x, y, z) \to (u_1, u_2, u_3)$ generalizing all the typical cases. Nice when orthogonal but requires serious algebra/geometry in complicated coordinate systems.