MATH 5410 LECTURE NOTES MULTI-DIMENSIONAL PDES: SPECIAL FUNCTIONS BESSEL FUNCTIONS ETC.

Topics covered

- References for looking up properties/numbers
- Useful properties of Bessel functions
  - First and second kind; behavior for small $|x|$
  - Bounded vs. unbounded solutions; zeros
  - Modified bessel functions (the 'negative eigenvalue' case)
- Legendre functions
- Reminder: Cauchy-Euler equations
- A condensed 'formula sheet' of properties

1. Special functions

When the eigenvalue ODE cannot be solved directly, we can appeal to theory to define solutions, then use other methods to derive relevant information.[1] Such functions arise as solutions to **eigenvalue problems** for PDEs like the heat equation in non-rectangular geometries (cylinder, sphere and so on).

There are several standard references for these 'special functions'. Be **very careful** when using these properties: make sure you are using the standard version of the function.

- The standard (trusted) book is Abramowitz & Stegun, *Handbook of mathematical functions*, available online as a pdf. The information here is correct.
- The standard on-line database is at `https://dlmf.nist.gov/`.
- Most (good) computing software has the standard special functions built in; for instance, `besselj(ord,x)` in Matlab for $J_\nu(x)$. Zero-finders for such functions are easy to find (e.g. on the Matlab file exchange)
- Wolfram has `BesselJ[ord,x]` and `BesselJZero[ord,n]` for J_ν and its zeros (etc.)

1.1. Scaling.
Often, it is necessary to **rescale** or **change variables** to convert the ODE into 'standard form', i.e. one of the standard already-solved problems with known solutions.

It is important to recognize which parameters affect the structure of the solution and which can be 'scaled out'. For example when solving the ODE

$$-y'' = m^2 y$$

we can set $\xi = mx$ (rescale) to get a 'standard' form

$$-y_{\xi\xi} = \xi \implies y = c_1\cos(\xi) + c_2\sin(\xi) = c_1\cos(mx) + c_2\sin(mx)$$

i.e. it suffices to derive properties of the equation (and the solution) for $m = 1$.

[1] Most of the tools come from **asymptotics** and **perturbation methods**; for the standard introduction, see Bender & Orszag's *asymptotic and perturbation methods for scientists and engineers.*

1.2. **Bessel functions.** The most notable example (appearing e.g. in cylindrical coordinates for the r-direction) is **Bessel's equation**

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0. \tag{1.1}$$

The number $\nu \geq 0$ is usually called the **order**. Since it is a second-order linear ODE, the general solution has the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for a pair of linarly independent solutions y_1, y_2. There is a standard choice for this basis (with nice properties). We write the solution as

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

where J_ν and Y_ν are 'Bessel functions' of the 'first' and 'second' kind. It is true that

$$Y_\nu = J_{-\nu} \text{ unless } 2\nu \text{ is an integer}$$

so for the most part, the two basis functions are J_ν and $J_{-\nu}$. The rule here is that $J_{-\nu}$ must be 'adjusted' in the special case where 2ν is an integer (in particular, for $\nu = 0$).

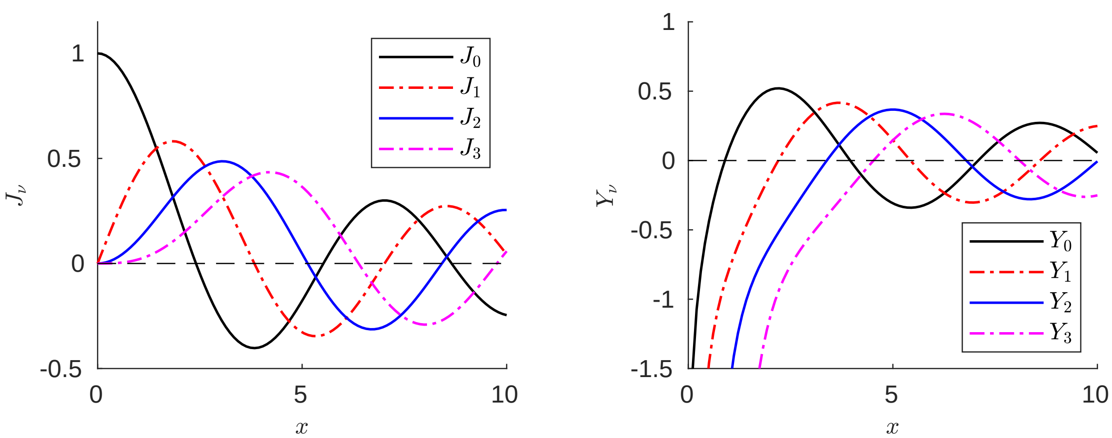
1.3. **Bounded vs. unbounded.** The behavior for small $|x|$ is as follows. If $\nu > 0$ then

$$\begin{aligned} &J_\nu(x) \sim Cx^\nu, \\ &Y_\nu(x) \sim Cx^{-\nu} \text{ as } |x| \to 0, \end{aligned}$$

and for zero-th order,

$$J_0(x) \sim 1, \quad Y_0(x) \sim C \log|x| \text{ as } |x| \to 0.$$

This means that if $\nu \geq 0$ then J_ν is bounded (as $x \to 0$) and Y_ν is **singuar** at $x = 0$, so if we need to apply a boundedness constraint then J_ν should be kept.



1.4. **Equivalent ODEs.** The eigenvalue probem

$$y'' + \frac{1}{x}y' + \left(\lambda - \frac{\nu^2}{x^2}\right)y = 0$$

for $\lambda > 0$ is equivalent to Bessel's equation by rescaling $\xi = x\sqrt{\lambda}$. When $\lambda < 0$ it is instead equivalent to the modified Bessel equation (see next page).

The **spherical Bessel equation**

$$\frac{1}{x^2}(x^2 y')' + \left(1 - \frac{\nu(\nu+1)}{x^2}\right) y = 0. \tag{1.2}$$

is equivalent to Bessel's equation of order $\nu + 1/2$ under the transform $w = x^{1/2} y$, i.e.

$$w(x) = x^{1/2} y(x) \implies w'' + \frac{1}{x} w' + \left(1 - \frac{(\nu+1/2)^2}{x^2}\right) w = 0.$$

Thus the 'spherical Bessel functions' solving (1.2) are

$$\frac{J_{\nu+1/2}}{\sqrt{x}}, \quad \frac{Y_{\nu+1/2}}{\sqrt{x}}$$

1.5. **Oscillation and zeros.** The Bessel function $J_\nu(x)$ (also Y_ν) has the property that

$$J_\nu(x) \text{ has a sequence of positive zeros } 0 < \gamma_{\nu,1} < \gamma_{\nu,2} < \cdots \tag{1.3}$$

due the decaying oscillations of the function for $x > 0$ (see graph). Equations like $J_\nu(a\sqrt{\lambda}) = 0$ can be solved in terms of these zeros (in this case, $(\gamma_n/a)^2$ for $n = 1, 2, \cdots$).

1.6. **Modified Bessel functions.** Note that Bessel's equation requires that the coefficient on the $1/x^2$ term is positive (ν^2). When solving for **negative eigenvalues**, one sometimes runs into the opposite case.
The **modified Bessel equation** has the form

$$y'' + \frac{1}{x} y' - \left(1 + \frac{\nu^2}{x^2}\right) y = 0 \tag{1.4}$$

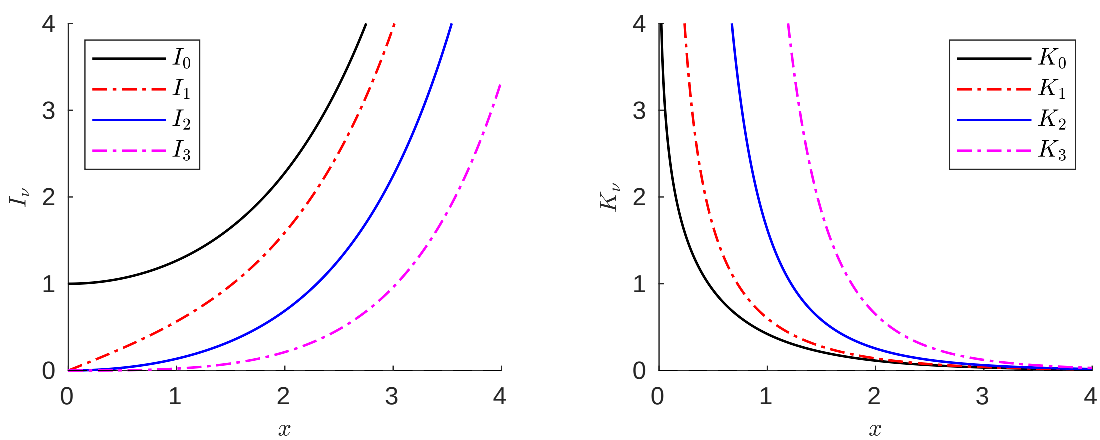
and the two **modified Bessel functions** of the first and second kind are I_ν and K_ν, with

$$y = c_1 I_\nu(x) + c_2 K_\nu(x).$$

Unlike the Bessel functions, the modified versions **do not oscillate** (see plot below):

$$I_\nu(x), K_\nu(x) > 0 \text{ for } x > 0.$$

Precisely, I_ν is increasing (and $I_\nu(0) = 0$); $K_\nu(x)$ is decreasing (and $K_\nu(x) \to \infty$ as $x \searrow 0$).



1.7. **Legendre 'polynomials'.** In spherical coordinates, we need to solve the problem

$$\frac{1}{\sin\phi}(\sin\phi\Phi')' + (\lambda - \frac{m^2}{\sin^2\phi})\Phi = 0.$$

Define the transformation

$$\xi = \cos\phi, \quad y(\xi) = \Phi(\phi).$$

Then

$$\frac{\partial}{\partial\phi} = \sin\phi\frac{\partial}{\partial\xi}, \qquad \sin^2\phi = 1 - \xi^2$$

we obtain the **(generalized) Legendre equation**

$$((1-x^2)y')' + (\lambda - \frac{m^2}{1-x^2})y = 0. \tag{1.5}$$

For a given n, this eigenvalue problem has bounded solutions only for eigenvalues

$$\lambda = n(n+1), \qquad n \text{ an integer } \geq m.$$

The eigenvalues/bounded eigenfunctions are the **Legendre functions**

$$\lambda = n(n+1),\ y = P_m^n(x), \quad n \geq m.$$

It is true that

$$m \text{ even } \implies P_m^n \text{ is a polynomial of deg. } m$$

in which case the function is called an **associated Legendre polynomial**.

Further relations can be derived using **Rodgrigues' formula**, which will not be detailed here (see Haberman, Chapter 7).

Properties of special functions

Bessel functions (first/second kind):

- Equation/solution ($\nu \geq 0$ is a constant: the 'order'):

$$y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2})y = 0 \implies y = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

- If $2\nu \notin \mathbb{Z}$ then $Y_\nu = J_{-\nu}$ (in particular, if $\nu = 0$)
- J_ν is bounded and Y_ν is unbounded (at zero). Precisely, if $\nu > 0$ then

$$J_\nu \sim C_\nu |x|^\nu \text{ as } x \to 0,$$
$$Y_\nu \sim |x|^{-\nu}$$

  and if $\nu = 0$then $J_\nu \sim 1$ and $Y_\nu \sim C \log |x|$.
- Values at zero (J_0 is 'normalized'):

$$J_0(0) = 1,\ J_\nu(0) = 0 \text{ for } \nu > 0,\ J'_\nu(0) = 0 \text{ for } \nu > 1,\ Y_\nu(0) = \infty$$

- Both J_ν and Y_ν oscillate for positive x; in particular

$$J_\nu(x) \implies \text{ positive zeros at } \gamma_{\nu,j} \text{ with } 0 < \gamma_{\nu,1} < \gamma_{\nu,2} < \cdots \to \infty.$$

  $J'_\nu(x)$ and $Y'_\nu(x)$ also have a sequence of positive zeros.

Modified Bessel functions: (Bessel, opposite sign)

$$y'' + \frac{1}{x}y' - (1 + \frac{\nu^2}{x^2})y = 0$$
$$\implies y = c_1 I_\nu(x) + c_2 K_\nu(x)$$

Both K and I are **positive** for $x > 0$ and

$$I_0(0) = 1,\ I_\nu(0) = 0 \text{ for } \nu > 0, \quad K(0) = \infty.$$

Related to J and Y by imaginary arguments, e.g. $I_n(x) = i^{-n} J_n(ix)$.

Bessel eigenvalue problem (both signs for λ):

$$y'' + \frac{1}{x}y' + (\lambda - \frac{\nu^2}{x^2})y = 0, \quad \lambda \geq 0 \implies y = c_1 J_\nu(x\sqrt{\lambda}) + c_2 Y_\nu(x\sqrt{\lambda})$$

$$y'' + \frac{1}{x}y' + (\lambda - \frac{\nu^2}{x^2})y = 0, \quad \lambda < 0 \implies y = c_1 I_\nu(x\sqrt{\lambda}) + c_2 K_\nu(x\sqrt{\lambda})$$

- Can check by other means that $\lambda \geq 0$ (Rayleigh quotient)
- $x(a) = 0$ or $x'(a) = 0 \implies$ only $\lambda \geq 0$ has solutions
- bounded at $x = 0 \implies$ no Y_ν, K_ν term

Spherical Bessel's equation

- Equivalent to Bessel with 'plus 1/2 order'

$$\frac{1}{x^2}(x^2 y')' + (\lambda - \frac{\nu(\nu+1)}{x^2})y = 0 \implies y = c_1 \frac{J_{\nu+1/2}(x\sqrt{\lambda})}{\sqrt{x}} + c_2 \frac{Y_{\nu+1/2}(x\sqrt{\lambda})}{\sqrt{x}}$$

- Special case (order 1/2 exactly):

$$\nu = 0 \implies y = c_1 \frac{\sin x\sqrt{\lambda}}{x} + c_2 \frac{\cos x\sqrt{\lambda}}{x}.$$

Legendre polynomials:

$$((1-x^2)y')' + (\lambda - \frac{n^2}{1-x^2})y = 0.$$

- Bounded solutions **only** for $\lambda = m(m+1)$ where $m \geq n$ is an integer.
- Bounded solution ('Legendre function/polynomial'): $y = P_n^m(x)$. Top letter: 'order' m. Bottom letter: eigenvalue index ($\lambda_n = n(n+1)$).
- If m is even then P_n^m is a polynomial of degree n.
- Given λ_n, total of $n+1$ eigenfunctions $P_n^0, \cdots P_n^n$
- Given order m, sequence $P_m^m, P_{m+1}^m, P_{m+2}^m, \cdots$ for eigenvalues $\lambda_m, \lambda_{m+1}, \cdots$

Other useful facts

Cauchy-Euler equations: For the ODE

$$y'' + \frac{p}{x}y' + \frac{q}{x^2}y'' = 0$$

with p, q, constants, guess a trial solution x^r:

$$x^r \text{ is a solution} \iff 0 = p(r) = r(r-1) + pr + q.$$

Three cases for the solution:

$$\begin{aligned} r_1 \neq r_2, \text{ real} &\implies y = c_1|x|^{r_1} + c_2|x|^{r_2} \\ r_1 = r_2 &\implies y = c_1|x|^{r_1} + c_2|x|^{r_1}\log|x| \\ r = a \pm \omega i &\implies y = c_1|x|^a\cos(\omega\log|x|) + c_2|x|^a\sin(\omega\log|x|). \end{aligned}$$

Indicial polynomial: For the ODE

$$y'' + \frac{p(x)}{x}y' + \frac{q(x)}{x^2}y'' = 0 \tag{1.6}$$

where p, q are continuous, define the **indicial polynomial**

$$f(r) = r(r-1) + p(0)r + q(0).$$

Its roots determine the behavior of the solutions as $x \to 0$:

$$\begin{aligned} r_1 \neq r_2, \text{ real} &\implies y_1 \sim |x|^{r_1},\ y_2 \sim |x|^{r_2} \\ r_1 = r_2 &\implies y_1 \sim |x|^{r_1},\ y_2 \sim |x|^{r_1}\log|x| \\ r = a \pm \omega i &\implies y_1 \sim |x|^a\cos(\log|x|),\ y_2 \sim |x|^a\sin(\log|x|) \end{aligned}$$

In short: near $x = 0$, (1.6) behaves 'like a Cauchy-Euler equation' with $p(x), q(x)$ replaced by $p(0)$ and $q(0)$. For the general theory (messy), see **Frobenius' method.**