MULTI-VARIABLE CALCULUS REVIEW/REFERENCE

Notation: Throughout, bold indicates vector quantities.

- $\int_\Omega (\cdots)\, dV$ is the integral over a region Ω (either volume or area, sometimes dA)
- $\partial\Omega$ = boundary of Ω and $\int_{\partial\Omega}(\cdots)\, dS$ is the surface integral over $\partial\Omega$.
- $\mathbf{x} = (x_1, \cdots, x_m)$ or $\mathbf{x} = (x, y)$ or $\mathbf{x} = (x, y, z)$ ($\mathbb{R}^2$ or $\mathbb{R}^3$)
- $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are standard unit vectors in $\mathbb{R}^3$ and $\mathbf{n}$ is the outward normal to Ω

1. Operators and identities

Operators defined in $\mathbb{R}^m$ (or $\mathbb{R}^3$ for curl):

$$\textbf{gradient:} \quad \nabla u = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_m}\right)$$

$$\textbf{Divergence:} \quad \nabla \cdot \mathbf{v} = \sum_{j=1}^{m} \frac{\partial v_j}{\partial x_j}$$

$$\textbf{Laplacian:} \quad \nabla \cdot (\nabla u) = \sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_j^2} \text{ (also } \Delta u \text{ or } \nabla^2 u)$$

$$\textbf{Curl } (\mathbb{R}^3)\textbf{:} \quad \nabla \times \mathbf{v} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{bmatrix}$$

$$\textbf{`planar' curl:} \quad \nabla \times (v_1, v_2, 0) = \left(\frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x}\right)\mathbf{k}$$

$$\textbf{directional derivative:} \quad \frac{\partial f}{\partial \mathbf{n}} = \nabla f \cdot \mathbf{n} \text{ (where } \mathbf{n} \text{ is a direction)}$$

Physical/intuitive interpretations:

- **flux** of a vector quantity $\mathbf{F}$ through a boundary $\partial\Omega$:

$$\text{flux (out)} = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\, dS$$

- Curl $\nabla \times \mathbf{v} \to$ local rotation of the vector field $\mathbf{v}$ (up to a factor of 1/2)
- Divergence $\nabla \cdot \mathbf{v} \to$ source of $\mathbf{v}$ ($\nabla \cdot \mathbf{v} > 0 \to$ source; $\nabla \cdot \mathbf{v} < 0 \to$ sink)

Useful formulas/identities: Let $f(\mathbf{x})$ be a scalar function, $\mathbf{v}, \mathbf{w}$ vector fields.

$$\nabla \cdot (f\mathbf{v}) = \mathbf{v} \cdot \nabla f + f\, \nabla \cdot \mathbf{v}$$

$$\nabla \times (f\mathbf{v}) = \nabla f \times \mathbf{v} + f\, \nabla \times \mathbf{v}$$

$$\nabla \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{w} \cdot \nabla + \nabla \cdot \mathbf{w})\mathbf{v} - (\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla)\mathbf{w}$$

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

2. Multi-variable calculus theorems

- **Gradient field:** A vector field is a gradient if and only if it is curl free:
$$\mathbf{v} = \nabla f \text{ for some } f \iff \nabla \times \mathbf{v} = 0$$
Also, $\mathbf{v} = \nabla f \iff \mathbf{v}$ is conservative (line integrals of $\mathbf{v}$ independent of path)
- **'Solenoidal' field:** A vector field is a curl if and only if it is divergence free:
$$\mathbf{v} = \nabla \times \mathbf{w} \text{ for some } \mathbf{w} \iff \nabla \cdot \mathbf{v} = 0$$
- In particular, it is **always true** that $\nabla \cdot (\nabla \times \mathbf{v}) = 0$ and $\nabla \times (\nabla f) = 0$
- For the general theorem and technical requirements: see **Helmholtz decomposition**.

Let $\mathbf{n} = (n^1, \cdots, n^m)$ and let $\Omega \subset \mathbb{R}^m$.

$$\textbf{Gradient theorem:} \quad \int_\Omega \frac{\partial u}{\partial x_i}\, dV = \int_{\partial\Omega} u n^i \, dS,$$
$$\text{(combined)} \quad \int_\Omega \nabla u \, dV = \int_{\partial\Omega} u\mathbf{n}\, dS$$

$$\boxed{\textbf{Divergence theorem:} \quad \int_\Omega \nabla \cdot \mathbf{v}\, dV = \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\, dS.}$$

(Integral of divergence in a region equals integral of flux out the surface.)

Integration by parts: (rule: replace $\partial/\partial x_i$ with n^i or $\nabla\cdot$ with $\mathbf{n}\cdot$ in boundary term)

$$\text{1d:} \quad \int_{[a,b]} \frac{df}{dx} g\, dx = fg\Big|_a^b - \int_{[a,b]} f\frac{dg}{dx}\, dx.$$
$$\text{multi-d:} \quad \int_\Omega \frac{\partial f}{\partial x_i} g\, dV = \int_{\partial\Omega} f g n^i \, dS - \int_\Omega f\frac{\partial g}{\partial x_i}\, dV$$
$$\implies \text{(e.g.)} \quad \int_\Omega \mathbf{v}\cdot\nabla f\, dV = \int_{\partial\Omega} (\mathbf{v}\cdot\mathbf{n}) f\, dS - \int_\Omega (\nabla\cdot\mathbf{v}) f\, dV$$

Green's formulas/identities:

$$\int_\Omega f\nabla^2 g\, dV = \int_{\partial\Omega} f\frac{\partial g}{\partial \mathbf{n}}\, dS - \int_\Omega \nabla f\cdot\nabla g\, dV$$
$$\int_\Omega (f\Delta g - g\Delta f)\, dV = \int_{\partial\Omega} f\frac{\partial g}{\partial \mathbf{n}} - g\frac{\partial f}{\partial \mathbf{n}}\, dS$$

Stokes' theorem: S = surface with edge curve Γ, $\mathbf{t}$ =tangent vector to Γ; then

$$\oint_\Gamma \mathbf{v}\cdot\mathbf{t}\, ds = \int_S (\nabla\times\mathbf{v})\cdot\mathbf{n}\, dS$$

Green's theorems (2d): Let $\Omega \subset \mathbb{R}^2$ with no holes and $\mathbf{v} = (v, w)$ and $\mathbf{x} = (x, y)$). Then

$$\oint_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\, ds = \int_\Omega \nabla\cdot\mathbf{v}\, dA \implies \oint_{\partial\Omega} -w\, dx + v\, dy = \int_\Omega \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\, dA$$
$$\oint_{\partial\Omega} \mathbf{v}\cdot\mathbf{t}\, ds = \int_\Omega (\nabla\times\mathbf{v})\cdot\mathbf{k}\, dA \implies \oint_{\partial\Omega} v\, dx + w\, dy = \int_\Omega \frac{\partial w}{\partial x} - \frac{\partial v}{\partial y}\, dA$$

3. Coordinate transforms

3.1. **Geometry.** Here $\mathbf{n}$ is an outward normal; $\hat{\mathbf{n}}$ is a unit (outward) normal.
normal vectors:

$$\textbf{surface (2d):}\ \mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)) \implies \mathbf{n} = \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v},\ \hat{\mathbf{n}} = \frac{\mathbf{n}}{\|\mathbf{n}\|}$$

$$\textbf{graph (1d):}\ (x, f(x)) \implies \hat{\mathbf{n}} = \frac{(-f_x, 1)}{\sqrt{1+f_x^2}}$$

$$\textbf{graph (2d):}\ (x, y, f(x,y)) = \implies \hat{\mathbf{n}} = \frac{(-f_x, -f_y, 1)}{\sqrt{1+f_x^2+f_y^2}}$$

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area integral: Let $\mathbf{x}(u,v) = x\mathbf{i} + y\mathbf{j} = (x(u,v), y(u,v))$ (coordinates (u,v))

$$dA = \ dx\,dy = |J|\,du\,dv, \quad J = \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} = \det \begin{bmatrix} x_u & y_u \\ x_v & y_v \end{bmatrix}$$

For instance for polar coordinates $x = r\cos\theta, y = r\sin\theta$:

$$J = x_r y_\theta - x_\theta y_r = (\cos\theta)(r\cos\theta) - (-r\sin\theta)(\sin\theta) = r$$
$$\implies dA = r\,dr\,d\theta$$

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volume integral: Let $\mathbf{x}(u,v,w) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x(u,v,w), y(u,v,w), z(u,v,w))$.

$$dV = dx\,dy\,dz = |J|\,du\,dv\,dw, \quad J = \frac{\partial \mathbf{x}}{\partial \mathbf{w}} \cdot \left(\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v}\right) = \det[\nabla x \,|\, \nabla y \,|\, \nabla z]$$

where $\nabla x = (x_u, x_v, x_w)^T$ etc.

Surface integral: Let $\mathbf{x}(u,v) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $x = x(u,v)$ etc.

$$dS = \left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right| du\,dv = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{matrix} \right\| du\,dv$$

with the $||$(matrix)$||$ denoting abs. of the determinant.

4. Formulas for cylindrical/spherical coordinates:

For more details, see any reference on multi-variable calculus (good tables are easy to find). **Caution:** be aware of the differing conventions for spherical coordinates!

Cylindrical: radius r, angle θ (in xy plane)

Coordinates: $$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \\ z = z \end{cases}, \qquad \theta \in [0, 2\pi],\ \ r \geq 0$$

Unit vectors: $$\begin{cases} \hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}, \\ \hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}, \\ \hat{z} = \hat{z} \end{cases}$$

volume: $dV = r\,dr\,d\theta\,dz$

cyl. surface (rad. a): $dS = a\,dz\,d\theta$

Gradient: $\nabla f = f_r\hat{r} + \frac{1}{r} f_\theta \hat{\theta} + f_z\hat{z}$

Divergence: $\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF^r) + \frac{1}{r}\frac{\partial F^\theta}{\partial \theta} + \frac{\partial F^z}{\partial z}$

Laplacian: $\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$

Spherical: radius ρ, azimuthal angle θ, polar angle ϕ (see note[1])

Coordinates: $$\begin{cases} x = \rho\sin\phi\cos\theta, \\ y = \rho\sin\phi\sin\theta, \\ z = \rho\cos\phi \end{cases} \qquad \theta \in [0, 2\pi],\ \ \phi \in [0, \pi], \rho \geq 0$$

Unit vectors: $$\begin{cases} \hat{\rho} = \sin\phi\cos\theta\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \sin\phi\,\hat{z}, \\ \hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{z} \\ \hat{\phi} = \cos\phi\cos\theta\,\hat{x} + \cos\phi\sin\theta\,\hat{y} - \sin\phi\,\hat{z} \end{cases}$$

volume: $dV = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

sphere surface (rad. a): $dS = a^2\sin\phi\,d\phi\,d\theta$

Gradient: $\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho\sin\phi}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi}$

Divergence: $\nabla \cdot \mathbf{F} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 F^\rho) + \frac{1}{\rho\sin\phi}\frac{\partial F^\theta}{\partial \theta} + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi F^\phi)$

Laplacian: $\nabla^2 f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2\frac{\partial f}{\partial \rho}) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial \phi}(\sin\phi\frac{\partial}{\partial \phi})f$

[1] Here ϕ is the angle of $\mathbf{x}$ from the z-axis and θ is the angle in the (x, y) plane. The opposite convention is also often used. Be careful (look for the $\cos\phi$ vs. $\cos\theta$ in the z-equation). Also watch for the range of ϕ and θ (one goes up to π, the other to 2π; sometimes one ranges from $-\pi$ to π instead.