MATH 5410 LECTURE NOTES
MULTI-DIMENSIONAL PDES: SEPARABLE PROBLEMS

Note: the discussion here makes use of the notes on 'special functions' posted separately for the problems in curved geometries (cylinder, sphere).

Topics covered

- Theory for the Laplacian (∇^2) in bounded domains
  - Partial separation for problems with time (heat, wave)
  - Helmholtz equation ($-\nabla^2\phi = \lambda\phi$)
  - L^2 theory (adjoint, orthogonality) in 2d
  - Sturm-Liouville theory for general domains
  - Rayleigh quotient (proving eigenvalue are > 0)
- Separable problems in multiple dimensions
  - Heat/wave equation in a disk (Bessel functions, cylinder
  - Sphere, radially symmetric (spherical Bessel functions)
  - Surface of a sphere (spherical harmonics, Legendre polynomials)
  - Laplace/heat equation in a sphere
  - Handling negative eigenvalues for Bessel's equation
- Special functions
  - Bessel functions (first/second kind and modified)
  - Properties: oscillation, small $|x|$ behavior
  - Legendre functions/polynomials

1. Separation of variables

Consider the heat equation in a bounded region $\Omega \subset \mathbb{R}^2$ in the plane:

$$\begin{aligned} u_t = \nabla^2 u, \quad & \mathbf{x} \in \Omega,\ t > 0 \\ &\text{(hom. BCs on } \partial\Omega) \quad \text{for } t > 0 \\ & u(\mathbf{x}, 0) = f(\mathbf{x}). \end{aligned} \tag{1.1}$$

Here we write $\mathbf{x} = (x, y)$ and $u = u(x, y, t)$ or $u(\mathbf{x}, t)$, depending on which is convenient.

To solve, we can use **partial separation** to separate the t-part from the spatial part, leading to ODEs in t and an eigenvalue problem in Ω. Look or a solution

$$u(\mathbf{x}, t) = T(t)\phi(\mathbf{x}) \tag{1.2}$$

Plug into the PDE (3.4) and separate t to get

$$\frac{T'(t)}{T(t)} = \frac{\nabla^2\phi(\mathbf{x})}{\phi(\mathbf{x})} = -\lambda$$

using the standard argument for SoV that

$$\text{function of } t = \text{function of } \mathbf{x} \implies \text{both equal to a constant.}$$

This leaves us with **Helmholtz' equation**

$$-\nabla^2\phi = \lambda\phi, \quad \mathbf{x} \in \Omega. \tag{1.3}$$

With boundary conditions on $\partial\Omega$, this is the eigenvalue problem for the (minus) Laplacian operator $-\nabla^2$ in the region Ω, for instance the Dirichlet problem

$$-\nabla^2\phi = \lambda\phi,\ \mathbf{x} \in \Omega, \qquad \phi = 0 \text{ for } \mathbf{x} \in \partial\Omega.$$

There are two main cases, depending on the applicability of further separation:

- Fully separable: We may separate (1.3) completely and seek solutions
$$\phi(x, y) = X(x)Y(y).$$
This works so long as the domain and boundary conditions are 'separable' in the sense that they **reduce to independent BCs for X and Y**. The result is a set of **one-dimensional** eigenvalue problems.
- Non-separable: No further separation is possible. Either the domain Ω is too complicated (no good cooordinate system) or the boundary conditions are not separable

In either case, it is useful to extend the 'L^2 theory' (Sturm-Liouville theory, adjoints, orthogonality and so on) to multiple dimensions.

2. Extending theory to 2D/3D

Let $\Omega \subset \mathbb{R}^n$ be a bounded region (in $\mathbb{R}^2$ or $\mathbb{R}^3$). Define the L^2 inner product on Ω as

$$\langle f, g\rangle = \int_\Omega f(\mathbf{x})g(\mathbf{x})\, dV = \int_\Omega f(x, y)g(x, y)\, dx\, dy. \tag{2.1}$$

Also define the operator (the Laplacian, with the convential minus sign)

$$Lu = -\nabla^2 u.$$

Recall Green's formula (from integrating by parts twice)

$$\int_\Omega v\nabla^2 u = \int_{\partial\Omega} v\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial v}{\partial \mathbf{n}}\, dS + \int_\Omega u\nabla^2 v\, dV$$

$$\implies \langle Lu, v\rangle = \int_{\partial\Omega} u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}}\, dS + \langle u, Lv\rangle. \tag{2.2}$$

In particular, suppose the boundary conditions are one of the standard types,

$$\alpha(x)u + \beta(x)\frac{\partial u}{\partial \mathbf{n}} = 0 \text{ for } \mathbf{x} \in \partial\Omega. \tag{2.3}$$

One common example is if u or $\partial u/\partial \mathbf{n}$ are zero on all points of the boundary.

Adjoint formula: Suppose $L = -\nabla^2$. Then, in general, we have **Green's formula**

$$\langle Lu, v\rangle = \int_{\partial\Omega} u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}}\, dS + \langle u, Lv\rangle$$

and with boundary conditions (2.3), L is self-adjoint:

$$\langle Lu, v\rangle = \langle u, Lv\rangle \text{ for all } u, v \text{ satisfying the BCs (2.3)}$$

2.1. **Eigenfunctions.** The eigenfunctions for L satisfy Helmholtz' equation

$$-\nabla^2\phi = \lambda\phi$$

plus whatever BCs are imposed (assume they are of the standard type (2.3)). We can easily show that eigenfunctions are orthogonal for distinct eigenvalues.

Proof. The proof is familiar. Let ϕ, ψ be eigenfunctions for eigenvalues λ and γ and let $\langle\cdot,\cdot\rangle$ denote the $L^2(\Omega)$ inner product (2.1). Then

$$\lambda\langle\phi,\psi\rangle = \langle L\phi,\psi\rangle = \langle\phi, L\psi\rangle = \gamma\langle\phi,\psi\rangle$$

so $\langle\phi,\psi\rangle = 0$ if $\lambda \neq \gamma$. Note that the boundary terms vanish since ϕ, ψ have hom. BCs. ☐

'Spectral' theory (extending Sturm-Liouville theory): The situation is more complicated than in 1d. For instance, we could have a ring in the plane with periodic boundary conditions, leading to multiple eigenfunctions per eigenvalue.

Aside from this, the result is similar to the theorem for regular SL operators, except that the eigenfunctions are often best indexed by a 'multi-index'

$$\mathbf{k} = (k_1, k_2)$$

(or whatever other symbols, e.g. $\mathbf{k} = (m, n)$). The result is as follows:

Eigenfunctions for $-\nabla^2$ in 2d/3d: Let $Lu = -\nabla^2 u$ with standard hom. BCs (2.3). Then there is a discrete set of eigenvalues $\lambda_{\mathbf{k}}$ and eigenfunctions $\phi_{\mathbf{k}}$ such that

i) The eigenvalues are bounded below (there is a smallest eigenvalue, possibly negative and $\lambda_{\mathbf{k}} \to \infty$ as $|\mathbf{k}| \to \infty$ (i.e. increasing in either index).

ii) Each eigenvalue has finite multiplicity (solution for each distinct eigenvalue is spanned by a finite number of eigenfunctions)

iii) The smallest eigenvalue has multiplicity one (so it has one eigenfunction)

iv) The eigenfunctions are orthogonal in the L^2 inner product (2.1) and are a basis for $L^2(\Omega)$, i.e. every L^2 function in Ω can be written in the form

$$f = \sum_{\mathbf{k}} c_{\mathbf{k}}\phi_{\mathbf{k}}(\mathbf{x}), \quad c_{\mathbf{k}} = \frac{\langle f, \phi_{\mathbf{k}}\rangle}{\langle\phi_{\mathbf{k}}, \phi_{\mathbf{k}}\rangle} = \frac{\int_\Omega f(\mathbf{x})\phi_{\mathbf{k}}(\mathbf{x})dV}{\int_\Omega \phi_{\mathbf{k}}(\mathbf{x})^2\, dV}.$$

With this result, we can then express functions f in terms of the eigenfunctions. For instance, for a region Ω in the plane, we might have a series like

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\phi_{mn}(x,y) \implies c_{mn} = \frac{\langle f, \phi_{mn}\rangle}{\|\phi_{mn}\|^2} = \frac{\int_\Omega f(x,y)\phi_{mn}(x,y)\,dx\,dy}{\int_\Omega \phi_{mn}(x,y)^2\,dx\,dy}$$

where $\|\phi\|^2 = \langle\phi,\phi\rangle$ is the square of the L^2 norm (easier to write than the inner product).

2.2. **Rayleigh quotient.** As in 1d, The Rayleigh quotient can be used to show all eigenvalues are positive. Consider, for instance, the 'Neumann' eigenvalue problem for $-\nabla^2$ in a bounded region $\Omega \subset \mathbb{R}^2$ (or $\mathbb{R}^3$):

$$-\nabla^2\phi = \lambda\phi \text{ in } \Omega\ , \tag{2.4}$$

$$\frac{\partial\phi}{\partial\mathbf{n}} = 0 \text{ for } \mathbf{x} \in \partial\Omega. \tag{2.5}$$

Take the inner product with ϕ of the eigenvalue equation (2.4) and integrate by parts once:

$$\begin{aligned} -\int_\Omega \phi\nabla^2\phi\, dV &= \lambda\int_\Omega \phi^2\, dV \\ \implies -\int_{\partial\Omega}\phi\frac{\partial\phi}{\partial\mathbf{n}}\, dS + \int_\Omega \nabla\phi\cdot\nabla\phi\, dV &= \lambda\int_\Omega\phi^2\, dV. \end{aligned}$$

This gives the general **Rayleigh quotient** formula (noting that $\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$)

$$\lambda = \frac{-\int_{\partial\Omega}\phi\frac{\partial\phi}{\partial\mathbf{n}}\, dS + \int_\Omega \|\nabla\phi\|^2\, dV}{\int_\Omega \phi^2\, dV}. \tag{2.6}$$

Using the Neumann BCs (2.5) we find that the boundary term is zero so

$$\lambda = \frac{\int_\Omega \|\nabla\phi\|^2\, dV}{\int_\Omega\phi^2\, dV} \geq 0$$

Furthermore, it follows from the above that

$$\lambda = 0 \iff \nabla\phi = \vec{0} \text{ for all } \mathbf{x}\in\Omega \iff \phi = \text{const}.$$

Thus we cannot exclude $\lambda = 0$. In fact, it is an eigenvalue with $\phi = $ const. (check explicitly!).

Another example: Consider the heat equation in a half-disk,

$$\begin{aligned} u_t = \nabla^2 u &= \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} \quad \mathbf{x}\in\Omega,\ t>0 \\ u(r,0,t) &= u(r,\pi,t) = 0 \\ -u_r(1,\theta,t) &= ku(1,\theta,t) \end{aligned} \tag{2.7}$$

where k is a constant. This describes a flux ku at the curved boundary. We therefore expect that if $k > 0$ (flux out), the eigenvalues should be positive (solutions decay) but not if $k < 0$.

$$\begin{aligned} \lambda\int_\Omega\phi^2\, dV &= -\int_{\partial\Omega}\phi\frac{\partial\phi}{\partial\mathbf{n}}\, dS + \int_\Omega\|\nabla\phi\|^2\, dV \\ &= -\int_0^\pi \phi\phi_r\Big|_{r=1}\, d\theta + \int_\Omega \|\nabla\phi\|^2\, dV \\ &= k\int_0^\pi (\phi(1,\theta))^2\, d\theta + \int_\Omega\|\nabla\phi\|^2\, dV \end{aligned}$$

so it is true that if $k > 0$ the eigenvalues are ≥ 0. Further, note that $\lambda = 0$ if and only if

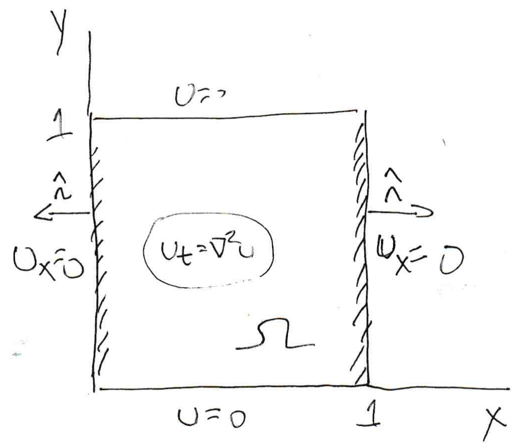
$$\phi(1,\theta) = 0 \text{ for } \theta\in[0,\pi], \quad \nabla\phi = 0 \text{ in } \Omega.$$

it follows that ϕ is constant; but then $\phi = 0$ on the bottom face implies $\phi = 0$ (so no zero eigenvalue) if $k \geq 0$. If $k < 0$ then the argument fails (see subsection 4.2).

3. Squares and disks

3.1. **Heat equation in a square (simplest case).** Let $\Omega = [0,1] \times [0,1]$ be a square of side length one, let $u = u(x,y,t)$ and suppose u solves the heat equation with Dirichlet BCs on the top/bottom faces and Neumann BCs on the left/right faces:

$$\begin{aligned} u_t = \nabla^2 u, \quad (x,y) \in \Omega,\ t > 0 \\ u(x,0,t) = u(x,1,t) = 0, \\ u_x(0,y,t) = u_x(1,y,t) = 0 \\ u(x,y,0) = f(x,y). \end{aligned} \tag{3.1}$$



Since the problem is homogeneous, separation of variables can be used for the whole problem (the inhomogeneous case will be treated after). Either way, we use the homogeneous problem to find the eigenfunctions/values first.

1) Partial separation: Separate out time by looking for a solution of the form

$$u(x,y,t) = T(t)\phi(x,y)$$

to obtain $T(t) = -\lambda T$ and Helmholtz' equation

$$-\nabla^2\phi = \lambda\phi. \tag{3.2}$$

2) Separate the eigenvalue problem: Look for an eigenfunction ϕ of the form

$$\phi(x,y) = X(x)Y(y). \tag{3.3}$$

Plug this into (3.2) to get (divide by XY)

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\lambda$$

$$\implies \frac{X''}{X} = -\lambda - \frac{Y''}{Y} = -\mu^2.$$

Note (shorcut): The 'constant' on the right has been written as μ^2, anticipating that it should be positive. Technically, we need to check later that it is positive (see part (3)).

For the BCs, plug the separated eigenfunction (3.3) into the BCs for u to get

$$u_x(0,y) = u_x(1,y) = 0 \implies X'(0) = X'(1) = 0$$
$$u(x,0) = u(x,1) = 0 \implies Y(0) = Y(1) = 0$$

which verifies that the BCs are separable: they become a pair of BCs for X and Y separately.

3) Solve each eig.problem: The 1d eigenvalue problems have the form

$$X'' + \mu^2 X = 0, \quad X'(0) = X'(1) = 0,$$
$$Y'' + \eta^2 Y = 0, \quad Y(0) = Y(1) = 0.$$

with $\eta^2 = \lambda - \mu^2$. **Note that** from earlier that both problems have no negative eigenvalues, so the use of μ^2 and η^2 is justified.[1]
Solving, we get

$$X_m(x) = \cos m\pi x, \quad \mu_m^2 = m^2\pi^2, \quad m \geq 0,$$
$$Y_n(y) = \sin n\pi y, \quad \eta_n^2 = n^2\pi^2, \quad n \geq 1.$$

Any product of X_m's with Y_n's will give an eigenfunction, so we have

$$\begin{cases} \phi_{mn}(x,y) = X_m(x)Y_n(y) \\ \lambda_{mn} = \mu_m^2 + \eta_n^2 = \pi^2(m^2+n^2) \end{cases} \qquad m \geq 0,\ n \geq 1.$$

Note on eigenvalues: Observe that

$$\lambda_{mn} > \lambda_{01} = \pi^2$$

so the eigenvalues are strictly positive (due to the Dirichlet part of the BCs). The Fredholm alternative then ensures a unique solution (not required but good to check).

4) Solve the PDE: The problem is homogeneous, so continue with SoV. From step (1),

$$T' = -\lambda T \implies T_{mn}(t) = c_{mn}e^{-\lambda_{mn}t}.$$

Thus the solution is

$$\begin{aligned} u(x,y,t) &= \sum_{m,n} T_{mn}(t)\phi_{mn}(x,y) \\ &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_{mn}e^{-\lambda_{mn}t}\phi_{mn}(x,y). \end{aligned}$$

To obtain the coefficients, take the $L^2(\Omega)$ inner product of the IC with ϕ_{mn}:

$$f(x,y) = u(x,y,0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_{mn}\phi_{mn}(x,y)$$

$$\langle \cdot, \phi_{mn}\rangle \implies c_{mn} = \frac{\langle f, \phi_{mn}\rangle}{\|\phi_{mn}\|^2} = \frac{\int_\Omega f(x,y)\phi_{mn}(x,y)\,dx\,dy}{\int_\Omega \phi_{mn}(x,y)^2\,dx\,dy}.$$

[1]We could have written the constants as, say, α, β and obtained $X'' + \alpha X = 0$ and $Y'' + \beta Y = 0$. Then we would solve the eigenvalue problems and conclude that α, β have to be positive, so $\alpha = \mu^2$ and $\beta = \eta^2$. A bit of intuition and/or Rayleigh quotient arguments help to avoid this extra work.

4b) Inhomogeneous procedure: Suppose the PDE/BCs were inhomogeneous, e.g.

$$\begin{aligned} u_t &= \nabla^2 u + s(x,y,t), \quad (x,y) \in \Omega,\ t > 0 \\ u(x,0,t) &= f_1(x,t), \quad u(x,1,t) = f_2(x,t), \\ u_x(0,y,t) &= 0, \quad u_x(1,y,t) = 0 \end{aligned} \tag{3.4}$$

The steps are as in 1d: write the source term as an eigenfunction series, then take $\langle \cdot, \phi_n \rangle$ and use the (self-)adjoint property to convert $\langle Lu, \phi \rangle \to \langle u, L \rangle \phi +$ bdry terms.
The main difference is that the boundary part is an integral over $\partial\Omega$ via Green's formula

To start (after completing Steps (1)-(3) to get the eigenfunctions/values), let

$$k_{mn} = \langle \phi_{mn}, \phi_{mn} \rangle = \|\phi_{mn}\|^2.$$

and write the source in terms as an eigenfunction series:

$$s(x,y,t) = \sum_{m,n} s_{mn}(t)\phi_{mn}(x,y), \qquad s_{mn}(t) = \frac{1}{k_{mn}} \langle s, \phi_{mn} \rangle.$$

Now take the $L^2(\Omega)$ inner product of the PDE with ϕ_{mn}:

$$\langle u_t, \phi_{mn} \rangle = \langle \nabla^2 u, \phi_{mn} \rangle + \langle s, \phi_{mn} \rangle$$

Use Green's formula (2.2) (the self-adjoint calculation for $L = -\nabla^2 u$) to get

$$\begin{aligned} k_{mn} c_{mn}(t) &= \int_{\partial\Omega} \phi_{mn} \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial \phi_{mn}}{\partial \mathbf{n}}\, dS + \langle u, \nabla^2 \phi_{mn} \rangle + k_n s_{mn} \\ &= \int_{\partial\Omega} \phi_{mn} \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial \phi_{mn}}{\partial \mathbf{n}}\, dS - \lambda_{mn} k_{mn} c_{mn}(t) + k_n s_{mn} \end{aligned}$$

so the coefficient ODE has the form

$$c_{mn}(t) + \lambda_{mn} c_{mn}(t) = b_{mn}(t) + s_{mn}(t).$$

To simplify the boundary term, first note that only the $y = 0$ and $y = 1$ parts of the integral are non-zero and that

$$\partial \phi_{mn} / \partial y = n\pi \cos(m\pi x) \cos(n\pi y).$$

Using this we get

$$\begin{aligned} b_{mn}(t) &= \frac{1}{k_{mn}} \int_0^1 \left(f_1(x,t) \frac{\partial \phi_{mn}}{\partial y}(x,0) - f_2(x,t) \frac{\partial \phi_{mn}}{\partial y}(x,1) \right) dx \\ &= \frac{n\pi}{k_{mn}} \int_0^1 (f_1(t) - f_2(t)) \cos(m\pi x)\, dx. \end{aligned}$$

Note that $\hat{n} = -\hat{y}$ at $y = 0$ and $\hat{n} = \hat{y}$ at $y = 1$.

3.2. **Heat equation in a disk (Bessel functions).** Let (r,θ) be polar coordinates and let the domain Ω be a circle of radius 1:

$$\Omega = \{(r,\theta) : r \le 1\}, \quad \partial\Omega = \{(r,\theta) : r = 1\}.$$

For simplicity, consider Dirichlet boundary conditions on $\partial\Omega$:

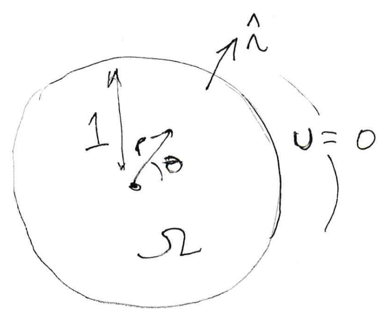
$$\begin{aligned} u_t &= \nabla^2 u, \quad \mathbf{x} \in \Omega,\ t > 0 \\ u(1,\theta,t) &= 0, \quad \theta \in [0,2\pi] \\ u(\mathbf{x},0) &= f(\mathbf{x}) \end{aligned} \tag{3.5}$$

with the additional implied boundary conditions

$$u \text{ is } 2\pi\text{-periodic in } \theta, \quad u \text{ is bounded in } \Omega.$$

Using the polar coordinates formula for ∇^2, the PDE is

$$u_t = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}.$$



1) Partial separation: Look for a separated solution

$$u(r,\theta,t) = T(t)\phi(r,\theta)$$

to obtain $T'(t) = -\lambda T(t)$ and Helmholtz' equation in polar coordinates:

$$\frac{1}{r}(r\phi_r)_r + \frac{1}{r^2}\phi_{\theta\theta} = -\lambda\phi.$$

Plugging into the boundary conditions, we get (writing the periodic boundary condition in the usual way for second-order equations)

$$\phi(R,\theta) = 0, \quad \phi \text{ bounded}, \quad \phi(r,0) = \phi(r,2\pi),\ \phi_\theta(r,0) = \phi_\theta(r,2\pi).$$

2) Separate eig. problems Separate fully:

$$\phi = R(r)g(\theta)$$

$$\implies \frac{1}{rR}(rR')' + \frac{1}{r^2}\frac{g''}{g} = -\lambda$$

to conclude that g''/g must be a constant μ^2. This leads to the eigenvalue problems

$$(g\text{-dir}) \qquad -g'' = \mu^2 g, \quad g(0) = g(2\pi),\ g'(0) = g'(2\pi) \tag{3.6}$$

$$(r\text{-dir}) \qquad -\frac{1}{r}(rR')' + \frac{\mu^2}{r^2}R = \lambda R, \quad R(1) = 0, \quad R \text{ bounded in } [0,1] \tag{3.7}$$

3) Solve the eig. problems: For g we have solved the problem (3.6) before; the previous calculations justify using μ^2 (eigenvalues are non-negative for (3.6)). The result is

$$\mu_m^2 = m^2, \quad g_m = \cos m\theta \text{ or } \sin m\theta, \quad \text{for } n \geq 0$$

noting that for $\mu_0 = 0$, there is only one eigenfunction: $g_0 = 1$. The 'or' indicates these are the two (orthogonal) eigenfunctions for the m-th eigenvalue.

For R, more work is needed. First, it is convenient to rescale by setting

$$\xi = r\sqrt{\lambda}.$$

The result is that (3.7) becomes **Bessel's equation** for $y(\xi) = R(\rho)$:

$$y'' + \frac{1}{\xi}y + \left(1 - \frac{m^2}{\xi^2}\right) y = 0.$$

(*see *notes on special functions*) The general solution for y is

$$y(\xi) = aJ_m(\xi) + bY_m(\xi) \implies R(r) = aJ_m(r\sqrt{\lambda}) + bY_m(r\sqrt{\lambda}).$$

where J_m and Y_m are the regular/modified Bessel functions of order m. We need two facts:

- Y_m is singular at $r = 0$ and J_m is bounded
- $J_m(x)$ has an infinite sequence of positive zeros γ_{mn} with

$$\gamma_{m1} < \gamma_{m2} < \gamma_{m3} < \cdots \to \infty.$$

Applying the BCs and the above facts, we get

$$R \text{ bounded in } [0,1] \implies R = J_m(r\sqrt{\lambda})$$
$$R(1) = 0 \implies J_m(\sqrt{\lambda}) = 0 \implies \lambda_{mn} = \gamma_{mn}^2.$$

Remark (order of solving): There is a chain of dependence; the result of the r-equation (m, n) depends on the θ-equation (m) through the values of λ_{mn}. In short the dependence is

$$\theta \to r \quad \text{(solve } \theta\text{-dir. first, then } r\text{)}.$$

The eigenfunctions/values for the full eigenvalue problem are then

$$\begin{cases} \phi_{mn}(r,\theta) = J_m(r\sqrt{\lambda_{mn}})(\cos m\theta \text{ or } \sin m\theta) \\ \lambda_{mn} = \gamma_{mn}^2 \end{cases}, \quad m \geq 0,\ n \geq 1. \tag{3.8}$$

Note that $\phi_{0n} = J_0(r\sqrt{\lambda_{0n}})$ (the $m = 0$ case has only one g eigenfunction). The 'or' here, as noted earlier, is just short for listing the two eigenfunctions per λ_{mn},

$$\phi_{mn}^a(r,\theta) = J_m(r\sqrt{\lambda_{mn}})\cos m\theta, \quad \phi_{mn}^b = J_m(r\sqrt{\lambda_{mn}})\sin m\theta$$

Practical note: For convenience, it is common (e.g. in in engineering applications) to write the θ part as a complex exponential, so

$$\phi_{mn}(r,\theta) = J_m(r\sqrt{\lambda_{mn}})e^{im\theta}$$

but now the coefficients in the full solution can be complex and we must take the real part to get a real solution.

4) Solve the PDE: The problem is homogeneous, so use SoV:

$$T' + \lambda T = 0 \implies T_{mn} = c_{mn} e^{-\lambda_{mn} t}.$$

The solution is then (with $b_{m0} = 0$ to avoid writing the $n = 0$ case separately)

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} e^{-\lambda_{mn} t} J_m(r\sqrt{\lambda_{mn}})(a_{mn}\cos m\theta + b_{mn}\sin m\theta)$$

Define symbols for the cos and sin eigenfunctions (which are orthogonal to each other):

$$\phi^c_{mn} = J_m(r\sqrt{\lambda_{mn}})\cos n\theta, \quad \phi^s_{mn} = J_m(r\sqrt{\lambda_{mn}})\sin n\theta$$

Taking the $L^2(\Omega)$ inner product of the IC with these eigenfunctions gives the coefficients:

$$u_0(r,\theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(r\sqrt{\lambda_{mn}})(a_n\cos n\theta + b_n \sin n\theta)$$

$$\implies a_{mn} = \frac{\langle u_0, \phi^c_{mn}\rangle}{\|\phi^c_{mn}\|^2}, \quad b_{mn} = \frac{\langle u_0, \phi^s_{mn}\rangle}{\|\phi^s_{mn}\|^2}.$$

In polar coordinates, L^2 inner product is

$$\langle f, g\rangle = \int_\Omega f(r,\theta)g(r,\theta) r\, dr\, d\theta.$$

Connection to SL theory: It is important to note that Sturm-Liouville theory applies to the 1d eigenvalue problems; the result here should be consistent. To check, by taking $\phi_{\ell n}$ and ϕ_{mn}, we find that

$$0 = \langle \phi_{mk}, \phi_{mn}\rangle \implies \int_0^1 J_m(r\sqrt{\lambda_{mk}})J_m(r\sqrt{\lambda_{mn}}) r\, dr = 0 \text{ if } k \neq n.$$

Note the factor of r from the area differential $dA = r\,dr\,d\theta$ here! The eigenvalue problem for R, (3.7), is a Sturm-Liouville problem with weight function $\sigma(r) = r$, so it is consistent. The 1d Sturm-Liouville theory also justifies the calculations done for the 1d eigenvalue problems.

Some analysis: We can show the eigenvalues are all positive using a Rayleigh quotient argument (subsection 2.2; detailed omitted here). It follows that

$$\lim_{t\to\infty} u(r,\theta,t) = \overline{u}(r,\theta)$$

for a steady state $\overline{u}$. It is not hard to check that this steady state is just $\overline{u} = 0$.[2]

Now after a long time, the term with the smallest λ will be much larger than the rest since each term decays exponentially with rate λ_{mn}. We know from the indexing that

$$\lambda_{m1} < \lambda_{m2} < \cdots$$

and one can then check that λ_{01} is the smallest of the λ_{m1}'s. It follows that

$$\max_{(r,\theta)\in\Omega} |u(r,\theta,t)| \sim Ce^{-\lambda_{01} t} \text{ for large } t.$$

This eigenvalue can be found numerically:

$$J_0(\sqrt{\lambda_{01}}) = 0 \implies \lambda_{01} = \gamma_{01}^2 \approx (2.405)^2 = 5.78.$$

[2]Check that zero is a solution; assume the solution to Laplace's equation $\nabla^2 u = 0$ is unique.

3.3. **Application: vibrating membrane.** Consider a thin membrane stretched over a disk of radius 1 (like a drum). The displacement $u(r, \theta, t)$ of the membrane is described by the wave equation:

$$\begin{aligned} u_{tt} &= c^2 \nabla^2 u, \quad \mathbf{x} \in \Omega, \\ u(1, \theta, t) &= 0 \\ u(\mathbf{x}, 0) &= f(\mathbf{x}) \end{aligned} \tag{3.9}$$

with $\nabla^2 u = \frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta}$. The solution via separation of variables is the same as the previous example, except that

$$T'' + c^2 \lambda T = 0 \implies T(t) = a \sin c\sqrt{\lambda} t + b \cos c\sqrt{\lambda} t.$$

The eigenfunctions/values are the same; $\lambda_{mn} = \gamma_{mn}^2$ $(J_m(\gamma_{mn}) = 0)$ and

$$\phi_{mn} = J_m(r\sqrt{\lambda_{mn}})(\cos m\theta \text{ or } \sin m\theta).$$

Assuming no $\cos c\sqrt{\lambda t}$ terms for simplicity, the solution is a superposition of modes

$$\sin(\sqrt{\lambda_{mn}} t) J_m(r\sqrt{\lambda_{mn}})(\cos m\theta \text{ or } \sin m\theta)$$

Each mode of vibration has **nodal sets**

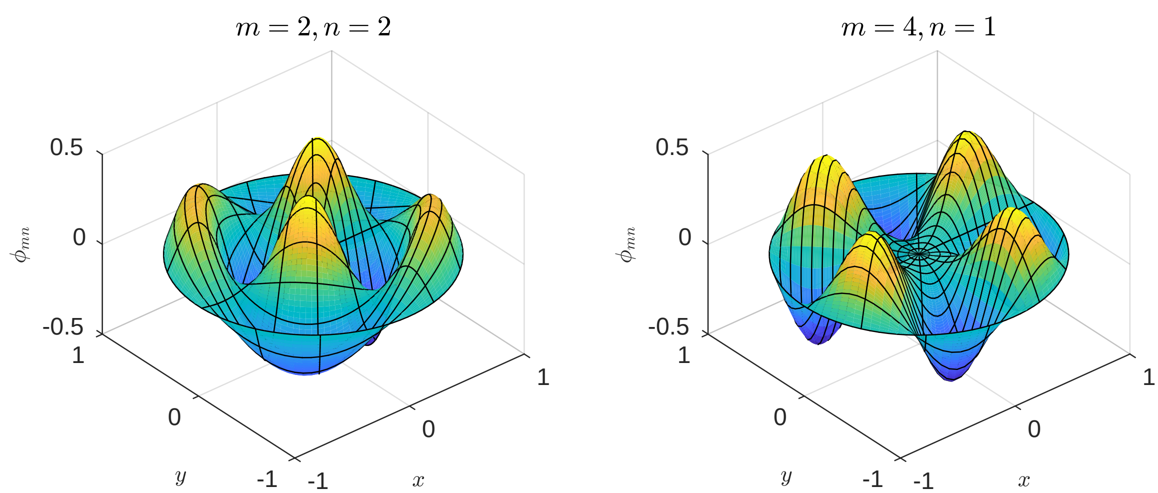
$$S_{mn} = \{(r, \theta) : \text{ the } m, n \text{ mode is zero for all } t\}$$

which is the set of points that do not move (for that mode). For these eigenfunctions, the nodal sets are concentric circles plus rays:

$$S_{mn} = \{\theta = 2\pi k/m, \ k = 0, \cdots, m-1\} \cup \{r = \gamma_{m\ell}/\sqrt{\lambda_{mn}}, \ \ell = 0, \cdots, n-1\}$$

i.e. $r\sqrt{\lambda_{mn}}$ must be one of the zeros of J_m less than the zero used to find λ_{mn}. The nodal sets can be visualized in experiments by placing particles like sand on the disk; they will collect at the nodes. Some modes are shown below; for more examples and nodal sets, see p320-321 of Haberman.

More complicated patterns can be found by imposing more complicated boundary conditions, changing the domain, exciting multiple modes and so on.



4. Cylinders and spheres

4.1. **Heat equation: half-cylinder.** Let (r,θ,z) be cylindrical coordinates and Ω be a half-cylinder of radius 1 and length L (the full cylinder case is similar):

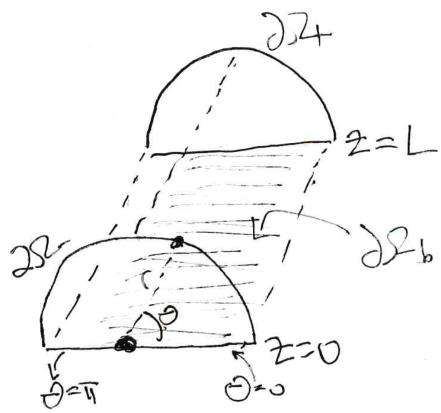
$$\Omega = \{(r,\theta,z) : r \le 1,\ 0 \le z \le L,\ 0 \le \theta \le \pi\}.$$

Decompose $\partial\Omega$ into four parts: the two ends, the flat bottom and circular part:

$$\partial\Omega_- = \{z=0\}, \quad \partial\Omega_+ = \{z = L\}, \quad \partial\Omega_c = \{r = 1, 0 \le \theta \le \pi\}, \quad \partial\Omega_b = \{\theta = 0 \text{ or } \pi\}.$$

Suppose that the cylinder is closed, so the flux through each boundary is zero. This Neumann problem is given by (writing out the $\partial u/\partial \mathbf{n} = 0$ terms on each face separately)

$$\begin{aligned}
u_t &= \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} \quad \mathbf{x} \in \Omega,\ t > 0\\
&u_z(r,\theta,0,t) = u_z(r,\theta,1,t) = 0\\
&u_r(1,\theta,z,t) = 0\\
&u_\theta(r,0,z,t) = u_\theta(r,\pi,z,t) = 0\\
\text{(IC:)}\ &u(r,\theta,z,0) = f(r,\theta,z).
\end{aligned} \tag{4.1}$$



1) Partial separation: Same as before; $u = T(t)\phi(r,\theta,z)$ and

$$T'(t) = -\lambda T, \quad -\nabla^2\phi = \lambda\phi.$$

2) Separated eig. problems: Look for a separated solution, now in three variables:

$$\phi = R(r)g(\theta)h(z).$$

Plug into the Helmholtz equation and divide by Rgh:

$$\frac{1}{r}\frac{(rR')'}{R} + \frac{1}{r^2}\frac{g''}{g} + \frac{h''}{h} = -\lambda.$$

First, this is has the form

$$(\text{function of } r,\theta) + (\text{function of } z) = \text{constant}$$

so it follows that (assuming the constant η^2 is positive)

$$h'' = -\eta^2 h, \qquad \frac{1}{r}\frac{(rR')'}{R} + \frac{1}{r^2}\frac{g''}{g} = -\lambda + \eta^2.$$

From here, separate r and θ (again, assuming the new constant is positive) to get

$$g'' = -\mu^2 g,$$

$$\frac{1}{r}(rR')' - \frac{\mu^2}{r^2}R - \eta^2 R = -\lambda R.$$

After plugging into the boundary conditions we obtain the eigenvalue problems

$$\begin{aligned} -h'' &= \eta^2 h, \quad h'(0) = h'(L) = 0 \\ -g'' &= \mu^2 g \quad g'(0) = g'(\pi) = 0 \\ -\frac{1}{r}(rR')' + \left(\frac{\mu^2}{r^2} + \eta^2\right) R &= \lambda R, \quad R'(0) = R'(1) = 0 \end{aligned}$$

The chain of dependence here is $\theta, z \to r$; the R-equation depends on the g, h equations.

3) Solve the eig. problems: The first two problems are the easy cases:

$$\begin{aligned} h_k &= \cos\frac{k\pi z}{L}, \quad \eta = k\pi/L, \quad k \geq 0 \\ g_m &= \cos m\theta, \quad \mu = m, \quad m \geq 0 \end{aligned}$$

To solve the R equation, note that it is essentially Bessel's equation:

$$R'' + \frac{1}{r}R' + \left((\lambda - \eta^2) - \frac{\mu^2}{r^2}\right) R = 0.$$

Note that if $\lambda < \eta^2$ then the equation is instead the modified Bessel's equation (wrong sign), which would yield no solutions (*see *notes on special functions*).
To rescale the problem, set $\alpha = \lambda - \eta^2 = \lambda - (k\pi/L)^2$ and

$$\xi = r\sqrt{\alpha}$$

so that the equation for $y(\xi) = R(r)$ becomes the standard Bessel equation

$$y'' + \frac{1}{\xi}y' + \left(1 - \frac{\mu^2}{\xi^2}\right) y = 0.$$

Using the bounded condition, we get $y = J_m(\xi)$ so

$$R(r) = J_m(r\sqrt{\lambda - (k\pi/L)^2}).$$

Applying the Neumann boundary condition we get the equation for the eigenvalues:

$$J_m'(\sqrt{\lambda - (k\pi/L)^2}) = 0. \tag{4.2}$$

The relevant properties are:

- $J_m'(0) = 0$; set $\gamma_{m0}' = 0$
- J_m' has a sequence of positive zeros (minima/maxima of J_m) γ_{mn}' where

$$0 = \gamma_{m0}' < \gamma_{m1}' < \gamma_{m2}' < \cdots \to \infty.$$

It follows from (4.2) that the solutions to the R-equation are (for $n \geq 0$)

$$\lambda_{kmn} = (k\pi/L)^2 + (\gamma_{mn}')^2, \quad R_n(r) = J_m(r\sqrt{\lambda_{kmn}}).$$

The eigenvalues/functions for the full problem are then $(k, m, n \geq 0)$

$$\begin{cases} \lambda_{kmn} = (k\pi/L)^2 + (\gamma_{mn}')^2, \\ \phi_{kmn}(r, \theta, z) = \cos\left(\frac{k\pi z}{L}\right)\cos(m\theta) J_m(r\sqrt{\lambda_{kmn}}) \end{cases}, \qquad k, m, n \geq 0. \tag{4.3}$$

Note that R_n depends on the solutions for h_k and g_m through the eigenvalue.

4) Solve the PDE: This part is the same as before.

$$T' = -\lambda T \implies T_{kmn} = c_{kmn} e^{-\lambda_{kmn} t}.$$

The solution is then

$$u(r,\theta,z,t) = \sum_{k,m,n\geq 0} c_{kmn} e^{-\lambda_{kmn} t} \phi_{kmn}(t) = \sum_{\mathbf{k}} c_{\mathbf{k}} e^{-\lambda_{\mathbf{k}} t} \phi_{\mathbf{k}}(t)$$

where $\mathbf{k} = (k,m,n)$ is a multi-index and the sum is over $\{k,m,n \geq 0\}$. The L^2 inner product for the half-cylinder is, in cylindrical coordinates,

$$\langle f, g\rangle = \int_\Omega fg\, dV = \int_0^L \int_0^\pi \int_0^1 f(r,\theta,z) g(r,\theta,z) r\, dr\, d\theta\, dz.$$

Taking this inner product with the IC we get

$$c_{kmn} = \frac{\langle f, \phi_{kmn}\rangle}{\|\phi_{kmn}\|^2} = \cdots .$$

Some analysis: In this case, because of the Neumann problem, $\lambda_{000} = 0$ is an eigenvalue. The eigenfunction is just a constant since

$$J(r\sqrt{\lambda - (k\pi/L)^2}) = J(0) = 1$$

All the other eigenvalues are positive (again, can be shown via the Rayleigh quotient). Thus

$$\lim_{t\to\infty} u(r,\theta,z,t) = c_{000}\phi_{000} = c_{000}.$$

The rate is given by the smallest positive eigenvalue. Since

$$\lambda_{kmn} = (k\pi/L)^2 + (\gamma'_{mn})^2$$

this occurs for $k = 0$ and (by the same argument as for the disk) for $m = 0$ and $n = 1$, so

$$\max_{(r,\theta,z)\in\Omega} |u(r,\theta,z,t)| \sim Ce^{-\lambda_{001} t}, \qquad \lambda_{001} = (\gamma'_{01})^2 \approx 14.7$$

after looking up zeros of J'_m ($\gamma'_{01} \approx 3.83$). The steady state value is just the $(0,0,0)$ coefficient in the solution, which is easy to compute since the eigenfunction is constant:

$$c_{000} = \frac{\int_\Omega f\, dV}{\int_\Omega 1\, dV} = \text{avg. value of } f \text{ in } \Omega.$$

The distribution of heat in the closed container converges to its average value as $t \to \infty$.

4.2. **A case with negative eigenvalues.** Suppose we have the following problem for the heat equation in a half-disk $\Omega = \{(r,\theta) : 0 \le \theta \le \pi,\ r \le 1\}$.

$$\begin{aligned} u_t = \nabla^2 u &= \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} \quad \mathbf{x} \in \Omega,\ t > 0 \\ u(r,0,t) &= u(r,\pi,t) = 0 \\ u_r(1,\theta,t) &= 2u(1,\theta,t) \end{aligned} \tag{4.4}$$

which models a problem with inflow on the curved part into the domain, proportional to u. We observed (example in subsection 2.2) that the Rayleigh quotient does **not** show the eigenvalues are positive, which suggests we must be on the lookout for negative ones.

By similar calculations to the disk with $\phi(r,\theta) = R(r)g(\theta)$ that

$$-g = \mu^2 g,\ g(0) = g(\pi) = 0 \implies g_m = \sin m\theta,\ \mu_m = m$$

$$-\frac{1}{r}(rR')' + \frac{m^2}{r^2}R = \lambda R, \quad R'(1) = 2R(1), \quad R \text{ bounded.}$$

For positive eigenvalues, set $y(\xi) = R(r)$ with $\xi = r\sqrt{\lambda}$ as before to get

$$y'' + \frac{1}{\xi}y + \left(1 - \frac{m^2}{\xi^2}\right)y = 0 \implies R(r) = J_m(r\sqrt{\lambda})$$

and then the boundary condition at 1 yields the equation for positive eigenvalues:

$$\sqrt{\lambda}J_m'(\sqrt{\lambda}) = 2J_m(\sqrt{\lambda}).$$

For **negative eigenvalues**, set $y(\xi) = R(r)$ with $\xi = r\sqrt{-\lambda}$ to get

$$y'' + \frac{1}{\xi}y + \left(1 + \frac{m^2}{\xi^2}\right)y = 0$$

This has modified Bessel functions I, K as solutions. Since K is not bounded,

$$y(\xi) = I_m(\xi) \implies R(r) = I_m(r\sqrt{-\lambda})$$

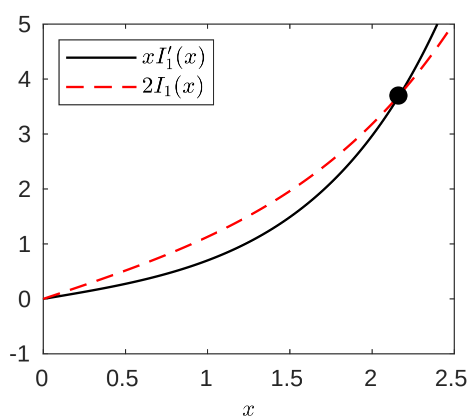
and then the BC at 1 yields (letting $\eta = \sqrt{-\lambda}$)

$$\eta I_m'(\eta) = 2I_m(\eta)$$

which has at most one positive solution. Here, of the possible values $m = 1, 2, \cdots$ only $m = 1$ has a solution, yielding (see plot)

$$\lambda_{1,-1} \approx -2.2, \quad R = I_1(r\sqrt{-\lambda_{1,-1}}).$$

There are no solutions for other values of m (not always true, e.g. $u_r = 4u$ gives a solution for $m = 2$ as well).



The solution to the PDE then has the form (letting $\lambda^* = -\lambda_{1,-1}$)

$$u(r,\theta,t) = c_{1,-1}e^{\lambda^* t}I_1(r\sqrt{\lambda^*})\sin\theta + \sum_{m,n\ge 1} c_{mn}e^{-\lambda_{mn}t}J_m(r\sqrt{\lambda_{mn}})\sin m\theta$$

so the solution **grows** exponentially with rate $\lambda^* \approx 2.2$ (unless the initial condition is orthogonal to this growing mode).

5. Laplace in a cylinder (more modified Bessel functions)

Modified Bessel functions also appear in solving Laplace's equation. To get the idea, we review the rectangle case first.

5.1. **Review (square):** Consider Laplace's equation in a square of side length 1,

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in [0, 1] \times [0, 1]$$

with Dirichlet BCs, homogeneous on the left/right faces:

$$u(0, y) = u(1, y) = 0, \quad u(x, 0) = f_1(x),\, u(x, 1) = f_2(x).$$

Look for a solution $u = X(x)Y(y)$ to get

$$\begin{aligned} -X'' &= \lambda X, \quad X(0) = X(1) = 0 \\ Y'' &= \lambda Y \end{aligned}$$

Note that the signs of λ are **opposite** for the two equations. The eigenvalue problem requires the 'right' sign for λ (positive):

$$\lambda > 0,\ -X'' = \lambda X \implies \text{oscillating solns.} \implies X_n = n\pi x$$

The oscillatory solution is needed to get eigenvalues out of $\sin\sqrt{\lambda} = 0$. The solution is then

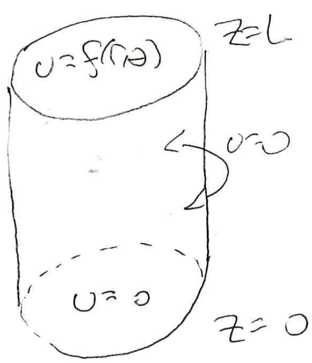
$$u = \sum_{n=1}^{\infty} \underbrace{Y_n(y)}_{\text{coeffs}} \underbrace{X_n(x)}_{\text{eig-funcs}}.$$

The ODE for the ceofficients is not an eigenvalue problem and it has the opposite sign:

$$Y'' = \lambda Y \implies \text{non-oscillating solns.} Y = ae^{\sqrt{\lambda}y} + be^{-\sqrt{\lambda}y}.$$

This suggests that we need to be able to solve the typical ODEs in both the positive eigenvalue cases (for the actual eigenfunctions) and the negative eigenvalue cases (for coefficients).

5.2. **Cylinder case 1:** (r, θ) **eigenfunctions.** Consider a cylinder of height L, radius 1 and Dirichlet BCs, inhomogeneous only at $z = L$:



$$\begin{aligned} 0 = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} &\quad \mathbf{x} \in \Omega, \\ u(1, \theta, z) = 0& \\ u(r, \theta, 0) = 0& \\ u(r, \theta, L) = f(r, \theta)& \end{aligned} \tag{5.1}$$

Since the BCs are inhomogeneous in z (but the others are homogeneous) we choose z as the 'coefficient' direction to separate out and seek a solution

$$u = \sum_{\mathbf{k}} c_{\mathbf{k}}(z)\phi_{\mathbf{k}}(r, \theta).$$

Look for (partially) separated solutions

$$u = h(z)\phi(r,\theta)$$

which gives

$$\text{(coeff ODE.) } h'' - \lambda h = 0, \qquad h(0) = 0$$

$$\text{(eigenvalue prob.) } \frac{1}{r}(r\phi_r)_r + \frac{1}{r^2}\phi_{\theta\theta} = -\lambda\phi, \quad \phi(1,\theta) = 0.$$

The BC at $z = L$ cannot be applied yet since it is inhomogeneous. The eigenvalue problem is the Dirichlet problem in a disk, solved by (3.8) in subsection 3.2:

$$\begin{cases} \lambda_{mn} = \gamma_{mn}^2, \quad \gamma_{mn} = n\text{-th pos. zero. of } J_m \\ \phi_{mn} = J_m(r\sqrt{\lambda_{mn}})(\cos m\theta \text{ or } \sin m\theta) \end{cases}, \qquad m \geq 0,\, n \geq 1.$$

Now solve for h, applying the one homogeneous BC at $z = 0$:

$$h'' - \lambda h = 0, \implies h = a\sinh(z\sqrt{\lambda}) + b\cosh(z\sqrt{\lambda})$$

$$h(0) = 0 \implies h = a\sinh(z\sqrt{\lambda}).$$

Since the BCs are homogeneous in the directions of the eigenfunctions, the full solution is a superposition of the separated solutions:

$$u(r,\theta,z) = \sum_{m\geq 0, n\geq 1} \sinh(z\sqrt{\lambda_{mn}})(a_{mn}\phi_{mn,c} + b_{mn}\phi_{mn,s})$$

Finally, apply the inhomogeneous BC at $z = L$ to get the coefficients:

$$u(r,\theta,1) = f(r,\theta)$$

$$\implies a_{mn} = \frac{1}{\sinh(\sqrt{\lambda_{mn}})} \frac{\langle f, \phi_{mn,c}\rangle}{\|\phi_{mn,c}\|^2}, \cdots \tag{5.2}$$

with a similar formula for b_{mn}. Here the inner product is the L^2 inner product in the cylinder,

$$\langle f, g\rangle = \int_0^L \int_0^{2\pi} \int_0^1 f\,g\,r\,dr\,d\theta\,dz.$$

Note on eigenvalues: The coefficients are all uniquely defined by (5.2) since the eigenvalues are positive. To verify this, we can use the Rayleigh quotient. The eigenvalue problem is

$$-\nabla^2\phi = \lambda\phi, \quad \phi(1,\theta) = 0$$

where ∇^2 is the Laplacian in the disk $D = \{(r,\theta) : r \leq 1\}$. Multiply by ϕ and IBP:

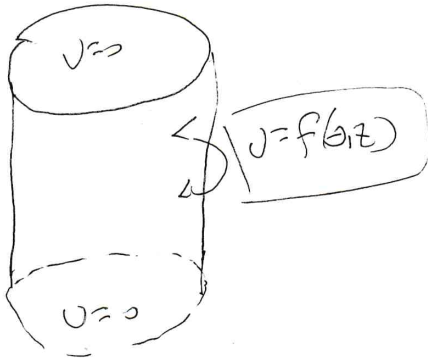
$$\int_D \nabla\phi\cdot\nabla\phi\,dA - \int_{\partial D} \phi\frac{\partial\phi}{\partial r}\,dS = \lambda\int_D \phi^2\,dA.$$

But $\phi = 0$ on ∂D (the boundary of the disk), so

$$\lambda = \frac{\int_D \|\nabla\phi\|^2\,dA}{\int_D \phi^2\,dA} \geq 0.$$

If $\lambda = 0$ then $\nabla\phi = 0$ in all of D, i.e. ϕ is constant. But $\phi = 0$ on the boundary, so this implies $\phi = 0$ everywhere. It follows that $\lambda \neq 0$.

5.3. **Cylinder case 2:** (θ, z) **eigenfunctions.** The process is the same as above, but the 'coefficient' and 'eigenfunction' directions are different. Consider



$$
\begin{aligned}
0 = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} \quad & \mathbf{x} \in \Omega, \\
u(1,\theta,z) = f(\theta,z) & \\
u(r,\theta,0) = 0 & \\
u(r,\theta,L) = 0 &
\end{aligned}
\tag{5.3}
$$

This time, we have inhomogeneous BCs in the r-direction, so we use r as the 'coefficient' direction and look for a solution

$$u = \sum_{\mathbf{k}} c_{\mathbf{k}}(r)\phi_{\mathbf{k}}(\theta, z).$$

Proceeding with SoV again, look for a partially separated solution

$$u = R(r)\phi(\theta, z)$$

and then separate again with $\phi = g(\theta)h(z)$.

Remark (the eigenvalues): The $1/r^2$ factor is a nuisance here. Define the 'θz' part of the Laplacian operator as

$$\tilde{\nabla} u := \frac{1}{r^2}u_{\theta\theta} + u_{zz}.$$

It is not quite true that $-\tilde{\nabla}\phi = \lambda\phi$ (i.e. the 'eigenfunctions' ϕ are not exactly solutions of Helmholtz' equation as in previous examples). Instead, we require that

$$-\tilde{\nabla}\phi = (\frac{\alpha}{r^2} + \beta)\phi$$

The solution procedure does not change; just the 'eigenvalue' λ now has two terms, one with a factor of $1/r^2$ and one without.

The standard procedure yields the 1d problems

$$
\begin{aligned}
-g'' &= \mu^2 g, \quad g \text{ is } 2\pi\text{-periodic} \\
-h'' &= \eta^2 h, \quad h(0) = h(L) = 0 \\
\frac{1}{r}(rR')' &- \left(\eta^2 + \frac{\mu^2}{r^2}\right) R = 0.
\end{aligned}
$$

After solving the θ, z eigenvalue problems (both standard) we get eigenfunctions

$$\phi_{mn}(\theta, z) = (\cos m\theta \text{ or } \sin m\theta)\sin \eta_n z, \qquad m \geq 0,\, n \geq 1$$

where $\eta_n = \frac{n\pi}{L}$. The ODE for R is Bessel's equation:

$$R'' + \frac{1}{r}R' + \left(-\eta_n^2 - \frac{m^2}{r^2}\right) R = 0$$

but with $\lambda = -\eta_n^2$ (negative, unlike before). Now, we transform with

$$\xi = \eta_n r, \quad y(\xi) = R(r)$$

to obtain the modified Bessel's equation

$$y'' + \frac{1}{\xi}y' + \left(-1 - \frac{m^2}{\xi^2}\right) y = 0.$$

Due to the different sign, the solution is instead in terms of modified Bessel functions:

$$R(r) = aK(r\eta_n) + bI(r\eta_n).$$

By the boundedness condition, the K term is zero, so

$$R = I_m(r\eta_n).$$

The full solution is a superposition of the separated solutions (since the BCs are homogeneous in the θ, z directions):

$$u(r,\theta,z) = \sum_{m,n} c_{mn}(r) I_m(r\eta_n)(a_{mn}\phi_{mn,c} + b_{mn}\phi_{mn,s})$$

Now all that is left is the inhomogeneous BC at $r = 1$, which determines the coefficients:

$$f(\theta, z) = u(1, \theta, z)$$
$$\implies a_{mn} = \frac{1}{I_m(\eta_n)} \frac{\langle f, \phi_{mn,c}\rangle}{\|\phi_{mn,c}\|^2}, \cdots .$$

Remark on coefficients: In general, it is true that

$$I(r) \to \infty \text{ as } r \to \infty$$
$$K(r) \to 0 \text{ as } r \to \infty$$

like the $e^{\pm x}$ solutions for Laplace's equation in a square. Since the domain is bounded, the fact that I diverges as $r \to \infty$ is not relevant here. If the domain were **outside** a cylinder, then K would be kept and I discarded instead.

6. Spherical harmonics

6.1. **Radial part (spherical symmetry).** A nice case where the eigenfunctions can be found explicitly. Consider **spherically symmetric** waves in a sphere of radius a with a reflecting boundary (waves reflect off the sphere surface). Let

$$\Omega = \text{sphere of radius } a, \quad u(r,\theta,\phi,t) = u(r,t).$$

Note that if $u = u(r,t)$ then only the r-derivative terms of $\nabla^2 u$ are non-zero. This leaves the 'spherically symmetric' wave equation

$$\begin{aligned} u_{tt} &= \frac{1}{r^2}(r^2 u_r)_r, \quad \mathbf{x} \in \Omega \\ u_r(a,t) &= 0 \\ u(r,0) &= f(r), \;\; u_t(r,0) = g(r). \end{aligned} \tag{6.1}$$

Separated equations: After separating with $u = R(r)T(t)$ we find

$$T''(t) + \lambda T = 0,$$

$$\frac{1}{r^2}(r^2 R')' = -\lambda R, \quad R'(a) = 0, \;\; R \text{ bounded in } [0,a]. \tag{6.2}$$

The eigenvalue ODE is a special case ($m = 0$) of the **spherical Bessel equation**

$$R'' + \frac{2}{r}R' + \left(\lambda - \frac{m(m+1)}{r^2}\right)R = 0, \qquad \text{(with } m = 0).$$

First note that when $\lambda = 0$ there is a bounded solution:

$$\lambda_0 = 0, \qquad R_0 = 1. \tag{6.3}$$

For non-zero eigenvalues, convert to Bessel's equation using the transformation[3]

$$R = y/\sqrt{r}.$$

After some unpleasant calculation, we find that y satisfies

$$y'' + \frac{1}{r}y' + \left(\lambda - \frac{(m+1/2)^2}{r^2}\right)y = 0.$$

Here $m = 0$ so this is Bessel's equation of order $1/2$. Now solve for $\lambda \neq 0$ to get

$$R(r) = c_1 \frac{J_{1/2}(r\sqrt{\lambda})}{\sqrt{r}} + c_2 \frac{Y_{1/2}(r\sqrt{\lambda})}{\sqrt{r}}.$$

By the properties of Bessel functions, $Y_{1/2}/\sqrt{r} \sim r^{-1/2}/r^{1/2} \sim 1/r$ as $r \to 0$ so it is unbounded and $J_{1/2}/\sqrt{r} \sim r^{1/2}/r^{1/2} \sim 1$ is bounded. Thus we must exclude the Y term (but can keep the J term) so

$$R(r) = \frac{J_{1/2}(r\sqrt{\lambda})}{\sqrt{r}}.$$

The positive eigenvalues λ satisfy

$$a\sqrt{\lambda}J'_{1/2}(a\sqrt{\lambda}) = \frac{1}{2}J_{1/2}(a\sqrt{\lambda}).$$

[3] This is one of many specific tricks for converting common differential equations to standard ones. Here, it is the rule that the spherical Bessel equation given m is the regular Bessel equation of order $m + 1/2$.

The solutions to this equation plus the zero eigenvalue (6.3) yield eigenvalues

$$0 = \lambda_0 < \lambda_1 < \cdots$$

and corresponding eigenfunctions

$$R_0(r) = 1, \quad R_n(r) = J_{1/2}(r\sqrt{\lambda_n})/\sqrt{r}, \; n \geq 1.$$

Simplification ('spherical' Bessel function of order zero): The Bessel functions $J_{1/2}$ that show up for this sphere problem are actually nice due to the identity

$$J_{1/2}(x) = \frac{\sin x}{\sqrt{x}}.$$

Using this identity, we get

$$R(r) = \frac{\sin r\sqrt{\lambda}}{r}$$

and the eigenfunctions/eigenvalues $\lambda_n > 0$ are given more simply by

$$R_0 = 1, \; R_n = \frac{\sin r\sqrt{\lambda_n}}{r}, \quad a\sqrt{\lambda} = \tan a\sqrt{\lambda}.$$

Solve the PDE: The solution to the wave equation is then

$$u(r,t) = \sum_{n=0}^{\infty} c_n(t) R_n(r)$$

where

$$c_n(t) = a_n \cos\sqrt{\lambda_n}t + b_n \sin\sqrt{\lambda_n}t.$$

Applying the initial conditions:

$$u(r,0) = f(r) \implies a_n = \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2} = \frac{\int_0^a \left[g(r)\sin(r\sqrt{\lambda_n})/r\right] r^2\,dr}{\int_0^a [\sin(r\sqrt{\lambda_n})/r] r^2\,dr}.$$

$$u_t(r,0) = g(r) \implies \sqrt{\lambda_n} b_n = \frac{\langle g, \phi_n \rangle}{\|\phi_n\|^2}$$

where the inner product is the L^2 inner product **in the sphere**. For functions of r only,

$$\langle f, g \rangle = \int_\Omega f(r)g(r)r^2 \sin\phi\,dr\,d\theta\,d\phi = 4\pi \int_0^a f(r)g(r)r^2\,dr$$

so for instance, the explicit formula for a_n is

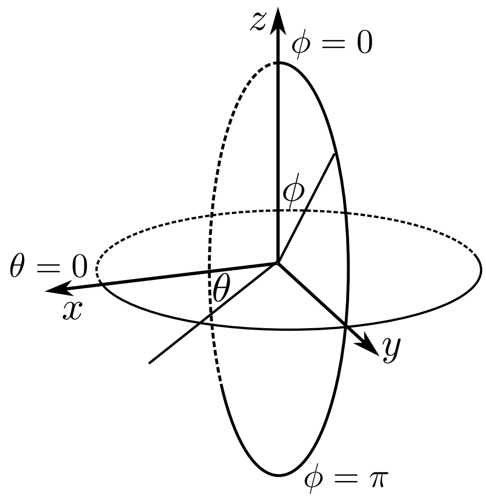
$$a_n = \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2} = \frac{\int_0^a \left[f(r)\sin(r\sqrt{\lambda_n})/r\right] r^2\,dr}{\int_0^a [\sin(r\sqrt{\lambda_n})/r]^2\, r^2\,dr}.$$

Note that we can interpret this as the inner product in $[0,a]$ with weight $\sigma = r^2$. This is the weight function for the 1d SL problem

$$-\frac{1}{r^2}(r^2 R')' = \lambda R$$

that defines the radially symmetric eigenfunctions. That is, our eigenfunctions R_n are orthogonal in the regular L^2 inner product on the sphere, or orthogonal in the $\sigma = r^2$ weighted inner product in the 1d interval $[0,a]$.

6.2. **Surface of a sphere: eigenfunctions.** Now we look for eigenfunctions of $-\nabla^2$ on the **surface** of a sphere of radius 1. The coordinate system is shown below; note that $\theta \in [0, 2\pi]$ is the angle in the x, y plane and $\phi \in [0, \pi]$ is the angle from the $+z$ axis.[4]



In spherical coordinates, the Laplacian is

$$\begin{aligned}\nabla^2 u =& \frac{1}{r^2}(r^2 u_r)_r + \frac{1}{r^2}\left(\frac{1}{\sin^2\phi}u_{\theta\theta} + \frac{1}{\sin\phi}(\sin\phi\, u_\phi)_\phi\right) \\ =& \frac{1}{r^2}(r^2 u_r)_r + \frac{1}{r^2}\nabla_s^2 u\end{aligned}$$

where $\nabla_s^2 u$ is the 'surface' Laplacian for the sphere,

$$\nabla_s^2 u = \frac{1}{\sin^2\phi}u_{\theta\theta} + \frac{1}{\sin\phi}(\sin\phi\, u_\phi)_\phi \tag{6.4}$$

We seek eigenvalues λ and eigenfunctions $Y(\theta, \phi)$ for ∇_s^2 (the choice of letter here is standard; note that Y is **not** a Bessel function). The eigenvalue problem is

$$\begin{gathered}\frac{1}{\sin^2\phi}Y_{\theta\theta} + \frac{1}{\sin\phi}(\sin\phi Y_\phi)_\phi = -\lambda Y \\ Y \text{ is } 2\pi\text{-periodic in } \theta.\end{gathered} \tag{6.5}$$

We solve this by looking for a fully separated eigenfunction:

$$\begin{gathered}Y = g(\theta)h(\phi) \\ \implies \frac{1}{\sin^2\phi}\frac{g''(\theta)}{g(\theta)} + \frac{1}{\sin\phi}\frac{(\sin\phi\, h')'}{h} = -\lambda\end{gathered}$$

which leads to the 1d eigenvalue problems

$$g'' + \mu^2 g = 0, \quad g(0) = g(2\pi), \quad g'(0) = g'(2\pi)$$

$$\frac{1}{\sin\phi}(\sin\phi h')' + (\lambda - \frac{\mu^2}{\sin^2\phi})h = 0. \tag{6.6}$$

The θ-equation is standard, yielding eigenfunctions/values

$$g_m = (\cos m\theta \text{ or } \sin m\theta), \quad \mu_m = m, \; m \geq 0.$$

[4]You'll often encounter the opposite convention in physics.

For the ϕ equation, convert into a standard form with the transformation

$$y(\xi) = h(\phi), \quad \xi = \cos\phi$$

which maps ϕ in $[0,\pi]$ to ξ in $[-1,1]$. The derivative transforms as

$$\frac{\partial}{\partial\phi} = \frac{\partial\xi}{\partial\phi}\frac{\partial}{\partial\xi} = -\sin\phi\frac{\partial}{\partial\xi}$$

so the eigenvalue problem (6.6) then becomes **Legendre's equation** for $y(\xi)$,

$$((1-\xi^2)y')' + (\lambda - \frac{m^2}{1-\xi^2})y = 0 \tag{6.7}$$

Note that this is a singular SL problem for a self-adjoint operator even though there are no BCs (as shown in past homework for $m = 0$). No boundary conditions are needed here. The standard result is that:

- A bounded solution exists if and only if $\lambda = n(n+1)$ where $n \in \mathbb{Z}$ and $n \geq m$
- These solutions are the 'Legendre polynomials'[5] $y(\xi) = P_n^m(\xi)$ for $n = m, m+1, \cdots$

Converting back to the angle ϕ, the eigenvalues/functions for the h-problem (6.6) are

$$\lambda_n = n(n+1), \quad h(\phi) = P_n^m(\cos\phi), \quad 0 \leq m \leq n.$$

Spherical harmonics: The eigenvalues λ and eigenfunctions $Y(\theta,\phi)$ for

$$-\nabla^2 Y = \lambda Y \text{ on the surface of a sphere of radius 1}$$

or explicitly,

$$\frac{1}{\sin^2\phi}Y_{\theta\theta} + \frac{1}{\sin\phi}(\sin\phi Y_\phi)_\phi = -\lambda Y$$

are the **spherical harmonics**

$$\begin{cases} Y_n^m(\theta,\phi) = P_n^m(\cos\phi)(\cos m\theta \text{ or } \sin m\theta) \\ \lambda_{mn} = n(n+1) \end{cases}, \quad 0 \leq m \leq n \tag{6.8}$$

For a given value of n, the eigenvalue $\lambda = n(n+1)$ has multiplicity $n+1$: there are $n+1$ eigenfunctions $P_n^0, \cdots P_n^n$ associated with λ.

Note that for a sphere of radius a, the result is the same except that $\lambda = n(n+1)/a^2$.

6.3. **Spherical harmonics (Laplace in a sphere).** Now we use the spherical harmonics to solve Laplace's equation for $u(r,\theta,\phi)$ in spherical coordinates. As an example, take Ω to be a sphere of radius A and impose a Dirichlet condition at the surface:

$$0 = \frac{1}{r^2}(r^2 u_r)_r + \frac{1}{r^2\sin^2\phi}u_{\theta\theta} + \frac{1}{r^2\sin\phi}(\sin\phi u_\phi)_\phi$$
$$u(1,\theta,\phi) = f(\theta,\phi)$$

Look for a separated solution

$$u = R(r)Y(\theta,\phi)$$

[5]Be careful: they are only polynomials for even values of m despite the name.

and plug into the equation to get

$$0 = \frac{(r^2 R_r)_r}{R} + \frac{\nabla_s^2 Y}{Y}$$

where ∇_s^2 is the 'surface' Laplacian (6.4). It follows that

$$\begin{aligned} -\nabla_s^2 Y &= \lambda Y, \qquad Y \text{ is } 2\pi\text{-periodic in } \theta \\ (r^2 R')' - \lambda R &= 0. \end{aligned}$$

The solutions for Y, from the previous section, are the spherical harmonics Y_m^n with eigenvalues $\lambda = n(n+1)$. The second equation is then

$$r^2 R'' + 2rR' - n(n+1)R = 0$$

which is a **Cauchy-Euler equation** with characteristic polynomial

$$p(\alpha) = \alpha(\alpha-1) + 2\alpha - n(n+1) = \alpha(\alpha+1) - n(n+1).$$

The zeros of this are n and $-(n+1)$ so the general solution is

$$R(r) = c_n r^n + d_n r^{-(n+1)}$$

but by the boundedness condition, $R(r) = r^n$. The full solution is then

$$u(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} r^n P_n^m(\cos\phi)(a_{mn}\cos m\theta + b_{mn}\sin m\theta).$$

Denote by $\langle \cdot, \cdot \rangle$ the L^2 inner product in the sphere,

$$\langle f, g\rangle = \int_\Omega f g\, r^2 \sin\phi \, dr\, d\theta\, d\phi,$$

and let for convenience let c and s subscripts denote the cos and sin parts of the spherical harmonic (so $Y_{n,c}^m = P_m^n(\cos\phi)\cos m\theta$).

Applying the initial condition we have

$$f(A,\theta,\phi) = u(A,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A^n \left(a_{mn} Y_{n,c}^m(\theta,\phi) + b_{mn} Y_{n,s}^m(\theta,\phi)\right)$$

$$\implies a_{mn} = \frac{\langle f, Y_{n,c}^m\rangle}{A^n \|Y_{n,c}^m\|^2}, \qquad b_{mn} = \frac{\langle f, Y_{n,s}^m\rangle}{A^n \|Y_{n,s}^m\|^2}$$

since the eigenfunctions $Y_{n,s}^m$'s and $Y_{n,c}^m$'s are all orthogonal.

Practical note: Often, the spherical harmonics are defined in complex form:

$$Y_n^m = P_n^m(\cos\phi)e^{im\theta}$$

and the complex coefficients of the solution

$$u = \sum_{0\le m\le n} c_{mn} Y_n^m(\theta,\phi)$$

are computed using the complex inner product $\langle f, g\rangle = \int_\Omega f\overline{g}\, dV$.

6.4. **For the heat equation.** Finally, consider the heat equation in a sphere Ω of radius A with Dirichlet boundary conditions:

$$\begin{aligned} u_t &= \nabla^2 u \quad \mathbf{x}\in\Omega,\ t>0 \\ u(A,\theta,\phi,t) &= 0 \\ u(r,\theta,\phi,0) &= f(r,\theta,\phi) \end{aligned} \tag{6.9}$$

where the Laplacian is (rewriting for convenience)

$$\nabla^2 u = \frac{1}{r^2}(r^2 u_r)_r + \frac{1}{r^2\sin^2\phi}u_{\theta\theta} + \frac{1}{r^2\sin\phi}(\sin\phi u_\phi)_\phi \quad = \frac{1}{r^2}(r^2 u_r)_r + \frac{1}{r^2}\nabla_s^2 u$$

Look for a separated solution (keeping the surface part together)

$$u = T(t)R(r)Y(\theta,\phi).$$

Plugging this in and separating we get

$$\frac{T'}{T} = \frac{1}{r^2}(r^2R')' + \frac{1}{r^2}\frac{\nabla_s^2 Y}{Y}.$$

After separation we get an ODE for T and eigenvalue problems for Y and R:

$$T' + \lambda T = 0,$$

$$\frac{1}{\sin^2\phi}Y_{\theta\theta} + \frac{1}{\sin\phi}(\sin\phi Y_\phi)_\phi = -\mu Y, \quad Y \text{ } 2\pi\text{-periodic in } \theta$$

$$\frac{1}{r^2}(r^2R')' + \left(\lambda - \frac{\mu}{r^2}\right)R = 0, \quad R(A) = 0$$

The chain of dependence here is

$$\theta \to \phi \to r$$

with the first two steps covered by the spherical harmonics computed earlier:

$$\mu_n = n(n+1), \quad Y_n^m = P_n^m(\cos\phi)(\cos(m\theta) \text{ or } \sin(m\theta)), \quad 0 \le m \le n$$

The R-equation was seen in the spherically symmetric problem; subsection 6.1).

By the transformation $R = y/\sqrt{r}$ we get Bessel's equation of order $n+1/2$:

$$y'' + \frac{1}{r}y' + \left(\lambda - \frac{(n+1/2)^2}{r^2}\right)y = 0.$$

Using the boundedness condition and the BC at $r = A$, the result is that

$$R_{mnk}(r) = J_{n+1/2}(r\sqrt{\lambda_{mnk}})/\sqrt{r}, \quad \lambda_{mnk} = \gamma_{n+1/2,k}^2/A^2$$

where $\gamma_{\nu,k}$ is the k-th positive zero of J_ν. This gives separated solutions

$$Z_{mnk} = \frac{1}{\sqrt{r}}J_{n+1/2}(r\sqrt{\lambda_{mnk}})Y_n^m$$

and the full solution is then

$$u = \sum_{k=1}^\infty\sum_{n=0}^\infty\sum_{m=0}^n c_{mnk}e^{-\lambda_{mnk}t}R_{mnk}(r)P_n^m(\cos\phi)(a_{mnk}\cos(m\theta) + b_{mnk}\sin(m\theta)).$$

The convergence rate as $t\to\infty$ can be determined by identifying the smallest eigenvalue. This value will be for $m = 0$, yielding $\nu = 1/2$ and $\lambda_{001} = \gamma_{1/2,1}^2/A^2 = \pi^2/A^2$.

7. Example: heat equation on a sphere

Example on the surface of a sphere, with more explicit details on the Legendre polnyomials.

Let $u(\theta, \phi, t)$ solve the heat equation on the surface of a sphere of radius 1:

$$u_t = \nabla_s^2 u, \quad \theta \in [0, 2\pi], \quad \phi \in [0, \pi], \quad t > 0$$

with initial condition

$$u(\theta, \phi, 0) = f(\phi).$$

In general, the solution would be

$$u = \sum_{n=0}^{\infty} \sum_{m=0}^{n} e^{-\lambda_n t} P_n^m(\cos\phi) \left(a_{mn} \cos m\theta + b_{mn} \sin m\theta\right).$$

However, the IC here has **cylindrical symmetry** (no θ dependence), so we expect that u will also have this symmetry, $u = u(\phi, t)$. In particular, note that $f(\phi)$ is orthogonal to the spherical harmonics for $m > 0$ (see box).

Note on orthogonality There are a few ways to view this claim. To check directly, compute the L^2 inner product on the sphere; for $m \geq 1$,

$$\langle f, Y_n^m \rangle = \left(\int_0^{\pi} P_n^m(\cos\phi) f(\phi) \sin\phi \, d\phi \right) \left(\int_0^{2\pi} 1 \cdot (\cos m\theta \text{ or } \sin m\theta) \, d\theta \right) = 0$$

by orthogonality of the θ eigenfunctions in $L^2[0, 2\pi]$.

That is, the inner product separates into the inner products for each 1d problem. Since 1 is an eigenfunction for the θ problem (for $m = 0$), it is orthogonal to all the others ($m \neq 0$) regardless of the ϕ part.

It follows that the solution also only contains these harmonics ($m = 0$), so

$$u = \sum_{n=0}^{\infty} c_n e^{-\lambda_n t} P_n^0(\cos\phi).$$

Let $h_n = P_n^0(\cos\phi)$ for convenience. The h_n's are orthogonal in $[0, \pi]$ with weight $\sigma = \sin\phi$:

$$\langle h_k, h_n \rangle_\sigma = \int_0^{\pi} P_k^0(\cos\phi) P_n^0(\cos\phi) \sin\phi \, d\phi = 0 \text{ for } k \neq n$$

which is the result given by 1d Sturm-Liouville theory. By the same theory, they are also a basis for functions of ϕ on $[0, \pi]$. This, again, is equivalent to using the L^2 inner product on the sphere.

Now from the initial condition,

$$f(\phi) = \sum_{n=0}^{\infty} c_n h_n(\phi) \implies c_n = \frac{\langle f, h_n \rangle_\sigma}{\langle h_n, h_n \rangle_\sigma} = \frac{\int_0^{\pi} f(\phi) P_n^0(\cos\phi) \sin\phi \, d\phi}{\int_0^{\pi} |P_n^0(\cos\phi)|^2 \, d\phi}.$$

Alternatively, one could regard $\langle f, h_n \rangle$ as the L^2 inner product on the sphere (the θ part just gives a factor of 2π that cancels out).